Linear Systems

Optimal and Robust Control

Alok Sinha

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To
The Loving Memory of My Mother, Grandparents,
and Uncle
And
My Father

A COMMON MODELING ERROR

Prakrieh kriyamanani
Gunaih karmani sarvasah
Ahamkararavimudhatma
Karta 'ham iti manyate

While all kinds of work are done by the modes of nature, he whose soul is bewildered by the self-sense thinks "I am the doer." (*The Bhagavad Gita: An English Translation*, by S. Radhakrishnan, George Allen and Unwin, 1971, page 143.)

A ROBUST AND OPTIMAL CONTROL ALGORITHM

Karmay eva 'dhikaras te
Ma phalesu kadacana
Ma karmaphalahetur bhur
Ma te sango 'stv akarmani

To action alone hast thou a right and never at all to its fruits; let not the fruits of action be thy motive; neither let there be in thee any attachment to inaction. (*The Bhagavad Gita: An English Translation*, by S. Radhakrishnan, George Allen and Unwin, 1971, page 119.)

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Contents

1 Introduction

1.1 OVERVIEW

This book contains materials on linear systems, and optimal and robust control, and is an outgrowth of two graduate level courses I have taught for many years at The Pennsylvania State University, University Park. The first course is on linear systems and optimal control, whereas the second course is on robust control. The unique feature of this book is that it presents the materials in a theoretically rigorous way while keeping the applications to practical problems in mind. Also, this is the first book containing H_∞ and sliding mode methods together.

The materials on linear systems include controllability, observability, and matrix fraction description. First, the concepts of state feedback control and observers are developed. Next, the optimal control is presented along with stochastic optimal control. Then, the lack of robustness of LQG control is discussed. This is followed by the presentation of robust control techniques. The derivation of H_∞ control theory is developed from the first principle. The sliding mode control of a linear system is presented. Then, it is shown how a blend of sliding mode control and H_∞ methods can enhance the robustness of the system.

One of the objectives is to make the book self contained as much as possible. For example, all the required concepts for stochastic processes are presented so that a student can understand LQG control without much prior background in stochastic processes. The book contains the presentation of theory with practical examples to illustrate the key theoretical concepts and to show their applications to practical problems. At the end of each chapter, exercise problems are included. The use of MATLAB software has been highlighted.

For my course on linear systems and optimal control, I have used the textbook by T. Kailath, *Linear Systems* (Prentice-Hall, 1980). This is a great book, and contains extensive amount of information on linear systems. For my course on robust control, I have used the textbook by K. Zhou and J. C. Doyle, *Essentials of Robust Control* (Prentice-Hall, 1998). This is also an excellent book. However, a typical engineering graduate student in most universities may find the materials in both these textbooks to be highly mathematical and, as a result, find them difficult to follow. Therefore, I have developed mathematical analyses in this book by keeping in view the background of a typical engineering student with a bachelor's degree. For example, the derivation of H_∞ does not require students to learn additional mathematical tools.

I have learned the linear system theory from the textbook of T. Kailath. Therefore, even though I have never met him, I would like to recognize T. Kailath as my virtual

teacher. During my sabbatical at MIT, I was fortunate to interact with M. Athans and attend his course on multivariable control systems, for which G. Stein presented excellent lectures on H_∞ control. I would also like to thank E. F. Crawley for arranging my sabbatical and providing me an opportunity to do research on sliding mode control at the MIT Space Engineering Research Center, where I was lucky to find David Miller who helped me implement my controller on the development model of the Middeck Active Control Experiment (MACE). Lastly, I would like to thank my wife, Hansa, and daughters, Divya and Swarna, for their support.

1.2 CONTENTS OF THE BOOK

In Chapter 2, methods to develop a state space realization from a SISO transfer function are presented, along with the concepts of controllability and observability. The connection between a minimal order of the SISO state space realization and simultaneous controllability and observability is presented. This is followed by the matrix fraction description of the MIMO system, and a method is developed to find a state space realization with the minimal order. Lastly, poles and zeros of a MIMO system are defined.

In Chapter 3, the design of a full state feedback control system is presented for a SISO system along with its impact on poles and zeros of the closed-loop system. Next, the full state feedback control system is presented for a MIMO system. The necessary conditions for the optimal control are then derived and used to develop the linear quadratic (LQ) control theory and the minimum time control.

In Chapter 4, methods to estimate states are developed on the basis of inputs and outputs of a deterministic and a stochastic system. Then, theories and examples of optimal state estimation and linear quadratic Gaussian control are presented.

In Chapter 5, the fundamental concepts of robust control are developed. The robustness of LQ and LQG control techniques developed in Chapter 3 and Chapter 4 are examined. Lastly, theories for H_2, H_∞, and μ techniques are presented along with Bode's sensitivity integrals and illustrative examples.

In Chapter 6, basic concepts of sliding modes are presented along with the sliding mode control of a linear system with full state feedback. Then, it is shown how H_∞ and sliding mode theories can be blended to control an uncertain linear system with full state feedback. Next, the sliding mode control of a deterministic linear system is developed with the feedback of estimated states. Lastly, the optimal sliding Gaussian (OSG) control theory is presented for a stochastic system.

2 State Space Description of a Linear System

First, methods to develop a state space realization from a SISO transfer function are presented, as well as the concepts of controllability and observability. The connection between a minimal order of the SISO state space realization and the simultaneous controllability and observability is presented. This is followed by the matrix fraction description (MFD) of the MIMO system, and a method is developed to find a state space realization with the minimal order. Lastly, poles and zeros of a MIMO system are defined.

2.1 TRANSFER FUNCTION OF A SINGLE INPUT/SINGLE OUTPUT (SISO) SYSTEM

The dynamics of a single input/single output (SISO) linear system can be represented in general by the following nth order differential equation:

$$\begin{gathered}\frac{d^n y}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+a_2\frac{d^{n-2}y}{dt^{n-2}}+\ldots+a_n y= \\ b_0\frac{d^n u}{dt^n}+b_1\frac{d^{n-1}u}{dt^{n-1}}+b_2\frac{d^{n-2}u}{dt^{n-2}}+\ldots+b_n u\end{gathered} \tag{2.1.1}$$

where $y(t)$ is the output and $u(t)$ is the input. The coefficients $a_1, a_2,\ldots, a_n$ and b_0, $b_1,\ldots, b_n$ are system parameters. These parameters are constants for a time-invariant system.

Taking the Laplace transformation of (2.1.1) and setting all initial conditions to be zero,

$$\frac{y(s)}{u(s)} = g(s) \tag{2.1.2}$$

where

$$g(s)=\frac{b_0 s^n+b_1 s^{n-1}+\ldots+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\ldots a_{n-1}s+a_n} \tag{2.1.3}$$

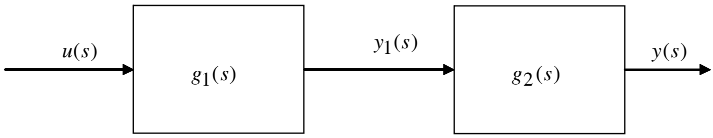

FIGURE 2.1 Cascaded linear systems.

The expression $g(s)$ is described as the transfer function with which the following facts can be attributed:

1. The transfer function is the ratio of Laplace transforms of the output and the input.
2. All initial conditions associated with the input and the output are taken to be zero.
3. The transfer function is only defined for a linear and time-invariant system.

Taking $u(t) = \delta(t)$, the unit impulse function, it can be seen that

$$g(t) = L^{-1}(g(s)) \tag{2.1.4}$$

is the unit impulse response function of the system. In other words, the transfer function of a linear time-invariant system is the Laplace transformation of the unit impulse response of the system.

To appreciate the usefulness of the transfer function approach, consider the system shown in Figure 2.1 in which the output of the first subsystem $y_1(s)$ is the input to the next subsystem. This is a typical situation found in the study or design of a control system. The output $y(s)$ and the input $u(s)$ are related as follows:

$$\frac{y(s)}{u(s)} = g_1(s)g_2(s) \tag{2.1.5}$$

In time domain, the output $y(t)$ is related to the input $u(t)$ via the convolution integral (Kuo, 1995). More specifically,

$$y(t) = \int_0^t g_2(t-\tau)y_1(\tau)d\tau \tag{2.1.6a}$$

and

$$y_1(t) = \int_0^t g_1(t-\tau)u(\tau)d\tau \tag{2.1.6b}$$

Therefore,

$$y(t) = \int_0^t g_2(t-\tau)\int_0^\tau g_1(\tau-\nu)u(\nu)d\nu d\tau \tag{2.1.7}$$

Comparing Equation 2.1.5 and Equation 2.1.7, it is obvious that the input–output relationship in *s*-domain is much simpler than that in time domain.

2.2 STATE SPACE REALIZATIONS OF A SISO SYSTEM

METHOD I

Define $\xi(t)$ such that

$$\frac{d^n\xi}{dt^n} + a_1\frac{d^{n-1}\xi}{dt^{n-1}} + a_2\frac{d^{n-2}\xi}{dt^{n-2}} + \ldots + a_n\xi = u(t) \tag{2.2.1}$$

Assuming that all initial conditions on $y(t)$ are zero and using the principle of superposition (Kailath, 1980), Equation 2.1.1 leads to

$$y(t) = b_0\frac{d^n\xi}{dt^n} + b_1\frac{d^{n-1}\xi}{dt^{n-1}} + b_2\frac{d^{n-2}\xi}{dt^{n-2}} + \ldots\ldots + b_n\xi \tag{2.2.2}$$

Although initial conditions on $y(t)$ and its higher derivatives have been taken to be zero, the Equation 2.2.2 is valid for nonzero initial conditions on $y(t)$ and its higher derivatives. The treatment of nonzero $y(0)$, $\dot{y}(0)$,…, and $\frac{d^n y}{dt^n}(0)$ is related to the **observability** issue and will be discussed later.

Define

$$\begin{aligned} x_1 &= \xi \\ x_2 &= \frac{d\xi}{dt} \\ x_3 &= \frac{d^2\xi}{dt^2} \\ &\vdots \\ x_n &= \frac{d^{n-1}\xi}{dt^{n-1}} \end{aligned} \tag{2.2.3}$$

Using (2.2.1) and (2.2.3), the following n first-order differential equations are obtained as follows:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_3$$

$$\vdots$$

$$\frac{dx_{n-1}}{dt} = x_n$$

$$\frac{dx_n}{dt} = -a_n x_1 - a_{n-1}x_2 - \ldots - a_1 x_n + u(t) \tag{2.2.4}$$

Using Equation 2.2.2, the output $y(t)$ is related to variables defined in (2.2.3) as follows:

$$y(t) = b_0 \frac{dx_n}{dt} + b_1 x_n + b_2 x_{n-1} + \ldots + b_n x_1 \tag{2.2.5}$$

Using (2.2.4),

$$y(t) = (b_n - b_0 a_n)x_1 + (b_{n-1} - b_0 a_{n-1})x_2 + \ldots.. + (b_1 - b_0 a_1)x_n + b_0 u(t) \tag{2.2.6}$$

The system represented by (2.2.4) and (2.2.6) can be realized using n analog integrators as shown in Figure 2.2. The variables $x_1, x_2, x_3, \ldots, x_n$ turn out to be outputs of integrators and are described as **state variables**. In matrix form, Equation 2.2.4 and Equation 2.2.6 are described as follows:

$$\frac{d\mathbf{x}}{dt} = A_c \mathbf{x}(t) + \mathbf{b}_c u(t) \tag{2.2.7}$$

and

$$y(t) = \mathbf{c}_c \mathbf{x}(t) + b_0 u(t) \tag{2.2.8}$$

where

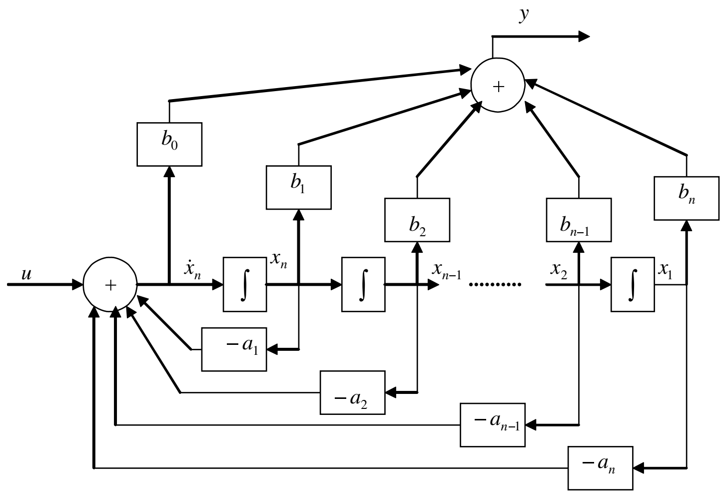

FIGURE 2.2 Analog computer simulation diagram (Method I).

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ -a_n & . & . & . & . & . & -a_2 & -a_1 \end{bmatrix} \tag{2.2.9}$$

$$\mathbf{b}_c = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix} \tag{2.2.10}$$

and

$$\mathbf{c}_c = [b_n - b_0 a_n \quad b_{n-1} - b_0 a_{n-1} \ldots b_1 - b_0 a_1] \tag{2.2.11}$$

Method II

Defining $p = \dfrac{d}{dt}$, Equation 2.1.1 can be written as

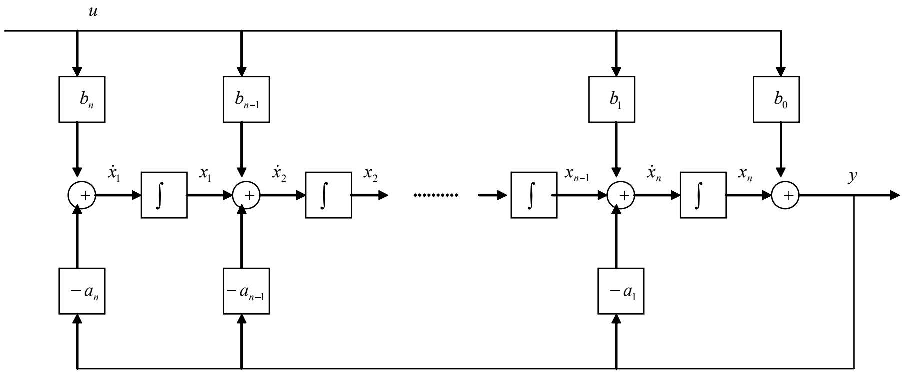

FIGURE 2.3 Analog computer simulation diagram (Method II).

$$p^n(y(t)-b_0u(t))+p^{n-1}(a_1y(t)-b_1u(t))+p^{n-2}(a_2y(t)-b_2u(t))+\ldots+ \\ (a_ny(t)-b_nu(t))=0 \quad (2.2.12)$$

Dividing (2.2.12) by p^n (Wiberg, 1971),

$$y(t)=b_0u(t)-\frac{a_1y(t)-b_1u(t)}{p}-\frac{a_2y(t)-b_2u(t)}{p^2}-\ldots-\frac{a_ny(t)-b_nu(t)}{p^n} \quad (2.2.13)$$

The analog computer simulation diagram corresponding to Equation 2.2.13 is shown in Figure 2.3.

Defining the outputs of integrators as state variables $x_1, x_2, x_3,\ldots, x_n$, the following relationships are obtained:

$$y(t)=x_n(t)+b_0u(t)$$

and

$$\frac{dx_1}{dt}=-a_ny(t)+b_nu(t)=-a_nx_n+(b_n-b_0a_n)u$$

$$\frac{dx_2}{dt}=-a_{n-1}y(t)+x_1(t)+b_{n-1}u(t)=-a_{n-1}x_n+x_1+(b_{n-1}-b_0a_{n-1})u$$

$$\vdots \quad (2.2.14)$$

$$\frac{dx_n}{dt}=-a_1y(t)+x_{n-1}(t)+b_1u(t)=-a_1x_n+x_{n-1}+(b_1-b_0a_1)u$$

Equations 2.2.14 can be put in the matrix form as follows:

$$\frac{d\mathbf{x}}{dt} = A_o\mathbf{x}(t) + \mathbf{b}_o u(t) \tag{2.2.15a}$$

and

$$y(t) = \mathbf{c}_o x(t) + b_0 u(t) \tag{2.2.15b}$$

where

$$A_o = \begin{bmatrix} 0 & 0 & . & . & . & -a_n \\ 1 & 0 & . & . & 0 & -a_{n-1} \\ 0 & 1 & . & . & 0 & -a_{n-2} \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & -a_1 \end{bmatrix} \tag{2.2.16}$$

$$\mathbf{b}_o = \begin{bmatrix} b_n - b_0 a_n \\ b_{n-1} - b_0 a_{n-1} \\ . \\ . \\ . \\ b_1 - b_o a_1 \end{bmatrix} \tag{2.2.17}$$

and

$$\mathbf{c}_o = \begin{bmatrix} 0 & 0 & . & . & 0 & 1 \end{bmatrix} \tag{2.2.18}$$

Notation

Often, a state space realization is represented by the symbol $\{A, \mathbf{b}, \mathbf{c}\}$ by which we mean the following:

$$\dot{\mathbf{x}} = A\mathbf{x}(t) + \mathbf{b}u(t) \tag{2.2.19a}$$

$$y(t) = \mathbf{c}\mathbf{x}(t) \tag{2.2.19b}$$

Example 2.1

$$g(s) = \frac{s^3 + 2s + 5}{s^3 + 2s^2 + 4s + 3} \tag{2.2.20}$$

Here, $b_0 = 1$. From Method I:

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -2 \end{bmatrix} \quad \mathbf{b}_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{c}_c = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \tag{2.2.21}$$

From Method II:

$$A_o = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -4 \\ 0 & 1 & -2 \end{bmatrix} \quad \mathbf{b}_o = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \mathbf{c}_o = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{2.2.22}$$

Properties of State Space Models

1. Duality

For any given realization $\{A, \mathbf{b}, \mathbf{c}\}$ of a system transfer function, there is a dual (Kailath, 1980) realization $\{A^T, \mathbf{c}^T, \mathbf{b}^T\}$.

It is interesting to note that

$$A_o = A_c^T,\ \mathbf{b}_o = \mathbf{c}_c^T \text{, and } \mathbf{c}_o = \mathbf{b}_c^T \tag{2.2.23}$$

Hence, the state space realization obtained by Method I is dual to that obtained by Method II, and *vice versa*.

2. Nonuniqueness of State Space Realization

Consider the following transformation:

$$\mathbf{x}(t) = T\overline{\mathbf{x}}(t) \tag{2.2.24}$$

where T is any nonsingular matrix. For the realization $\{A, \mathbf{b}, \mathbf{c}\}$,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b}u(t) \tag{2.2.25}$$

$$y(t) = \mathbf{c}\mathbf{x}(t) \tag{2.2.26}$$

Substituting (2.2.24) into (2.2.25) and (2.2.26),

$$\frac{d\bar{\mathbf{x}}}{dt} = \bar{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}u(t) \tag{2.2.27}$$

$$y(t) = \bar{\mathbf{c}}\bar{x}(t) \tag{2.2.28}$$

where

$$\bar{A} = T^{-1}AT \ , \quad \bar{\mathbf{b}} = T^{-1}\mathbf{b} \ , \quad \text{and} \quad \bar{\mathbf{c}} = \mathbf{c}T \tag{2.2.29}$$

Hence, a new realization, $\{\bar{A}, \bar{\mathbf{b}}, \bar{\mathbf{c}}\}$, has been obtained. As the choice of the nonsingular matrix T is arbitrary, there are clearly many realizations or nonunique state space realizations corresponding to a given transfer function (Kailath, 1980). In matrix theory, the transformation (2.2.24) is known as similarity transformation, and A and $\bar{A}$ are called **similar matrices**.

2.3 SISO TRANSFER FUNCTION FROM A STATE SPACE REALIZATION

Taking the Laplace transformation of equations (2.2.19),

$$y(s) = \mathbf{c}(sI - A)^{-1}\mathbf{x}(0) + \mathbf{c}(sI - A)^{-1}\mathbf{b}u(s) \tag{2.3.1}$$

For the definition of a transfer function, $\mathbf{x}(0) = 0$. Therefore,

$$g(s) = \frac{y(s)}{u(s)} = \mathbf{c}(sI - A)^{-1}\mathbf{b} \tag{2.3.2}$$

Equation 2.3.2 can be expressed as

$$g(s) = \frac{\mathbf{c}Adj(sI - A)\mathbf{b}}{\det(sI - A)} \tag{2.3.3}$$

where

$$Adj(sI - A) = A^{n-1} + (s + a_1)A^{n-2} + \ldots + (s^{n-1} + a_1 s^{n-2} + \ldots + a_{n-1})I_n \tag{2.3.4}$$

and

$$\det(sI_n - A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1}s + a_n \tag{2.3.5}$$

Further, it can be shown that

$$\mathbf{c}Adj(sI - A)\mathbf{b} = b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_{n-1}s + b_n \tag{2.3.6}$$

The results (2.3.5) and (2.3.6) are also true for $\bar{A}$, $\bar{\mathbf{b}}$, and $\bar{\mathbf{c}}$. Hence, the transfer function is unique for all **similar** state space realizations.

2.4 SOLUTION OF STATE SPACE EQUATIONS

2.4.1 Homogeneous Equation

Consider the solution of Equation 2.2.19 with $u(t) = 0$; i.e.,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \;; \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.4.1}$$

Taking the Laplace transformation of (2.4.1),

$$\mathbf{x}(s) = (sI - A)^{-1}\mathbf{x}_0 \tag{2.4.2}$$

Now,

$$(sI - A)^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} = \frac{1}{s}\left(I + \frac{A}{s} + \frac{A^2}{s^2} + \ldots\right) \tag{2.4.3}$$

Taking the inverse Laplace transformation of (2.4.3),

$$L^{-1}(sI - A)^{-1} = I + At + A^2\frac{t^2}{2} + \ldots; \quad t \geq 0 \tag{2.4.4}$$

The matrix exponential is defined as follows:

$$e^{At} = L^{-1}[(sI - A)^{-1}] \tag{2.4.5}$$

Lastly, taking the inverse Laplace transformation of (2.4.2),

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 \ ; \quad t \geq 0 \tag{2.4.6}$$

The matrix e^{At} is also known as the **state transition matrix** because it takes the initial state $\mathbf{x}_0$ to a state $\mathbf{x}(t)$ in time t.

Properties of e^{At}

1. $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$ (2.4.7)
2. $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$ (2.4.8)
3. If e^{At} is nonsingular, $(e^{At})^{-1} = e^{-At}$ (2.4.9)

EXAMPLE 2.2

$$A = \begin{bmatrix} -4 & 3 \\ -1 & 0 \end{bmatrix} \ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.4.10}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{(s+1)(s+3)} & \frac{3}{(s+1)(s+3)} \\ -\frac{1}{(s+1)(s+3)} & \frac{s+4}{(s+1)(s+3)} \end{bmatrix} \tag{2.4.11}$$

$$L^{-1}[(sI - A)^{-1}] = e^{At} = \begin{bmatrix} 1.5e^{-3t} - 0.5e^{-t} & -1.5e^{-3t} + 1.5e^{-t} \\ 0.5e^{-3t} - 0.5e^{-t} & -0.5e^{-3t} + 1.5e^{-t} \end{bmatrix} \tag{2.4.12}$$

Then, Equation 2.4.6 yields

$$\mathbf{x}(t) = \begin{bmatrix} -1.5e^{-3t} + 1.5e^{-t} \\ -0.5e^{-3t} + 1.5e^{-t} \end{bmatrix}; \quad t \geq 0 \tag{2.4.13}$$

2.4.2 INHOMOGENEOUS EQUATION

Taking the Laplace transformation of (2.2.19),

$$\mathbf{x}(s) = (sI - A)^{-1}\mathbf{x}(0) + (sI - A)^{-1}\mathbf{b}u(s) \tag{2.4.14}$$

Utilizing the definition (2.4.5), the inverse Laplace transformation of (2.4.14) yields

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}\mathbf{b}u(\tau)d\tau \; ; \; t \geq 0 \tag{2.4.15}$$

Example 2.3

Let

$$A = \begin{bmatrix} -4 & 3 \\ -1 & 0 \end{bmatrix} \; \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{; and } u(t) = 1 \text{ for } t \geq 0 \tag{2.4.16}$$

$$e^{A(t-\tau)}\mathbf{b} = \begin{bmatrix} 1.5e^{-3(t-\tau)} - 0.5e^{-(t-\tau)} \\ 0.5e^{-3(t-\tau)} - 0.5e^{-(t-\tau)} \end{bmatrix} \tag{2.4.17}$$

Then

$$\int_0^t e^{A(t-\tau)}\mathbf{b}u(\tau)d\tau = \begin{bmatrix} 1.5e^{-3t}\int_0^t e^{3\tau}d\tau - 0.5e^{-t}\int_0^t e^{\tau}d\tau \\ 0.5e^{-3t}\int_0^t e^{3\tau}d\tau - 0.5e^{-t}\int_0^t e^{\tau}d\tau \end{bmatrix} = \begin{bmatrix} -0.5e^{-3t} + 0.5e^{-t} \\ -\frac{1}{3} - \frac{0.5}{3}e^{-3t} + 0.5e^{-t} \end{bmatrix} \tag{2.4.18}$$

Hence, from (2.4.15),

$$\mathbf{x}(t) = \begin{bmatrix} -2e^{-3t} + 2e^{-t} \\ -\frac{1}{3} - \frac{2}{3}e^{-3t} + 2e^{-t} \end{bmatrix}; \quad t \geq 0 \tag{2.4.19}$$

2.5 OBSERVABILITY AND CONTROLLABILITY OF A SISO SYSTEM

2.5.1 Observability

The state space model $\{A, \mathbf{b}, \mathbf{c}\}$ can be developed on the basis of the governing differential Equation 2.1.1 or the transfer function (2.1.3). If the initial conditions on the output and its derivatives are nonzero, how would one get correct values of initial states? To answer this question, consider the output Equation 2.2.19b,

$$y(t) = \mathbf{cx}(t)$$

Differentiating it and using (2.2.19a),

$$\frac{dy}{dt} = \mathbf{c}A\mathbf{x}(t) + \mathbf{cb}u(t) \tag{2.5.1}$$

Continuing this differentiation process,

$$\begin{aligned} \frac{d^2y}{dt^2} &= \mathbf{c}A^2\mathbf{x}(t) + \mathbf{c}A\mathbf{b}u(t) + \mathbf{cb}\frac{du}{dt} \\ &\vdots \\ \frac{d^{n-1}y}{dt^{n-1}} &= \mathbf{c}A^{n-1}\mathbf{x}(t) + \mathbf{c}A^{n-2}\mathbf{b}u(t) + \mathbf{c}A^{n-3}\mathbf{b}\frac{du}{dt} + \ldots + \mathbf{cb}\frac{d^{n-1}u}{dt^{n-1}} \end{aligned} \tag{2.5.2}$$

Representing (2.5.1) and (2.5.2) in matrix form,

$$\mathbf{y}(t) = \Theta\mathbf{x}(t) + \mathrm{T}\mathbf{u}(t) \tag{2.5.3}$$

where

$$\mathbf{y}(t) = \begin{bmatrix} y & \frac{dy}{dt} & . & . & \frac{d^{n-1}y}{dt^{n-1}} \end{bmatrix}^T \tag{2.5.4}$$

$$\Theta = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}A \\ \mathbf{c}A^2 \\ . \\ . \\ . \\ \mathbf{c}A^{n-1} \end{bmatrix} \tag{2.5.5}$$

$$\mathbf{u}(t) = \begin{bmatrix} u & \frac{du}{dt} & . & . & \frac{d^{n-1}u}{dt^{n-1}} \end{bmatrix}^T \tag{2.5.6}$$

$$T = \begin{bmatrix} 0 & 0 & 0 & . & . & 0 \\ \mathbf{c}b & 0 & 0 & . & . & 0 \\ \mathbf{c}Ab & \mathbf{c}b & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mathbf{c}A^{n-2}b & . & . & . & \mathbf{c}b & 0 \end{bmatrix} \quad (2.5.7)$$

From (2.5.3),

$$\Theta\mathbf{x}(0) = \mathbf{y}(0) - \mathbf{Tu}(0) \quad (2.5.8)$$

Therefore,

$$\mathbf{x}(0) = \Theta^{-1}\mathbf{y}(0) - \Theta^{-1}\mathbf{Tu}(0) \quad (2.5.9)$$

provided the matrix Θ is nonsingular. Hence, the condition for initial states to be calculated from the initial values of input, output, and their derivatives is that the matrix Θ should be nonsingular. The matrix Θ is known as the *observability matrix*.

The solution (2.4.15) of the state space equation indicates that $\mathbf{x}(t)$ can be calculated for any $u(t)$ if the initial value $\mathbf{x}(0)$ is known.

Observability of State Space Realization Using Method I

Using (2.5.5),

$$\Theta = \begin{bmatrix} b_n - b_0 a_n & . & . & . & . & b_1 - b_0 a_1 \\ -a_n(b_1 - b_0 a_1) & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \quad (2.5.10)$$

The determinant and hence the singularity of the observability matrix will depend on system parameters. Hence, the realization may or may not be observable.

Observability of State Space Realization Using Method II

Using (2.5.5),

$$\Theta = \begin{bmatrix} 0 & 0 & 0 & . & . & 1 \\ 0 & 0 & . & . & 1 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 0 & 0 & . & . & 0 \end{bmatrix} \tag{2.5.11}$$

It can be easily seen that

$$\det \Theta = -1 \text{ or } +1 \tag{2.5.12}$$

Therefore, Θ is nonsingular irrespective of the system parameter values. Hence, this state space realization is always observable.

Example 2.4

Consider a spring-mass-damper system Figure 2.4, with the following differential equation:

$$m\ddot{x} + \alpha\dot{x} + \beta x = f(t) \tag{2.5.13}$$

where $f(t)$ is the applied force. Dividing (2.5.13) by m,

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{\beta}{m}x = \frac{f(t)}{m} = u(t) \tag{2.5.14}$$

Defining states as

$$x_1 = x \quad \text{and} \quad x_2 = \dot{x} \tag{2.5.15a,b}$$

state equations are

$$\dot{\mathbf{x}} = A\mathbf{x}(t) + \mathbf{b}u(t) \tag{2.5.16}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 \\ -\beta/m & -\alpha/m \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.5.17}$$

Case I: Position Output

If the position of the mass, x, is measured by a sensor,

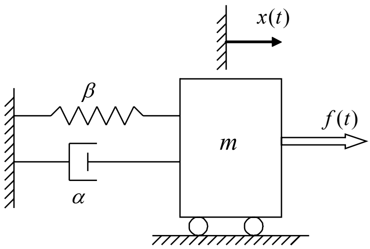

FIGURE 2.4 A spring-mass-damper system.

$$y(t) = \mathbf{c}\mathbf{x}(t) \tag{2.5.18}$$

where

$$\mathbf{c} = [1 \quad 0] \tag{2.5.19}$$

Then the observability matrix is

$$\Theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.5.20}$$

Hence, the state space realization is observable for all values of spring stiffness and damping constant.

Case II: Velocity Output

If the velocity of the mass, $\dot{x}$, is measured by a sensor,

$$y(t) = \mathbf{c}\mathbf{x}(t) \tag{2.5.21}$$

where

$$\mathbf{c} = [0 \quad 1] \tag{2.5.22}$$

Then the observability matrix is

$$\Theta = \begin{bmatrix} 0 & 1 \\ -\beta / m & -\alpha / m \end{bmatrix} \tag{2.5.23}$$

Therefore,

$$\det \Theta = \beta / m \tag{2.5.24}$$

Hence, the state space realization is not observable if the spring stiffness is zero, i.e., if there is no spring in the system. This is an interesting result because the displacement can be obtained by integrating velocity:

$$x_1(t) = x_1(0) + \int_0^t x_2(t)dt \tag{2.5.25}$$

However, the initial condition $x_1(0)$ is needed. In the absence of a spring, the loss of observability implies that $x_1(0)$ cannot be obtained, and the position of the system cannot be observed.

2.5.2 Controllability

The linear differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b}u(t) \tag{2.5.26}$$

is controllable if and only if it can be transferred from *any* initial state to *any* final state in a *finite* time.

Therefore, for the linear and time-invariant system (2.5.26), the issue is to find $u(t)$; $t_0 \le t \le t_f$, which will take the system from any $\mathbf{x}(t_0)$ to any $\mathbf{x}(t_f)$ in a finite time $t_f - t_0$.

Using the solution of (2.5.26),

$$\mathbf{x}(t_f) = e^{A(t_f - t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t_f} e^{A(t_f - \tau)}\mathbf{b}u(\tau)d\tau \tag{2.5.27}$$

This is an integral equation because the unknown function $u(.)$ appears under an integral sign. Define a matrix $P(t_f, t_0)$ as follows:

$$P(t_f, t_0) = \int_{t_0}^{t_f} e^{A(t_f - \tau)}\mathbf{b}\mathbf{b}^T e^{A^T(t_f - \tau)} d\tau \tag{2.5.28}$$

This matrix $P(t_f, t_0)$ is known as the *controllability Gramian*.

The solution of (2.5.27) is found (Friedland, 1985) to be

$$u(\tau) = \mathbf{b}^T e^{A^T(t_f-\tau)} P^{-1}(t_f, t_0)[\mathbf{x}(t_f) - e^{A(t_f-t_0)}\mathbf{x}(t_0)] \tag{2.5.29}$$

This result can be verified by substituting (2.5.29) into (2.5.27). Hence, the control input $u(.)$ can be found if and only if the matrix $P(t_f, t_0)$ is nonsingular.
Defining a new variable $\nu = t_f - t_0$,

$$P(t_f, t_0) = \int_0^{t_f-t_0} e^{A\nu}\mathbf{b}\mathbf{b}^T e^{A^T\nu} d\nu \tag{2.5.30}$$

Note that $P(t_f, t_0) = P(t_f - t_0)$; i.e., $P(t_f, t_0)$ only depends on $t_f - t_0$ for a linear and time-invariant system.

A test for linear independence of any functions $\{\ell_i(\tau);\ 0 \le \tau \le t_f - t_0, i = 1, 2, ..., n\}$ is that their Gramian matrix G (Appendix A) is nonsingular where

$$G = \int_0^{t_f-t_0} \underline{\ell}(\tau)\underline{\ell}^T(\tau) d\tau \quad \text{and} \quad \underline{\ell}^T(\tau) = \begin{bmatrix} \ell_1(\tau) & . & . & . & \ell_n(\tau) \end{bmatrix} \tag{2.5.31}$$

Equation 2.5.30 can be written as

$$P(t_f, t_0) = \int_0^{t_f-t_0} e^{A\nu}\mathbf{b}(e^{A\nu}\mathbf{b})^T d\nu \tag{2.5.32}$$

Comparing (2.5.31) and (2.5.32), nonsingularity of the matrix P implies that elements of the vector $e^{A\nu}\mathbf{b}$ are linearly independent over $(0, t_f - t_0)$. Therefore, the pair $\{A, \mathbf{b}\}$ is controllable over $(0, t_f - t_0)$ if and only if the elements of the vector $e^{A\nu}\mathbf{b}$ are linearly independent.

Example 2.5

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.5.33}$$

Find e^{At} :

$$sI - A = \begin{bmatrix} s & -1 \\ 4 & s \end{bmatrix} \tag{2.5.34}$$

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s}{s^2+4} & \dfrac{1}{s^2+4} \\ \dfrac{-4}{s^2+4} & \dfrac{s}{s^2+4} \end{bmatrix} \tag{2.5.35}$$

$$e^{At} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} \cos 2t & 0.5 \sin 2t \\ -2 \sin 2t & \cos 2t \end{bmatrix} \tag{2.5.36}$$

Find $e^{A\nu}\mathbf{b}$:

$$e^{A\nu}\mathbf{b} = \begin{bmatrix} \cos 2\nu & \dfrac{\sin 2\nu}{2} \\ -2 \sin 2\nu & \cos 2\nu \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{\sin 2\nu}{2} \\ \cos 2\nu \end{bmatrix} \tag{2.5.37}$$

From the controllability Gramian,

$$e^{A\nu}\mathbf{b}\mathbf{b}^T e^{A^T\nu} = \begin{bmatrix} \dfrac{\sin^2 2\nu}{4} & \dfrac{\sin 2\nu \cos 2\nu}{2} \\ \dfrac{\sin 2\nu \cos 2\nu}{2} & \cos^2 2\nu \end{bmatrix} \tag{2.5.38}$$

Let

$$t_0 - t_f = \Delta t \tag{2.5.39}$$

Hence,

$$P(t_f, t_0) = \int_0^{\Delta t} e^{A\nu}\mathbf{b}\mathbf{b}^T e^{A^T\nu} d\nu$$

$$= \begin{bmatrix} \dfrac{\Delta t - (\sin(4\Delta t))/4}{8} & \dfrac{-\cos(4\Delta t) + 1}{16} \\ \dfrac{-\cos(4\Delta t) + 1}{16} & \dfrac{\Delta t + (\sin(4\Delta t))/4}{2} \end{bmatrix} \tag{2.5.40}$$

$$\det(P(\Delta t)) = \frac{(\Delta t)^2 - 0.25 \sin^2(2\Delta t)}{16} \tag{2.5.41}$$

Theorem

The system is completely controllable if and only if the **controllability matrix**

$$C = [\mathbf{b} \quad A\mathbf{b} \quad A^2\mathbf{b} \quad . \quad . \quad A^{n-1}\mathbf{b}] \tag{2.5.42}$$

is nonsingular.

Proof

Step I: Consider that the matrix P, Equation 2.5.28, is singular. In this case, elements of the vector $e^{A\nu}\mathbf{b}$ are linearly dependent. As a result, there exists a nonzero vector q such that

$$z(t) = \mathbf{b}^T e^{A^T t}\mathbf{q} = 0 \quad \text{for } 0 \le t \le t_f - t_0 \tag{2.5.43}$$

Differentiating (2.5.43),

$$\begin{aligned}
\dot{z}(t) &= \mathbf{b}^T A^T e^{A^T t}\mathbf{q} = 0 \\
\ddot{z}(t) &= \mathbf{b}^T (A^T)^2 e^{A^T t}\mathbf{q} = 0 \\
&\vdots \\
\frac{d^{n-1}}{dt^{n-1}} z(t) &= \mathbf{b}^T (A^T)^{n-1} e^{A^T t}\mathbf{q} = 0
\end{aligned} \tag{2.5.44}$$

Putting (2.5.43) and (2.5.44) in matrix form,

$$\begin{bmatrix} \mathbf{b}^T \\ \mathbf{b}^T A^T \\ . \\ . \\ \mathbf{b}^T (A^T)^{n-1} \end{bmatrix} e^{A^T t}\mathbf{q} = 0 \tag{2.5.45}$$

Using the definition (2.5.42),

$$C^T e^{A^T t}\mathbf{q} = 0 \tag{2.5.46}$$

Let

$$C^T = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & . & . & \mathbf{r}_n \end{bmatrix} \quad (2.5.47)$$

and

$$e^{A^T t}\mathbf{q} = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ . \\ . \\ \alpha_n(t) \end{bmatrix} \quad (2.5.48)$$

Hence, from (2.5.46) – (2.5.48),

$$\alpha_1(t)\mathbf{r}_1 + \alpha_2(t)\mathbf{r}_2 + \ldots + \alpha_n(t)\mathbf{r}_n = 0 \quad (2.5.49)$$

Hence, columns of C^T are linearly dependent. As $rank(C) = rank(C^T)$, the matrix C is singular.

Step II: Consider that C is singular. We will show that the matrix P, Equation 2.5.28, is singular in this case.

$$e^{A\nu} = I + A\nu + \frac{A^2\nu^2}{2!} + \ldots + \frac{A^{n-1}\nu^{n-1}}{(n-1)!} + \frac{A^n\nu^n}{n!} + \ldots \quad (2.5.50)$$

Recall the Cayley–Hamilton theorem:

$$A^n = -a_1A^{n-1} - a_2A^{n-2} - \ldots - a_nI_n \quad (2.5.51)$$

where

$$\det(sI - A) = s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_n \quad (2.5.52)$$

Using (2.5.50) and (2.5.51),

$$e^{A\nu} = If_1(\nu) + Af_2(\nu) + \ldots + A^{n-1}f_n(\nu) \quad (2.5.53)$$

where $f_1(\nu), \ldots, f_n(\nu)$ are functions of ν.

Hence,

$$e^{A\nu}\mathbf{b} = C\mathbf{f}(\nu) \tag{2.5.54}$$

where

$$\mathbf{f}(\nu) = \begin{bmatrix} f_1(\nu) \\ f_2(\nu) \\ . \\ . \\ f_n(\nu) \end{bmatrix} \tag{2.5.55}$$

Therefore, from (2.5.30),

$$P(t_f, t_0) = C[\int_0^{t_f - t_0} \mathbf{f}(\nu)\mathbf{f}^T(\nu)d\nu]C^T \tag{2.5.56}$$

Note that $rank(AB) \le \min\{rank(A), rank(B)\}$. Therefore, the rank of $P(t_f, t_0)$ is going to be less than n because it has been assumed that the matrix C is singular.

Controllability of the State Space Realization Obtained Using Method I

Using (2.5.42),

$$C = \begin{bmatrix} 0 & 0 & 0 & . & . & 1 \\ 0 & 0 & . & . & 1 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & 1 & -a_1 & . & . & . \\ 1 & -a_1 & -a^2 + a_1 & . & . & . \end{bmatrix} \tag{2.5.57}$$

It can be shown that

$$\det C = -1 \text{ or } +1 \tag{2.5.58}$$

Therefore, the state space realization obtained using Method I is always controllable.

Controllability of the State Space Realization Obtained Using Method II

Using (2.5.42),

$$C^T = \begin{bmatrix} b_n - b_0 a_n & . & . & . & . & b_1 - b_0 a_1 \\ -a_n(b_1 - b_0 a_1) & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \quad (2.5.59)$$

The determinant and hence the singularity of the controllability matrix will depend on system parameters. Hence, the realization may or may not be controllable.

Example 2.6

Consider the tank system shown in Figure 2.5, where u_1 and u_2 are input mass flow rate to tank 1 and tank 2, respectively. Let p_1 and p_a be the pressure at the left end of the pipe and the atmospheric pressure, respectively. Then,

$$p_1 - p_a = \rho g h_1 \quad (2.5.60)$$

where ρ is the fluid density.

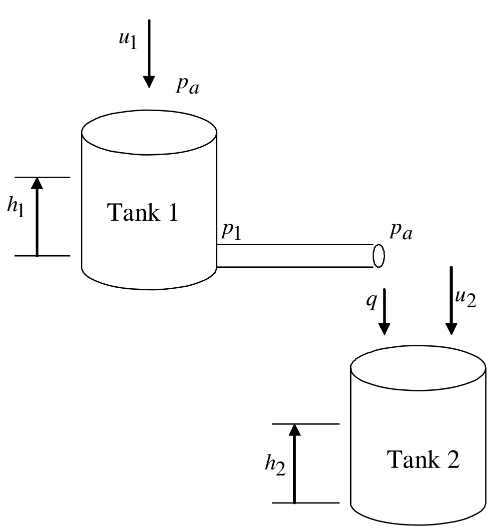

FIGURE 2.5 A tank-pipe system.

Considering a laminar pipe flow,

$$q = \frac{p_1 - p_a}{R_L} \tag{2.5.61}$$

where q and R_L are the volumetric flow rate through the pipe and the flow resistance, respectively. From the law of conservation of mass,

$$\rho A_1 \frac{dh_1}{dt} = u_1 - \rho q \tag{2.5.62}$$

$$\rho A_2 \frac{dh_2}{dt} = u_2 + \rho q \tag{2.5.63}$$

where A_1 and A_2 are cross-sectional areas of tanks 1 and 2, respectively. Using (2.5.60) and (2.5.61),

$$\begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = \begin{bmatrix} -\alpha_1 & 0 \\ \alpha_2 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} u_2 \tag{2.5.64}$$

where

$$\alpha_1 = \frac{\rho g}{A_1 R_L}, \quad \alpha_2 = \frac{\rho g}{A_2 R_L}, \quad \beta_1 = \frac{1}{\rho A_1}, \quad \text{and} \quad \beta_2 = \frac{1}{\rho A_2} \tag{2.5.65}$$

The controllability matrix with respect to input u_1 is

$$C = \begin{bmatrix} \beta_1 & -\alpha_1\beta_1 \\ 0 & \alpha_2\beta_1 \end{bmatrix} \tag{2.5.66}$$

Hence, the system (2.5.64) is controllable with respect to the input u_1. It is obvious that both states h_1 and h_2 are influenced by the input u_1. And, the controllability matrix with respect to the input u_2 is

$$C = \begin{bmatrix} 0 & 0 \\ \beta_2 & 0 \end{bmatrix} \tag{2.5.67}$$

Hence, the system (2.5.64) is not controllable with respect to the input u_2. It is obvious that the state h_1 cannot be influenced by the input u_2.

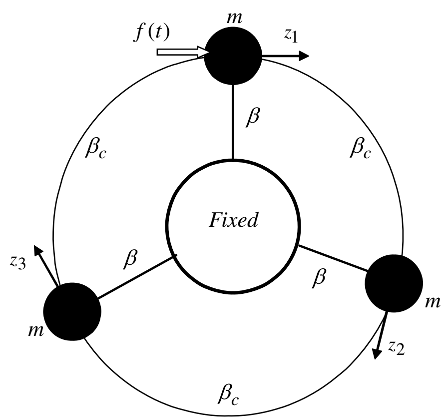

FIGURE 2.6 A periodic spring-mass system.

Example 2.7: A Periodic Structure

For the system in Figure 2.6, differential equations of motion are

$$m\ddot{z}_1 + \beta z_1 + \beta_c(z_1 - z_3) + \beta_c(z_1 - z_2) = f(t) \tag{2.5.68}$$

$$m\ddot{z}_2 + \beta z_2 + \beta_c(z_2 - z_1) + \beta_c(z_2 - z_3) = 0 \tag{2.5.69}$$

$$m\ddot{z}_3 + \beta z_3 + \beta_c(z_3 - z_2) + \beta_c(z_3 - z_1) = 0 \tag{2.5.70}$$

Define

$$\mathbf{x} = [z_1 \quad z_2 \quad z_3 \quad \dot{z}_1 \quad \dot{z}_2 \quad \dot{z}_3]^T \tag{2.5.71}$$

and

$$u(t) = \frac{f(t)}{m} \tag{2.5.72}$$

Then state equations are

$$\dot{\mathbf{x}} = A\mathbf{x}(t) + \mathbf{b}u(t) \tag{2.5.73}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -p & q & q & 0 & 0 & 0 \\ q & -p & q & 0 & 0 & 0 \\ q & q & -p & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad (2.5.74a,b)$$

$$p = \frac{\beta + 2\beta_c}{m} \quad \text{and} \quad q = \frac{2\beta_c}{m} \qquad (2.5.75a,b)$$

The controllability matrix is

$$C = \begin{bmatrix} 0 & 1 & 0 & -p & 0 & p^2+2q^2 \\ 0 & 0 & 0 & q & 0 & q^2-2pq \\ 0 & 0 & 0 & p & 0 & q^2-2pq \\ 1 & 0 & -p & 0 & p^2+2q^2 & 0 \\ 0 & 0 & q & 0 & q^2-2pq & 0 \\ 0 & 0 & q & 0 & q^2-2pq & 0 \end{bmatrix} \qquad (2.5.76)$$

Therefore,

$$\det C = \det \begin{bmatrix} 1 & -p & p^2+2q^2 & 0 & 0 & 0 \\ 0 & q & q^2-2pq & 0 & 0 & 0 \\ 0 & p & q^2-2pq & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -p & p^2+2q^2 \\ 0 & 0 & 0 & 0 & q & q^2-2pq \\ 0 & 0 & 0 & 0 & q & q^2-2pq \end{bmatrix} \qquad (2.5.77)$$

$$\det C = \det \begin{bmatrix} 1 & -p & p^2+2q^2 \\ 0 & q & q^2-2pq \\ 0 & q & q^2-2pq \end{bmatrix} . \det \begin{bmatrix} 1 & -p & p^2+2q^2 \\ 0 & q & q^2-2pq \\ 0 & q & q^2-2pq \end{bmatrix} = 0 \qquad (2.5.78)$$

In other words, the state space realization is not controllable. Because of symmetry, the influence of input on states z_2 and z_3 is identical. As a result, the input cannot take the system from any initial states to any arbitrary final states.

2.6 SOME IMPORTANT SIMILARITY TRANSFORMATIONS

2.6.1 Diagonal Form

Consider the similarity transformation (2.2.24). Find a nonsingular matrix T such that

$$\bar{A} = A_d = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{2.6.1}$$

In other words,

$$T^{-1}AT = A_d \tag{2.6.2}$$

or

$$AT = TA_d \tag{2.6.3}$$

Define

$$T = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & . & . & \mathbf{t}_n \end{bmatrix} \tag{2.6.4}$$

where $\mathbf{t}_i$ is the ith column of the matrix T. Hence, from (2.6.3) and (2.6.4),

$$A\mathbf{t}_i = \lambda_i \mathbf{t}_i \tag{2.6.5}$$

Therefore, λ_i and $\mathbf{t}_i$ must be the eigenvalue and the corresponding eigenvector of the matrix A, respectively. The matrix T will be nonsingular if and only if A has n independent eigenvectors. In this context, the following two properties are stated:

If A has n distinct eigenvalues, there exists n independent eigenvectors.
If an eigenvalue of A is repeated r times, r independent eigenvectors corresponding to this eigenvalue will be found provided:

$$rank(A - \lambda_i I) = n - r \tag{2.6.6}$$

Example 2.8

$$A = \begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{bmatrix} \tag{2.6.7}$$

Eigenvalues of the matrix A are 3, 6, and 6; i.e., the eigenvalue 6 is repeated twice.

$$A-6I=\begin{bmatrix}-1 & -1 & -1\\ -1 & -1 & -1\\ -1 & -1 & -1\end{bmatrix} \tag{2.6.8}$$

Therefore, $rank(A-6I)=1$, the condition (2.6.6) is satisfied, and the matrix A can be digonalized.

2.6.2 Controllability Canonical Form

Given a realization $\{A,\mathbf{b},\mathbf{c}\}$, find the similarity transformation matrix T such that

$$A_c=T^{-1}AT, \qquad \mathbf{b}_c=T^{-1}\mathbf{b}, \quad \text{and} \quad \mathbf{c}_c=\mathbf{c}T \tag{2.6.9a,b,c}$$

Note that Equation 2.2.29 has been used here.

From (2.6.9a),

$$TA_c=AT \tag{2.6.10}$$

Let

$$T=\begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2 & . & . & \mathbf{t}_n\end{bmatrix} \tag{2.6.11}$$

Now,

$$AT=[A\mathbf{t}_1 \quad A\mathbf{t}_2 \quad A\mathbf{t}_3 \quad . \quad . \quad A\mathbf{t}_n] \tag{2.6.12}$$

and using (2.2.9),

$$TA_c=[-a_n\mathbf{t}_n \quad \mathbf{t}_1-a_{n-1}\mathbf{t}_n \quad \mathbf{t}_2-a_{n-2}\mathbf{t}_n \quad . \quad . \quad \mathbf{t}_{n-1}-a_1\mathbf{t}_n] \tag{2.6.13}$$

From (2.6.10), (2.6.12), and (2.6.13),

$$A\mathbf{t}_1=-a_n\mathbf{t}_n$$

$$A\mathbf{t}_2=\mathbf{t}_1-a_{n-1}\mathbf{t}_n$$

$$A\mathbf{t}_3=\mathbf{t}_2-a_{n-2}\mathbf{t}_n$$

$$\vdots$$

$$A\mathbf{t}_n = \mathbf{t}_{n-1} - a_1\mathbf{t}_n \tag{2.6.14}$$

As $\mathbf{b}_c = [0 \quad 0 \quad . \quad . \quad 0 \quad 1]^T$, Equation 2.6.9b yields

$$\mathbf{t}_n = \mathbf{b} \tag{2.6.15}$$

Solving equations (2.6.14) from the last equation backward,

$$\mathbf{t}_{n-1} = (A + a_1 I)\mathbf{b} = A\mathbf{b} + a_1\mathbf{b}$$

$$\mathbf{t}_{n-2} = A(A + a_1 I)\mathbf{b} + a_2\mathbf{b} = A^2\mathbf{b} + a_1 A\mathbf{b} + a_2\underline{\mathbf{b}} \tag{2.6.16}$$

$$\vdots$$

$$\mathbf{t}_2 = A^{n-2}\mathbf{b} + a_1 A^{n-3}\mathbf{b} + ... + a_{n-2}\mathbf{b}$$

$$\mathbf{t}_1 = A^{n-1}\mathbf{b} + a_1 A^{n-2}\mathbf{b} + ... + a_{n-1}\mathbf{b}$$

Representing (2.6.16) in the matrix form,

$$T = CQ \tag{2.6.17}$$

where

$$Q = \begin{bmatrix} a_{n-1} & a_{n-2} & . & . & a_1 & 1 \\ a_{n-2} & a_{n-3} & . & . & 1 & 0 \\ . & . & . & . & . & . \\ a_1 & 1 & 0 & . & . & 0 \\ 1 & 0 & 0 & . & . & 0 \end{bmatrix} \tag{2.6.18}$$

The matrix Q is always nonsingular. Therefore, the matrix T is nonsingular if and only if the controllability matrix C is nonsingular.

In summary, any given realization can be converted to the canonical form $\{A_c, \mathbf{b}_c, \mathbf{c}_c\}$, provided the controllability matrix is nonsingular.

2.7 SIMULTANEOUS CONTROLLABILITY AND OBSERVABILITY

The realization $\{A_c, \mathbf{b}_c, \mathbf{c}_c\}$ is always controllable. On the other hand, the realization $\{A_o, \mathbf{b}_o, \mathbf{c}_o\}$ is always observable. Under what conditions are $\{A_c, \mathbf{b}_c, \mathbf{c}_c\}$ and

$\{A_o, \mathbf{b}_o, \mathbf{c}_o\}$ observable and controllable, respectively? To answer this question, consider the following theorems (Kailath, 1980).

Theorem 1

A realization is both observable and controllable if and only if there are no common factors between the numerator and the denominator of the transfer function.

Theorem 2

Observability and controllability properties are preserved under similarity transformations.

Therefore, $\{A_c, \mathbf{b}_c, \mathbf{c}_c\}$ is guaranteed to be observable provided there are no common factors between the numerator and the denominator of the transfer function. Similarly, $\{A_o, \mathbf{b}_o, \mathbf{c}_o\}$ is guaranteed to be controllable provided there are no common factors between the numerator and the denominator of the transfer function. Furthermore, if one can find one realization that is both observable and controllable, all realizations are both observable and controllable. If there are no common factors between numerator and denominator, any realization can be converted to either $\{A_c, \mathbf{b}_c, \mathbf{c}_c\}$ or $\{A_o, \mathbf{b}_o, \mathbf{c}_o\}$.

2.7.1 Observability of State Space Realization Using Method I

Without any loss of generality, assume that $b_0 = 0$. Therefore,

$$\mathbf{c}_c = \begin{bmatrix} b_n & b_{n-1} & . & . & b_1 \end{bmatrix} \tag{2.7.1}$$

Let $\mathbf{e}_i$ be the ith row of the identity matrix I_n. Then, it can be easily shown (Kailath, 1980) that

$$\mathbf{e}_i A_c = \mathbf{e}_{i+1}\ ; \quad 1 \le i \le n-1 \tag{2.7.2}$$

$$\mathbf{e}_n A_c = \begin{bmatrix} -a_n & -a_{n-1} & . & . & -a_1 \end{bmatrix} \tag{2.7.3}$$

Consider

$$b(A_c) = b_1 A_c^{n-1} + b_2 A_c^{n-2} + \ldots + b_{n-1} A_c + b_n I_n \tag{2.7.4}$$

Then,

$$\mathbf{e}_1 b(A_c) = b_1 \mathbf{e}_1 A_c^{n-1} + b_2 \mathbf{e}_1 A_c^{n-2} + \ldots + b_{n-1} \mathbf{e}_1 A_c + b_n \mathbf{e}_1 I_n \tag{2.7.5}$$

Because of (2.7.2),

$$\begin{aligned}\mathbf{e}_1 A_c &= \mathbf{e}_2 \\ \mathbf{e}_1 A_c^2 &= \mathbf{e}_2 A_c = \mathbf{e}_3 \\ &\vdots \\ \mathbf{e}_1 A_c^{n-1} &= \mathbf{e}_n\end{aligned} \tag{2.7.6}$$

From (2.7.5) and (2.7.6),

$$\mathbf{e}_1 b(A_c) = b_1 \mathbf{e}_n + b_2 \mathbf{e}_{n-1} + \ldots + b_{n-1}\mathbf{e}_2 + b_n \mathbf{e}_1 = \mathbf{c}_c \tag{2.7.7}$$

From (2.7.2) and (2.7.7),

$$\mathbf{e}_2 b(A_c) = \mathbf{e}_1 A_c b(A_c) = \mathbf{e}_1 b(A_c) A_c = \mathbf{c}_c A_c \tag{2.7.8}$$

In a similar manner,

$$\mathbf{e}_3 b(A_c) = \mathbf{c}_c A_c^2, \ldots, \mathbf{e}_n b(A_c) = \mathbf{c}_c A_c^{n-1} \tag{2.7.9}$$

Therefore, the observability matrix is

$$\Theta_c = \begin{bmatrix} \mathbf{c}_c \\ \mathbf{c}_c A_c \\ \cdot \\ \cdot \\ \mathbf{c}_c A_c^{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \cdot \\ \cdot \\ \mathbf{e}_n \end{bmatrix} b(A_c) = I_n b(A_c) = b(A_c) \tag{2.7.10}$$

Note that $b(A_c)$ is a polynomial in A_c. Therefore, it can easily be shown that

$$b(A_c)\mathbf{p} = b(\lambda)\mathbf{p} \tag{2.7.11}$$

where λ and $\mathbf{p}$ are the eigenvalue and the associated eigenvector of A_c. In other words, the eigenvalue and the associated eigenvector of $b(A_c)$ are $b(\lambda)$ and $\mathbf{p}$, respectively. From (2.7.4),

$$b(\lambda) = b_1 \lambda^{n-1} + b_2 \lambda^{n-2} + \ldots\ldots + b_n \tag{2.7.12}$$

As the determinant of a matrix equals the product of its eigenvalues, Equation 2.7.10 yields

$$\det(\Theta_c) = \det(b(A_c)) = \prod_{i=1}^{n} b(\lambda_i) \tag{2.7.13}$$

Therefore, the determinant of the observability matrix will be zero if and only if λ_i, an eigenvalue of A_c, satisfies $b(\lambda_i) = 0$; i.e., λ_i is also a zero of the transfer function. In other words, the state space realization obtained using Method I is observable if and only if all poles and zeros of the transfer function are distinct.

Example 2.9

The transfer function (2.2.20) can be written as

$$g(s) = 1 + h(s) \tag{2.7.14}$$

where

$$h(s) = \frac{-2s^2 + 2s + 2}{s^3 + 2s^2 + 4s + 3} \tag{2.7.15}$$

It can be easily seen that $\{A_c, \mathbf{b}_c, \mathbf{c}_c\}$ and $\{A_o, \mathbf{b}_o, \mathbf{c}_o\}$, Equation 2.2.21 and Equation 2.2.22, correspond to the transfer function $h(s)$. Poles of $h(s)$ are 1, and $-0.5 \pm 1.6583j$, whereas zeros are 0.618 and 1.618. In other words, there are no common factors between the numerator and denominator. Therefore, any state space realization is guaranteed to be both observable and controllable. As a result, $\{A_c, \mathbf{b}_c, \mathbf{c}_c\}$ will be observable, and $\{A_o, \mathbf{b}_o, \mathbf{c}_o\}$ will be controllable.

Example 2.10

From Figure 2.7,

$$\frac{y(s)}{u(s)} = \frac{s+1}{s(s+4)} \tag{2.7.16}$$

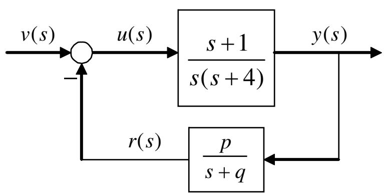

FIGURE 2.7 A feedback system.

$$\frac{r(s)}{y(s)} = \frac{p}{s+q}, \tag{2.7.17}$$

$$u(s) = v(s) - r(s) \tag{2.7.18}$$

For the transfer function (2.7.16), Method I yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.7.19}$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.7.20}$$

And, for the transfer function (2.7.17),

$$\dot{r} + qr(t) = py(t) \tag{2.7.21}$$

Combining (2.7.19) and (2.7.21) and using (2.7.18) and (2.7.20), the state space realization is obtained:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & -1 \\ p & p & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v(t) \tag{2.7.22}$$

$$y(t) = [1 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ r \end{bmatrix} \tag{2.7.23}$$

Controllability:

$$C = \begin{bmatrix} 0 & 1 & -4 \\ 1 & -4 & 16-p \\ 0 & p & p-4p-pq \end{bmatrix} \tag{2.7.24}$$

$$\det C = p(1-q) \tag{2.7.25}$$

The realization is not controllable when $p = 0$ or $q = 1$.

Observability:

$$O = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & -1 \\ -p & 12-p & 3+q \end{bmatrix} \tag{2.7.26}$$

$$\det O = 3(1-q) \tag{2.7.27}$$

The realization is not observable when $q = 1$.

Note that

$$\frac{y(s)}{v(s)} = \frac{(s+1)(s+q)}{s(s+3)(s+q)+p(s+1)} \tag{2.7.28}$$

When $p = 0$ or $q = 1$, $(s+q)$ or $(s+1)$ is a common factor between the numerator and denominator of the transfer function (2.7.28), respectively. Therefore, a state space realization cannot be both observable and controllable when $p = 0$ or $q = 1$. Here, the realization is not controllable when $p = 0$, and is neither observable nor controllable when $q = 1$.

2.8 MULTIINPUT/MULTIOUTPUT (MIMO) SYSTEMS

The transfer function matrix of a MIMO system is defined as

$$\mathbf{y}(s) = G(s)\mathbf{u}(s) \tag{2.8.1}$$

where $\mathbf{y}(s)$ and $\mathbf{u}(s)$ are $px1$ and $mx1$ vectors, respectively.

$$\mathbf{y}(s) = \begin{bmatrix} y_1(s) \\ y_2(s) \\ . \\ . \\ y_p(s) \end{bmatrix} \quad \text{and} \quad \mathbf{u}(s) = \begin{bmatrix} u_1(s) \\ u_2(s) \\ . \\ . \\ u_m(s) \end{bmatrix} \tag{2.8.2a,b}$$

Accordingly, the matrix $G(s)$ is of order pxm. As an example, for a 2-input/2-output system,

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+2} \\ \dfrac{s+5}{(s+1)^2} & \dfrac{1}{s+3} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} \tag{2.8.3}$$

Taking the least common multiple of denominators of elements of the matrix $G(s)$,

$$G(s) = \frac{1}{(s+1)^2(s+2)(s+3)} N(s) \tag{2.8.4}$$

where

$$N(s) = \begin{bmatrix} (s+1)(s+2)(s+3) & (s+1)^2(s+3) \\ (s+2)(s+3)(s+5) & (s+1)^2(s+2) \end{bmatrix} \tag{2.8.5}$$

The elements of the matrix $N(s)$ are polynomials in s. Hence, $N(s)$ will be called a polynomial matrix. It can also be expressed as follows:

$$N(s) = s^3 N_1 + s^2 N_2 + sN_3 + N_4 \tag{2.8.6}$$

where

$$N_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad N_2 = \begin{bmatrix} 6 & 5 \\ 10 & 4 \end{bmatrix}; \quad N_3 = \begin{bmatrix} 11 & 7 \\ 31 & 5 \end{bmatrix}; \quad N_4 = \begin{bmatrix} 6 & 3 \\ 30 & 2 \end{bmatrix} \tag{2.8.7}$$

In general, a pxm transfer function matrix $G(s)$ is represented as

$$G(s) = \frac{N(s)}{d(s)} \tag{2.8.8}$$

where

$$d(s) = s^r + d_1 s^{r-1} + \ldots + d_r \tag{2.8.9}$$

and

$$N(s) = s^{r-1} N_1 + s^{r-2} N_2 + \ldots + sN_{r-1} + N_r \tag{2.8.10}$$

$N_1, N_2, \ldots, N_r$ are pxm matrices.

Definitions

1. A rational transfer function matrix $G(s)$ is said to be proper if

$$\lim_{s \to \infty} G(s) < \infty \tag{2.8.11}$$

   and strictly proper if

$$\lim_{s\to\infty} G(s) = 0 \tag{2.8.12}$$

2. A vector of polynomials is called a polynomial vector. The degree of a polynomial vector equals the highest degree of all the entries of the vector. For example, the polynomial vector

$$\begin{bmatrix} s^4+1 \\ s^2+s+1 \\ s+7 \end{bmatrix} \tag{2.8.13}$$

has degree = 4.

Lemma

If $G(s)$ is a strictly proper (proper) transfer function matrix and

$$G(s) = N(s)D^{-1}(s) \tag{2.8.14}$$

then every column (Kailath, 1980) of $N(s)$ has degree strictly less than (less than or equal to) that of the corresponding column of $D(s)$.

Definition: Column-Reduced Matrix

Let

$$k_i = \text{ the degree of } i\text{th column of } m \times m \text{ matrix } D(s) \tag{2.8.15}$$

Then, it can be easily seen that

$$\deg \ \det D(s) \le \sum_{i=1}^{m} k_i \tag{2.8.16}$$

Inequality holds when there are cancellations of terms in the expansion of det $D(s)$. A matrix $D(s)$ for which the equality sign holds is called a *column reduced* matrix (Kailath, 1980).

Example 2.11

$$D(s) = \begin{bmatrix} s^4+s & s^2+2 \\ s^2+s+1 & 1 \end{bmatrix} \tag{2.8.17}$$

Here,

$$k_1 = 4 \quad \text{and } k_2 = 2 \tag{2.8.18}$$

and

$$\det D(s) = -s^3 - 3s^2 - s - 2 \tag{2.8.19}$$

In other words,

$$\deg \ \det D(s) < \sum_{i=1}^{2} k_i \tag{2.8.20}$$

Therefore, the matrix *D(s)* is not column reduced.

In general, a polynomial matrix $D(s)$ can be expressed (Kailath, 1980) as

$$D(s) = D_{hc} S(s) + L(s) \tag{2.8.21}$$

where

$$S(s) = diag[s^{k_1} \quad s^{k_2} \quad . \quad . \quad s^{k_m}] \tag{2.8.22}$$

$$\begin{aligned} D_{hc} &= \text{the highest-column-degree coefficient matrix,} \\ &\text{or the leading (column) coefficient matrix of } D(s) \end{aligned} \tag{2.8.23}$$

$$L(s): \text{remaining terms} \tag{2.8.24}$$

It can be shown that

$$\det D(s) = (\det D_{hc}) s^{\sum k_i} + \text{ terms of lower degrees in } s \tag{2.8.25}$$

When $\det D_{hc} \neq 0$,

$$\deg \ \det D(s) = \sum_{i=1}^{m} k_i \tag{2.8.26}$$

and the matrix $D(s)$ is column reduced.

Facts

A nonsingular polynomial matrix is column reduced if and only if its leading (column) coefficient matrix is nonsingular (Kailath, 1980).

If D(s) is column reduced, then $G(s) = N(s)D^{-1}(s)$ is strictly proper (proper) if and only if (Kailath, 1980) each column of N(s) has degree less than (less than or equal to) the degree of the corresponding column of D(s).

2.9 STATE SPACE REALIZATIONS OF A TRANSFER FUNCTION MATRIX

Two methods, similar to those for a SISO system (Section 2.2), will be presented.

METHOD I

Define

$$\xi(s) = \frac{1}{d(s)}\mathbf{u}(s) \tag{2.9.1}$$

Using Equation 2.8.9,

$$(s^r + d_1 s^{r-1} + \ldots + d_r)\xi(s) = \mathbf{u}(s) \tag{2.9.2}$$

In time-domain,

$$\frac{d^r}{dt^r}\xi(t) + d_1\frac{d^{r-1}}{dt^{r-1}}\xi(t) + \ldots + d_r\xi(t) = \mathbf{u}(t) \tag{2.9.3}$$

Define

$$\mathbf{x}_1 = \xi\ ; \quad \mathbf{x}_2 = \frac{d}{dt}\xi\ , \ldots, \ \mathbf{x}_r = \frac{d^{r-1}}{dt^{r-1}}\xi \tag{2.9.4}$$

Equation 2.9.3 and Equation 2.9.4 can be written as

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_3 \\ &\ \vdots \\ \dot{\mathbf{x}}_{r-1} &= \mathbf{x}_r \end{aligned} \tag{2.9.5}$$

$$\dot{\mathbf{x}}_r = -d_1\mathbf{x}_r - d_2\mathbf{x}_{r-1} - \ldots\ldots - d_r\mathbf{x}_1 + \mathbf{u}(t)$$

In matrix form,

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \\ \cdot \\ \cdot \\ \dot{\mathbf{x}}_{r-1} \\ \dot{\mathbf{x}}_r \end{bmatrix} = \begin{bmatrix} 0 & I_m & 0 & \cdot & \cdot & 0 \\ 0 & 0 & I_m & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & I_m \\ -d_rI_m & -d_{r-1}I_m & \cdot & \cdot & \cdot & -d_1I_m \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdot \\ \cdot \\ \mathbf{x}_{r-1} \\ \mathbf{x}_r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ I_m \end{bmatrix} \mathbf{u}(t) \tag{2.9.6}$$

From (2.8.1), (2.8.8), and (2.9.1),

$$\mathbf{y}(s) = N(s)\xi(s) \tag{2.9.7}$$

or

$$\mathbf{y}(s) = (s^{r-1}N_1 + s^{r-2}N_2 + \ldots + sN_{r-1} + N_r)\xi(s) \tag{2.9.8}$$

In time-domain,

$$\mathbf{y}(t) = N_1\frac{d^{r-1}}{dt^{r-1}}\xi(t) + N_2\frac{d^{r-2}}{dt^{r-2}}\xi(t) + \ldots + N_r\xi(t) \tag{2.9.9}$$

Using the definition (2.9.4),

$$\mathbf{y}(t) = \begin{bmatrix} N_r & N_{r-1} & \cdot & \cdot & N_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdot \\ \cdot \\ \mathbf{x}_r \end{bmatrix} \tag{2.9.10}$$

Equation 2.9.6 and Equation 2.9.10 comprise a state space realization:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned} \tag{2.9.11}$$

where

$$A = \begin{bmatrix} 0 & I_m & 0 & . & . & 0 \\ 0 & 0 & I_m & 0 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & I_m \\ -d_r I_m & -d_{r-1} I_m & . & . & . & -d_1 I_m \end{bmatrix} \tag{2.9.12}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ I_m \end{bmatrix}; \quad C = \begin{bmatrix} N_r & N_{r-1} & . & . & N_1 \end{bmatrix} \tag{2.9.13a,b}$$

The state vector $\mathbf{x}$ is described as

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ . \\ . \\ \mathbf{x}_r \end{bmatrix} \tag{2.9.14}$$

As the dimension of each of the block elements $\mathbf{x}_i$ is $mx1$,

$$\text{Number of states, } n = mr \tag{2.9.15}$$

Method II

From (2.8.1) and (2.8.8),

$$\mathbf{y}(s) = \frac{N(s)}{d(s)}\mathbf{u}(s) \tag{2.9.16}$$

$$(s^r + d_1 s^{r-1} + \ldots + d_r)\mathbf{y}(s) = (s^{r-1}N_1 + s^{r-2}N_2 + \ldots + sN_{r-1} + N_r)\mathbf{u}(s) \tag{2.9.17}$$

Dividing both sides by s^r,

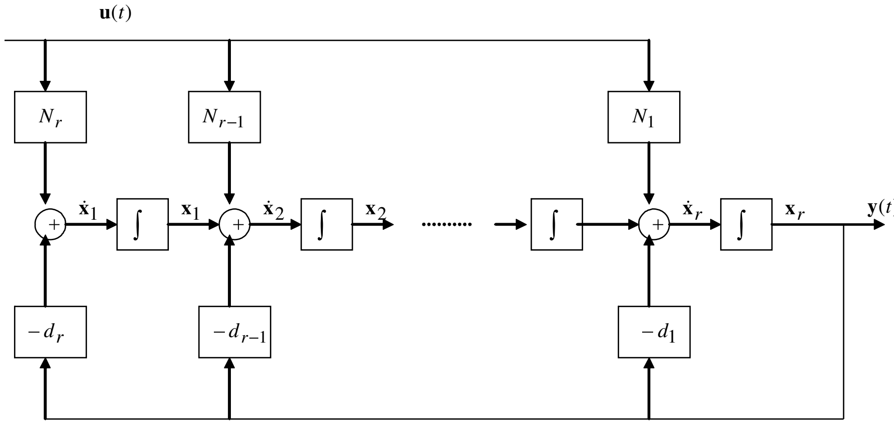

FIGURE 2.8 MIMO analog computer simulation diagram (Method II).

$$\left(1+\frac{d_1}{s}+\ldots\ldots+\frac{d_r}{s^r}\right)\mathbf{y}(s)=\left(\frac{N_1}{s}+\frac{N_2}{s^2}+\ldots+\frac{N_{r-1}}{s^{r-1}}+\frac{N_r}{s^r}\right)\mathbf{u}(s) \tag{2.9.18}$$

This equation can be written as

$$\begin{aligned}\mathbf{y}(s)=&-\frac{d_1}{s}\mathbf{y}(s)-\ldots-\frac{d_r}{s^r}\mathbf{y}(s)\\&+\frac{N_1}{s}\mathbf{u}(s)+\frac{N_2}{s^2}\mathbf{u}(s)+\ldots.+\frac{N_{r-1}}{s^{r-1}}\mathbf{u}(s)+\frac{N_r}{s^r}\mathbf{u}(s)\end{aligned} \tag{2.9.19}$$

On the basis of Equation 2.9.19, the simulation diagram is constructed as shown in Figure 2.8. Defining outputs of integrators as p-dimensional state variable vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots\ldots, \mathbf{x}_r$, the following state space model is obtained:

$$\mathbf{y}(t)=\mathbf{x}_r(t) \tag{2.9.20}$$

and

$$\begin{aligned}\dot{\mathbf{x}}_1&=-d_r\mathbf{y}(t)+N_r\mathbf{u}(t)=-d_r\mathbf{x}_r(t)+N_r\mathbf{u}(t)\\\dot{\mathbf{x}}_2&=-d_{r-1}\mathbf{y}(t)+\mathbf{x}_1(t)+N_{r-1}\mathbf{u}(t)=-d_{r-1}\mathbf{x}_r(t)+\mathbf{x}_1(t)+N_{r-1}\mathbf{u}(t)\\&\vdots\\\dot{\mathbf{x}}_r&=-d_1\mathbf{y}(t)+\mathbf{x}_{r-1}(t)+N_1\mathbf{u}(t)=-d_1\mathbf{x}_r(t)+\mathbf{x}_{r-1}(t)+N_1\mathbf{u}(t)\end{aligned} \tag{2.9.21}$$

Equation 2.9.20 and Equation 2.9.21 comprise the state space model:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t) \tag{2.9.22}$$

where

$$A = \begin{bmatrix} 0 & 0 & . & . & 0 & -d_r I_p \\ I_p & 0 & . & . & 0 & -d_{r-1} I_p \\ 0 & I_p & 0 & . & 0 & -d_{r-2} I_p \\ 0 & 0 & I_p & . & . & -d_{r-3} I_p \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & I_p & -d_1 I_p \end{bmatrix} \tag{2.9.23}$$

$$C = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ I_m \end{bmatrix}^T ; \quad B = \begin{bmatrix} N_r & N_{r-1} & . & . & N_1 \end{bmatrix}^T \tag{2.9.24a,b}$$

The state vector $\mathbf{x}$ is described as

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ . \\ . \\ \mathbf{x}_r \end{bmatrix} \tag{2.9.25}$$

As the dimension of each of the block elements $\mathbf{x}_i$ is $px1$,

$$\text{the number of states, } n = pr \tag{2.9.26}$$

2.10 CONTROLLABILITY AND OBSERVABILITY OF A MIMO SYSTEM

Definition 1

A multiinput state space realization is controllable if and only if the controllability matrix

$$C = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \tag{2.10.1}$$

is of full rank.

Definition 2

A multioutput state space realization is observable if and only if the observability matrix

$$\Theta = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} \tag{2.10.2}$$

is of full rank.

2.10.1 Controllability and Observability of Methods I and II Realizations

Method I Realization

$$C = \begin{bmatrix} B & AB & A^2B. & . & A^rB:\dots & A^{n-1}B \end{bmatrix} \tag{2.10.3}$$

where $n = mr$. The dimension of the matrix C is $n \times mn$. Using (2.10.3), (2.9.12), and (2.9.13a),

$$C = \begin{bmatrix} 0 & 0 & . & . & I_m: & \\ 0 & 0 & . & I_m & *: & \\ . & . & . & . & .: & \\ 0 & I_m & . & * & *: & \\ I_m & * & . & * & *: & \end{bmatrix} \tag{2.10.4}$$

It is obvious that the $mr \times mr$ submatrix shown in (2.10.4) is nonsingular. Hence, the matrix C is of full rank, i.e.,

$$\text{Rank}\ (C) = n \tag{2.10.5}$$

Hence, the realization obtained using Method I is guaranteed to be controllable. However, it need not be always observable.

Method II Realization

From (2.10.2),

$$\Theta = \begin{bmatrix} C \\ CA \\ . \\ CA^{r} \\ \cdots\cdots \\ . \\ . \\ CA^{n-1} \end{bmatrix} \tag{2.10.6}$$

where $n = pr$. The dimension of the matrix Θ is $n \times pn$. Using (2.10.6), (2.9.23), and (2.9.24a),

$$\Theta = \begin{bmatrix} 0 & 0 & . & 0 & I_m \\ 0 & 0 & . & I_m & * \\ . & . & . & . & . \\ 0 & I_m & . & * & * \\ I_m & * & . & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & \\ & & & & \end{bmatrix} \tag{2.10.7}$$

It is obvious that the $pr \times pr$ submatrix shown in (2.10.7) is nonsingular. Hence, the matrix Θ is of full rank, i.e.,

$$\text{Rank}(\Theta) = n \tag{2.10.8}$$

Hence, the realization obtained using Method II is guaranteed to be observable. However, it need not be always controllable.

2.11 MATRIX-FRACTION DESCRIPTION (MFD)

Given a transfer function matrix, two methods for obtaining state space realizations have been obtained in a manner analogous to what we did for SISO systems. The state space realization from Method I is always controllable, whereas the state space realization from Method II is always observable. Furthermore, the number of states from the two methods is different when the number of outputs does not equal number of inputs. As a result, following questions arise:

1. When will a state space realization be both controllable and observable?
2. How do we obtain a state space realization with the minimum number of states?

To answer these questions in a general way, the MFD of a transfer function is introduced. Equation 2.8.8 can also be written as

$$G(s) = N_R(s)D_R^{-1}(s) \tag{2.11.1}$$

where

$$N_R(s) = N(s) \quad \text{and} \quad D_R(s) = d(s)I_m \tag{2.11.2}$$

Alternatively,

$$G(s) = D_L^{-1}(s)N_L(s) \tag{2.11.3}$$

where

$$N_L(s) = N(s) \quad \text{and} \quad D_L(s) = d(s)I_p \tag{2.11.4}$$

Equation 2.11.1 and Equation 2.11.3 are known as right and left MFD, respectively. Any transfer function matrix can be represented by either right or left MFD. It should be noted that $D_R(s)$ and $D_L(s)$ can be viewed as denominator polynomial matrices. Similarly, $N_R(s)$ and $N_L(s)$ can be viewed as numerator polynomial matrices.

2.11.1 Degree of a Square Polynomial Matrix and Greatest Common Right Divisor (*GCRD*)

The *degree of a square polynomial matrix* is equal to the degree of the determinant of the polynomial matrix (Kailath, 1980).
Therefore,

$$deg\ D_R(s) = deg\ det\ D_R(s) = mr \tag{2.11.5}$$

and

$$deg\ D_L(s) = deg\ det\ D_L(s) = pr \tag{2.11.6}$$

It is interesting to note that the number of states in Method I realization equals $deg\ D_R(s)$, whereas the number of states in Method II realization equals $deg\ D_L(s)$. In fact, Method I and Method II are connected to the right and left MFD, respectively. Furthermore, having expressed the transfer function matrix as a right MFD, a state space realization can be developed to have the number of states equal $deg\ D_R(s)$. The same result also holds for left MFD.

For the transfer function matrix in Equation 2.8.3, a right MFD is constructed using (2.11.2):

$$G(s) = \begin{bmatrix} (s+1)(s+2)(s+3) & (s+1)^2(s+3) \\ (s+2)(s+3)(s+5) & (s+1)^2(s+2) \end{bmatrix} \times$$

$$\begin{bmatrix} (s+1)^2(s+2)(s+3) & 0 \\ 0 & (s+1)^2(s+2)(s+3) \end{bmatrix}^{-1} \tag{2.11.7}$$

$$= N_1(s)D_1^{-1}(s)$$

where

$$deg\ D_1(s) = 8$$

Another right MFD can also be constructed:

$$G(s) = \begin{bmatrix} (s+1) & (s+3) \\ (s+5) & (s+2) \end{bmatrix} \begin{bmatrix} (s+1)^2 & 0 \\ 0 & (s+2)(s+3) \end{bmatrix}^{-1} = N_2(s)D_2^{-1}(s) \tag{2.11.8}$$

where

$$deg\ D_2(s) = 4$$

The diagonal elements of $D_2(s)$ are obtained by taking the least common multiple of each column separately. It can be easily verified that

$$N_2(s) = N_1(s)W^{-1}(s) \quad \text{and} \quad D_2(s) = D_1(s)W^{-1}(s) \tag{2.11.9}$$

where

$$W(s) = \begin{bmatrix} (s+2)(s+3) & 0 \\ 0 & (s+1)^2 \end{bmatrix} \tag{2.11.10}$$

Here, $W(s)$ is a nonsingular polynomial matrix, but $W^{-1}(s)$ is not a polynomial matrix. When $W^{-1}(s)$ is postmultiplied with $N_1(s)$, another polynomial matrix $N_2(s)$ is obtained. Therefore, the polynomial matrix $W(s)$ is called a **right divisor** of $N_1(s)$. Using this definition, the polynomial matrix $W(s)$ is also a **right divisor** of $D_1(s)$. In other words, $W(s)$ is a **common right divisor** (*crd*) of both $N_1(s)$ and $D_1(s)$. From (2.11.9),

$$D_2(s)W(s) = D_1(s) \tag{2.11.11}$$

From the procedure of matrix multiplication and the definition of the determinant,

$$deg\ D_1(s) = deg\ D_2(s)W(s) = deg\ D_2(s) + deg\ W(s) \tag{2.11.12}$$

Thus, finding a *crd* of numerator and denominator polynomial matrices, the degree of the denominator polynomial matrix (hence, the number of states) can be decreased by the degree of the *crd*. Therefore, the minimal order of state space realization will be obtained (Kailath, 1980) by finding **the greatest common right divisor** (gcrd). If the degree of a *gcrd* is zero, the number of states cannot be reduced any further. A polynomial square matrix is said to be **unimodular** if its degree equals zero. A polynomial matrix is unimodular if and only if its inverse is also a polynomial matrix.

Nonuniqueness of a gcrd

Let $W_g(s)$ be a gcrd of $N(s)$ and $D(s)$. Then,

$$\tilde{N}(s) = N(s)W_g^{-1}(s) \tag{2.11.13}$$

$$\tilde{D}(s) = D(s)W_g^{-1}(s) \tag{2.11.14}$$

where $\tilde{N}(s)$ and $\tilde{D}(s)$ are polynomial matrices. It can be verified that $U_a(s)W_g(s)$ is also a *gcrd* where $U_a(s)$ is any unimodular matrix.

Definitions

1. Two polynomial matrices $N(s)$ and $D(s)$ with the same number of columns will be said to be relatively right prime (or right coprime) if they only have unimodular *crd*.
2. An MFD $G(s) = N(s)D^{-1}(s)$ will be said to be irreducible if $N(s)$ and $D(s)$ are right coprime.

2.11.2 Elementary Row and Column Operations

A general procedure for obtaining a *gcrd* is based on elementary operations on polynomial matrices defined as follows (Kailath, 1980; Rugh, 1993):

1. Interchange of any two rows or columns
2. Addition to any row (column) by a polynomial multiple of any other column (row)
3. Multiplying any row or column by any nonzero real or complex number

These elementary operations are represented by elementary matrices. Postmultiplication of a polynomial matrix by an elementary matrix results in elementary row operation, whereas premultiplication results in elementary column operation. As an example, consider the following polynomial matrix:

$$P(s)=\begin{bmatrix} s+1 & s^2+s+1 & s \\ 1 & 2s+3 & s+5 \\ s+3 & s(s+1) & 1 \end{bmatrix} \tag{2.11.15}$$

Then, various elementary operations are described as follows:

Interchange of first and second rows:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} P(s)=\begin{bmatrix} 1 & 2s+3 & s+5 \\ s+1 & s^2+s+1 & s \\ s+3 & s(s+1) & 1 \end{bmatrix} \tag{2.11.16}$$

Interchange of first and second columns:

$$P(s)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} =\begin{bmatrix} s^2+s+1 & s+1 & s \\ 2s+3 & 1 & s+5 \\ s(s+1) & s+3 & 1 \end{bmatrix} \tag{2.11.17}$$

Addition of s times first row to the third row:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & 0 & 1 \end{bmatrix} P(s)=\begin{bmatrix} s+1 & s^2+s+1 & s \\ 1 & 2s+3 & s+5 \\ s(s+1)+s+3 & s(s^2+s+1)+s(s+1) & s^2+1 \end{bmatrix} \tag{2.11.18}$$

Addition of s times first column to the third column:

$$P(s)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & 0 & 1 \end{bmatrix} = \begin{bmatrix} s+1 & s^2+s+1 & s(s+1)+s \\ 1 & 2s+3 & s+s+5 \\ s+3 & s(s+1) & s(s+3)+1 \end{bmatrix} \tag{2.11.19}$$

Scaling of first row by 2:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P(s) = \begin{bmatrix} 2(s+1) & 2(s^2+s+1) & 2s \\ 1 & 2s+3 & s+5 \\ s+3 & s(s+1) & 1 \end{bmatrix} \tag{2.11.20}$$

Scaling of first column by 2:

$$P(s)\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (s+1) & (s^2+s+1) & 2s \\ 1 & 2s+3 & 2(s+5) \\ s+3 & s(s+1) & 2 \end{bmatrix} \tag{2.11.21}$$

All the matrices by which $P(s)$ is pre- or postmultiplied are elementary matrices that have the following properties:

1. The inverse of an elementary matrix is also an elementary matrix.
2. Determinants of elementary matrices are nonzero constants and hence independent of s. In other words, elementary matrices are **unimodular** (see Section 2.11.1).

2.11.3 Determination of a *gcrd*

Given $m \times m$ and $p \times m$ polynomial matrices $D(s)$ and $N(s)$, find elementary row operations or equivalently a unimodular matrix $U(s)$ such that at least the last p rows on the right-hand side of following equation are zero, i.e.,

$$U(s)\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix} \tag{2.11.22}$$

Then the square matrix $R(s)$ is a *gcrd* (Kailath, 1980).

Example 2.12

Find a *gcrd* of

$$N(s)=\begin{bmatrix}(s+1)(s+5) & 0\\ 0 & (s+2)(s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix} \tag{2.11.23}$$

and

$$D(s)=\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & (s+4)(s+5)\end{bmatrix} \tag{2.11.24}$$

Solution

$$\begin{bmatrix}D(s)\\ N(s)\end{bmatrix}=\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & (s+4)(s+5)\\ (s+1)(s+5) & 0\\ 0 & (s+2)(s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix} \tag{2.11.25}$$

Step I: Subtract fourth row from the first row.

$$U_1(s)\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & (s+4)(s+5)\\ (s+1)(s+5) & 0\\ 0 & (s+2)(s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix}=\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & (s+4)(s+5)\\ (s+1)(s+5) & 0\\ 0 & -2(s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix} \tag{2.11.26}$$

$$U_1(s)=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix} \tag{2.11.27}$$

Step II: Divide the fourth row by –1/2.

$$U_2(s)\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & (s+4)(s+5)\\ (s+1)(s+5) & 0\\ 0 & -2(s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix}=\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & (s+4)(s+5)\\ (s+1)(s+5) & 0\\ 0 & (s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix} \quad (2.11.28)$$

$$U_2(s)=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1/2 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix} \quad (2.11.29)$$

Step III: Subtract (s + 4) times the fourth row from the second row.

$$U_3(s)\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & (s+4)(s+5)\\ (s+1)(s+5) & 0\\ 0 & (s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix}=\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & 0\\ (s+1)(s+5) & 0\\ 0 & (s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix} \quad (2.11.30)$$

$$U_3(s)=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -(s+4) & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix} \quad (2.11.31)$$

Step IV: Subtract 0.5 times the first row from 0.5 times the third row.

$$U_4(s)\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & 0\\ (s+1)(s+5) & 0\\ 0 & (s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix}=\begin{bmatrix}(s+3)(s+5) & 0\\ 0 & 0\\ (s+5) & 0\\ 0 & (s+5)\\ (s+2)(s+3) & (s+1)(s+4)\end{bmatrix} \quad (2.11.32)$$

$$U_4(s)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\-1/2&0&1/2&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\tag{2.11.33}$$

Step V: Subtract $(s + 3)$ times the third row from the first row.

$$U_5(s)\begin{bmatrix}(s+3)(s+5)&0\\0&0\\(s+5)&0\\0&(s+5)\\(s+2)(s+3)&(s+1)(s+4)\end{bmatrix}=\begin{bmatrix}0&0\\0&0\\(s+5)&0\\0&(s+5)\\(s+2)(s+3)&(s+1)(s+4)\end{bmatrix}\tag{2.11.34}$$

$$U_5(s)=\begin{bmatrix}1&0&-(s+3)&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\tag{2.11.35}$$

Step VI: Replace fifth row by adding $(s + 3)/3$ times the third row and $-1/3$ times the fifth row.

$$U_6(s)\begin{bmatrix}0&0\\0&0\\(s+5)&0\\0&(s+5)\\(s+2)(s+3)&(s+1)(s+4)\end{bmatrix}=\begin{bmatrix}0&0\\0&0\\(s+5)&0\\0&(s+5)\\(s+3)&-(s+1)(s+4)/3\end{bmatrix}\tag{2.11.36}$$

$$U_6(s)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&(s+3)/3&0&-1/3\end{bmatrix}\tag{2.11.37}$$

Step VII: Add (s + 4)/3 times the fourth row and to the fifth row.

$$U_7(s)\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ (s+5) & 0 \\ 0 & (s+5) \\ (s+3) & -(s+1)(s+4)/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ (s+5) & 0 \\ 0 & (s+5) \\ (s+3) & 4(s+4)/3 \end{bmatrix} \quad (2.11.38)$$

$$U_7(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (s+4)/3 & 1 \end{bmatrix} \quad (2.11.39)$$

Step VIII: Subtract the third row from the fifth row.

$$U_8(s)\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ (s+5) & 0 \\ 0 & (s+5) \\ (s+3) & 4(s+4)/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ (s+5) & 0 \\ 0 & (s+5) \\ -2 & 4(s+4)/3 \end{bmatrix} \quad (2.11.40)$$

$$U_8(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \quad (2.11.41)$$

Step IX: Replace fifth row by the fourth row — 3/4 times the fifth row.

$$U_9(s)\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ (s+5) & 0 \\ 0 & (s+5) \\ -2 & 4(s+4)/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ (s+5) & 0 \\ 0 & (s+5) \\ 3/2 & 1 \end{bmatrix} \quad (2.11.42)$$

$$U_9(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3/4 \end{bmatrix} \tag{2.11.43}$$

Step X: Subtract 2(s + 5)/3 times the fifth row from the third row.

$$U_{10}(s) \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ (s+5) & 0 \\ 0 & (s+5) \\ 3/2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -2(s+5)/3 \\ 0 & (s+5) \\ 3/2 & 1 \end{bmatrix} \tag{2.11.44}$$

$$U_{10}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2(s+5)/3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.11.45}$$

Step XI: Add 2/3 times the fourth row to the third row.

$$U_{11}(s) \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -2(s+5)/3 \\ 0 & (s+5) \\ 3/2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & (s+5) \\ 3/2 & 1 \end{bmatrix} \tag{2.11.46}$$

$$U_{11}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.11.47}$$

Step XII: Interchanging rows,

$$U_{12}(s)\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & (s+5) \\ 3/2 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & 1 \\ 0 & (s+5) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.11.48}$$

$$U_{12}(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.11.49}$$

In summary, with reference to Equation 2.11.22,

$$U(s) = U_{12}(s)U_{11}(s)\ldots U_2(s)U_1(s) \tag{2.11.50}$$

and

$$R(s) = \begin{bmatrix} 3/2 & 1 \\ 0 & (s+5) \end{bmatrix} \tag{2.11.51}$$

The square matrix $R(s)$ is a *gcrd* of N(s) and D(s).

2.12 MFD OF A TRANSFER FUNCTION MATRIX FOR THE MINIMAL ORDER OF A STATE SPACE REALIZATION

Let a transfer function be expressed as

$$G(s) = N(s)D^{-1}(s) \tag{2.12.1}$$

First, find $W_g(s)$, a *gcrd* of $N(s)$ and $D(s)$. Then, obtain the following matrices:

$$\bar{N}(s) = N(s)W_g^{-1}(s) \tag{2.12.2}$$

and

$$\bar{D}(s) = D(s)W_g^{-1}(s) \tag{2.12.3}$$

The transfer function matrix can also be written as

$$G(s) = \bar{N}(s)\bar{D}^{-1}(s) \tag{2.12.4}$$

where

$$deg\ \bar{D}(s) = deg\ D(s)\ deg\ W_g(s) \tag{2.12.5}$$

Now, a state space realization based on Equation 2.12.4 will have a minimum number of states (Kailath, 1980) equal to $deg\ \bar{D}(s)$.

Example 2.13

Let the transfer function matrix be defined as

$$G(s) = \begin{bmatrix} \frac{s+1}{s+3} & 0 \\ 0 & \frac{s+2}{s+4} \\ \frac{s+2}{s+5} & \frac{s+1}{s+5} \end{bmatrix} \tag{2.12.6}$$

With respect to (2.12.1), $N(s)$ and $D(s)$ are given by (2.11.23) and (2.11.24). From (2.11.51), a *gcrd* of $N(s)$ and $D(s)$ is

$$W_g(s) = \begin{bmatrix} 3/2 & 1 \\ 0 & (s+5) \end{bmatrix} \tag{2.12.7}$$

Therefore,

$$\bar{N}(s) = N(s)W_g^{-1}(s) = \begin{bmatrix} \frac{2(s+1)(s+5)}{3} & -\frac{2(s+1)}{3} \\ 0 & (s+2) \\ \frac{2(s+2)(s+3)}{3} & \frac{s}{3} \end{bmatrix} \tag{2.12.8}$$

and

$$\bar{D}(s) = D(s)W_g^{-1}(s) = \begin{bmatrix} \dfrac{2(s+3)(s+5)}{3} & -\dfrac{2(s+3)}{3} \\ 0 & (s+4) \end{bmatrix} \quad (2.12.9)$$

Because $deg\ \bar{D}(s)$ = 3, the minimal number of states equals 3.

2.13 CONTROLLER FORM REALIZATION FROM A RIGHT MFD

2.13.1 State Space Realization

Consider a strictly proper right MFD (Kailath, 1980)

$$G(s) = N(s)D^{-1}(s) \quad (2.13.1)$$

Rewrite (2.8.1) as

$$D(s)\xi(s) = \mathbf{u}(s) \quad (2.13.2a)$$

$$\mathbf{y}(s) = N(s)\xi(s) \quad (2.13.2b)$$

The matrix D(s) can be written as Equation 2.8.21 where

$$L(s) = D_{lc}\Psi(s) \quad (2.13.3)$$

Here, D_{lc} is the lower-column-degree coefficient matrix of $D(s)$ and

$$\Psi^T(s) =$$

$$\begin{bmatrix} s^{k_1-1} & . & . & s & 1 & 0 & . & . & . & 0 & . & . & 0 & . & . & . & 0 \\ 0 & . & . & . & 0 & s^{k_2-1} & . & . & s & 1 & . & . & 0 & . & . & . & 0 \\ & & . & & & & & . & & & . & . & & & . & & \\ & & . & & & & & . & & & . & . & & & . & & \\ 0 & . & . & . & 0 & 0 & . & . & . & 0 & . & . & s^{k_m-1} & . & . & s & 1 \end{bmatrix} \quad (2.13.4)$$

Note that the dimension of the matrix $\Psi^T(s)$ is $mx(k_1 + k_2 + \ldots k_m)$. Furthermore, the degree of the ith column of N(s) will be at the most $k_i - 1$. Therefore, a matrix N_{lc} can be defined such that

$$\mathbf{y}(s) = N(s)\xi(s) = N_{lc}\Psi(s)\xi(s) \tag{2.13.5}$$

From (2.8.21) and (2.13.2a),

$$\left[D_{hc}S(s) + D_{lc}\Psi(s)\right]\xi(s) = \mathbf{u}(s) \tag{2.13.6}$$

Assuming that the matrix D(s) is column reduced,

$$S(s)\xi(s) = -D_{hc}^{-1}D_{lc}\mathbf{q}(s) + D_{hc}^{-1}\mathbf{u}(s) \tag{2.13.7}$$

where

$$\mathbf{q}(s) = \Psi(s)\xi(s) \tag{2.13.8}$$

Equation 2.13.8 is rewritten as

$$\mathbf{q}(s) = \Psi(s)S^{-1}(s)\mathbf{v}(s) \tag{2.13.9}$$

where

$$\mathbf{v}(s) = S(s)\xi(s) \tag{2.13.10}$$

Using (2.13.4), Equation 2.13.9 yields

$$\mathbf{q}(s) = \begin{bmatrix} s^{-1} & . & . & s^{-(k_1-1)} & s^{-k_1} & 0 & . & . & . & 0 & . & . & 0 & . & . & . & 0 \\ 0 & . & . & . & 0 & s^{-1} & . & . & s^{-(k_2-1)} & s^{-k_2} & . & . & 0 & . & . & . & 0 \\ & & . & & & & . & & & & . & . & & & . & & \\ & & . & & & & . & & & & . & . & & & . & & \\ 0 & . & . & . & 0 & 0 & . & . & . & 0 & . & . & s^{-1} & . & . & s^{-(k_n-1)} & s^{-k_n} \end{bmatrix}^T \times \begin{bmatrix} v_1(s) \\ v_2(s) \\ . \\ . \\ v_m(s) \end{bmatrix} \tag{2.13.11}$$

Let

$$s^{-1}v_1(s) = x_1, \ldots, s^{-(k_1-1)}v_1(s) = x_{k_1-1},\ s^{-k_1}v_1(s) = x_{k_1}$$

$$s^{-1}v_2(s) = x_{k_1+1}(s), \ldots, s^{-(k_2-1)}v_2(s) = x_{k_1+k_2-1}(s), s^{-(k_2-1)}v_2(s) = x_{k_1+k_2}(s)$$
$$\ldots$$
$$\ldots \tag{2.13.12}$$
$$s^{-1}v_m(s) = x_{k_1+\ldots+k_{m-1}+1}(s), \ldots, s^{-1}v_m(s) = x_{k_1+\ldots+k_{m-1}+k_m-1}(s)$$
$$s^{-1}v_m(s) = x_{k_1+\ldots+k_{m-1}+k_m}(s)$$

In this case,

$$\dot{x}_1 = v_1, \dot{x}_2 = x_1, \ldots, \dot{x}_{k_1} = x_{k_1-1}$$
$$\dot{x}_{k_1+1} = v_2, \dot{x}_{k_1+2} = x_{k_1+1}, \ldots, \dot{x}_{k_1+k_2} = x_{k_1+k_2-1}$$
$$\ldots \tag{2.13.13}$$
$$\ldots$$
$$\dot{x}_{k_1+\ldots+k_{m-1}+1} = v_m, \dot{x}_{k_1+\ldots+k_{m-1}+2} = x_{k_1+\ldots+k_{m-1}+1}, \ldots, \dot{x}_{k_1+\ldots+k_m} = x_{k_1+\ldots+k_m-1}$$

Define the state vector $\mathbf{x}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 & . & x_{k_1} & . & x_{k_1+k_2} & . & . & . & . & x_{k_1+\ldots+k_{m-1}+1} & . & x_{k_1+\ldots+k_m} \end{bmatrix}^T \tag{2.13.14}$$

Then, Equations 2.13.13 can be written as

$$\dot{\mathbf{x}} = A_c^0\mathbf{x}(t) + B_c^0\mathbf{v}(t) \tag{2.13.15}$$

where

$$A_c^0 = block\ diagonal \left\{ \begin{bmatrix} 0 & 0 & . & 0 & 0 \\ 1 & 0 & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 1 & 0 \end{bmatrix}; k_i x k_i; i = 1, 2, \ldots, m \right\} \tag{2.13.16}$$

and

$$[B_c^0]^T = block\ diagonal \{ [1 \quad 0 \quad . \quad 0]; 1xk_i; i = 1, 2, \ldots, m \} \tag{2.13.17}$$

Utilizing (2.13.9) and (2.13.12), it can be easily seen that

$$\mathbf{q}(s) = \mathbf{x}(s) \tag{2.13.18}$$

Therefore, from (2.13.7) and (2.13.10),

$$\mathbf{v}(t) = -D_{hc}^{-1} D_{lc}\mathbf{x}(t) + D_{hc}^{-1}\mathbf{u}(t) \tag{2.13.19}$$

Substituting (2.13.19) into (2.3.15), the state equations are obtained:

$$\dot{\mathbf{x}} = A_c\mathbf{x}(t) + B_c\mathbf{u}(t) \tag{2.13.20}$$

where

$$A_c = A_c^0 - B_c^0 D_{hc}^{-1} D_{lc} \tag{2.13.21}$$

and

$$B_c = B_c^0 D_{hc}^{-1} \tag{2.13.22}$$

From (2.13.5), (2.13.14), and (2.13.18), output equations are given as

$$\mathbf{y}(t) = C_c\mathbf{x}(t) \tag{2.13.23}$$

where

$$C_c = N_{lc} \tag{2.13.24}$$

Example 2.14

Find the minimal state space realization of transfer function matrix $G(s)$, Equation 2.12.6.

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} - H(s) \tag{2.13.25}$$

where

$$H(s) = \begin{bmatrix} \frac{2}{s+3} & 0 \\ 0 & \frac{2}{s+4} \\ \frac{3}{s+5} & \frac{4}{s+5} \end{bmatrix} \tag{2.13.26}$$

Note that *H*(*s*) is a strictly proper MFD and can be expressed as

$$H(s) = N_H(s) D_H^{-1}(s) \tag{2.13.27}$$

where

$$N_H(s) = \begin{bmatrix} 2(s+5) & 0 \\ 0 & 2(s+5) \\ 3(s+3) & 4(s+4) \end{bmatrix} \tag{2.13.28}$$

and

$$D_H(s) = \begin{bmatrix} (s+3)(s+5) & 0 \\ 0 & (s+4)(s+5) \end{bmatrix} \tag{2.13.29}$$

A *gcrd* of $N_H(s)$ and $D_H(s)$ is again found as

$$W_h(s) = \begin{bmatrix} 3/2 & 1 \\ 0 & s+5 \end{bmatrix} \tag{2.13.30}$$

Then,

$$\bar{N}_H(s) = N_H(s) W_h^{-1}(s) = \begin{bmatrix} \frac{4(s+5)}{3} & -\frac{4}{3} \\ 0 & 2 \\ 2(s+3) & 2 \end{bmatrix} \tag{2.13.31}$$

and

$$\bar{D}_H(s) = D_H(s)W_h^{-1}(s) = \begin{bmatrix} \frac{2(s+3)(s+5)}{3} & -\frac{2(s+3)}{3} \\ 0 & (s+4) \end{bmatrix} \quad (2.13.32)$$

For $\bar{D}_H(s)$ in Equation 2.13.32, $k_1 = 2$ and $k_2 = 1$. And,

$$D_{hc} = \begin{bmatrix} 2/3 & -2/3 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D_{lc} = \begin{bmatrix} 16/3 & 10 & -2 \\ 0 & 0 & 4 \end{bmatrix} \quad (2.13.32)$$

$$A_c^0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B_c^0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.13.33)$$

For $\bar{N}_H(s)$ in Equation 2.13.31,

$$N_{lc} = \begin{bmatrix} 4/3 & 20/3 & -4/3 \\ 0 & 0 & 2 \\ 2 & 6 & 2 \end{bmatrix} \quad (2.13.34)$$

From (2.13.21), (2.13.22), and (2.13.24),

$$A_c = \begin{bmatrix} -8 & -15 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} \; B_c = \begin{bmatrix} 1.5 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C_c = N_{lc} \quad (2.13.35)$$

2.13.2 Similarity Transformation to Convert Any State Space Realization {*A*,*B*,*C*} to the Controller Form Realization

Given *any* state space realization $\{A,B,C\}$, the objective is to find the nonsingular matrix T such that

$$T^{-1}AT = \hat{A}_c \quad \text{and} \quad T^{-1}B = \hat{B}_c \quad (2.13.36)$$

where the structures of $\hat{A}_c$ and $\hat{B}_c$ match those of A_c and B_c given by Equation 2.13.21 and Equation 2.3.22, respectively.

TABLE 2.1
Search for Independent Vectors

$\mathbf{b}_1$ X	$\mathbf{b}_2$ X	$\mathbf{b}_3$ X
$A\mathbf{b}_1$ X	$A\mathbf{b}_2$ X	$A\mathbf{b}_3$ X
$A^2\mathbf{b}_1$ 0	$A^2\mathbf{b}_2$ 0	$A^2\mathbf{b}_3$ 0
$A^3\mathbf{b}_1$	$A^3\mathbf{b}_2$	$A^3\mathbf{b}_3$
$A^4\mathbf{b}_1$	$A^4\mathbf{b}_2$	$A^4\mathbf{b}_3$
$A^5\mathbf{b}_1$	$A^5\mathbf{b}_2$	$A^5\mathbf{b}_3$

First, consider the controllability matrix

$$C = [B \quad AB \quad A^2B \quad . \quad . \quad A^{n-1}B] \tag{2.13.37}$$

The order of the matrix is $n \times nm$. Out of these nm column vectors of the matrix, only n column vectors are independent. To find these n independent column vectors, Table 2.1 is created for the state equations with three inputs and six states as an example. The search for independent columns is conducted (Kailath, 1980) row-wise from left to right, and a sign "X" is introduced to indicate that it is independent of all the previous vectors that have been found to be independent. The sign "0" is introduced to indicate that the associated vector is not independent of previously found independent columns. There is no need to consider vectors in the table column below the vector with the "0" sign because they are guaranteed to be dependent on previously found independent vectors due to the Cayley–Hamilton theorem. Next, each vector with the sign "0" is expanded in terms of independent vectors, i.e.,

$$A^2\mathbf{b}_i = \sum_{j=1}^{3} \alpha_{ji}\mathbf{b}_j + \sum_{j=1}^{3} \beta_{ji}A\mathbf{b}_j \; ; \quad i = 1, 2, \text{ and } 3 \tag{2.13.38}$$

where α_j and β_j are coefficients to be determined. Using the matrix notation, Equation 2.13.38 can be written as

$$C_I[\alpha_{1i} \quad \alpha_{2i} \quad \alpha_{3i} \quad \beta_{1i} \quad \beta_{2i} \quad \beta_{3i}]^T = A^2\mathbf{b}_i \tag{2.13.39}$$

where

$$C_I = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3\end{bmatrix} \tag{2.13.40}$$

Solving (2.13.39),

$$[\alpha_{1i} \quad \alpha_{2i} \quad \alpha_{3i} \quad \beta_{1i} \quad \beta_{2i} \quad \beta_{3}i]^T = C_I^{-1}A^2\mathbf{b}_i \tag{2.13.41}$$

Then, for each vector with the sign “0”, terms having A are collected on the left side and the matrix A is factored out as follows:

$$A\mathbf{t}_{i2} = \sum_{j=1}^{3} \alpha_{ji}\mathbf{b}_j \tag{2.13.42}$$

where

$$\mathbf{t}_{i2} = [A\mathbf{b}_i - \sum_{j=1}^{3} \beta_{ji}\mathbf{b}_j] \tag{2.13.43}$$

Again, for $\mathbf{t}_{i2}$, terms having A are collected on the left side and the matrix A is factored out as follows:

$$A\mathbf{t}_{i1} = \mathbf{t}_{i2} + \sum_{j=1}^{3} \beta_{ji}\mathbf{b}_j \tag{2.13.44}$$

where

$$\mathbf{t}_{i1} = \mathbf{b}_i \tag{2.13.45}$$

Lastly, the transformation matrix T is defined as

$$T = [\mathbf{t}_{11} \quad \mathbf{t}_{12} \quad \mathbf{t}_{21} \quad \mathbf{t}_{22} \quad \mathbf{t}_{31} \quad \mathbf{t}_{32}] \tag{2.13.46}$$

2.14 POLES AND ZEROS OF A MIMO TRANSFER FUNCTION MATRIX

2.14.1 Smith Form

For any pxm polynomial matrix $P(s)$, unimodular matrices $U(s)$ and $V(s)$ can be found (Kailath, 1980) such that

$$U(s)P(s)V(s) = \Lambda(s) \tag{2.14.1}$$

where

$$\Lambda(s) = \begin{bmatrix} \lambda_1(s) & 0 & . & 0 & 0 & . & 0 & 0 \\ 0 & \lambda_2(s) & . & 0 & 0 & . & 0 & 0 \\ . & . & . & 0 & 0 & . & 0 & 0 \\ 0 & . & . & \lambda_r(s) & 0 & . & 0 & 0 \\ 0 & 0 & . & 0 & 0 & . & 0 & 0 \\ 0 & 0 & . & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \end{bmatrix} \quad (2.14.2)$$

r is the normal rank of $P(s)$, and $\lambda_i(s);\ i = 1,2,\ldots,r$ are unique monic polynomials obeying the division property

$$\lambda_i(s) \big| \lambda_{i+1}(s); \quad i = 1,2,\ldots,r-1 \quad (2.14.3)$$

The pxm polynomial matrix $\Lambda(s)$ is known as the Smith form of $P(s)$. The notation (2.14.3) indicates that the polynomial $\lambda_i(s)$ is divisible by the polynomial $\lambda_{i+1}(s)$.

2.14.2 Smith–McMillan Form

A transfer function matrix $G(s)$ can be written (Kailath, 1980) as

$$G(s) = \frac{1}{d(s)} N(s) \quad (2.14.4)$$

where $d(s)$ is the monic least common multiple of the denominators of elements of $G(s)$. From (2.14.4),

$$d(s)G(s) = N(s) \quad (2.14.5)$$

Expressing $N(s)$ in the Smith form (2.14.1),

$$N(s) = U(s)\Lambda(s)V(s) \quad (2.14.6)$$

where $U(s)$ and $V(s)$ are unimodular matrices. From (2.14.5) and (2.14.6),

$$U^{-1}(s)G(s)V^{-1}(s) = \frac{1}{d(s)} \Lambda(s) \quad (2.14.7)$$

As the inverse of a unimodular matrix is also a unimodular matrix, any transfer function matrix can be converted to the form on the right-hand side of (2.14.7) via elementary row and column operations. Now, from (2.14.2), nonzero diagonal elements of the matrix on the right-hand side of (2.14.7) will be

$$\frac{\lambda_i(s)}{d(s)}\ ; \quad i=1,2,\ldots,r \tag{2.14.8}$$

where r is the normal rank of $G(s)$.

Eliminating common factors between $\lambda_i(s)$ and $d(s)$,

$$\frac{\lambda_i(s)}{d(s)}=\frac{\varepsilon_i(s)}{\psi_i(s)} \tag{2.14.9}$$

There are no common factors between $\varepsilon_i(s)$ and $\psi_i(s)$. Define

$$M(s)=$$

$$\begin{bmatrix} \varepsilon_1(s)/\psi_1(s) & 0 & . & 0 & 0 & . & 0 & 0 \\ 0 & \varepsilon_2(s)/\psi_2(s) & . & 0 & 0 & . & 0 & 0 \\ . & . & . & 0 & 0 & . & 0 & 0 \\ 0 & . & . & \varepsilon_r(s)/\psi_r(s) & 0 & . & 0 & 0 \\ 0 & 0 & . & 0 & 0 & . & 0 & 0 \\ 0 & 0 & . & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \end{bmatrix} \tag{2.14.10}$$

Then, from (2.14.7) and (2.14.10),

$$U_1^{-1}(s)G(s)U_2^{-1}(s)=M(s) \tag{2.14.11}$$

In other words, a transfer function matrix $G(s)$ can be converted to have the structure of the matrix $M(s)$ via elementary row and column operations. Polynomials in $M(s)$ satisfy the following properties:

$$\psi_{i+1}(s)\big|\psi_i(s)\ ; \quad i=1,2,\ldots,r-1 \tag{2.14.12}$$

$$\varepsilon_i(s)\big|\varepsilon_{i+1}(s)\ ;\ i=1,2,\ldots,r-1 \tag{2.14.13}$$

$$d(s)=\psi_1(s) \tag{2.14.14}$$

Notations (2.14.12) and (2.14.13) indicate that polynomials $\psi_{i+1}(s)$ and $\varepsilon_i(s)$ are divisible by polynomials $\psi_i(s)$ and $\varepsilon_{i+1}(s)$, respectively.

2.14.3 Poles and Zeros via Smith–McMillan Form

Poles of a multivariable transfer function $G(s)$ are roots of denominator polynomials $\psi_i(s)$ in the Smith–McMillan form $M(s)$, Equation 2.4.10. Similarly, zeros of a multivariable transfer function $G(s)$ are roots of numerator polynomials $\varepsilon_i(s)$ in the Smith–McMillan form $M(s)$.

Example 2.15

Find poles and zeros of the following transfer function matrix:

$$G(s) = \frac{1}{s^2 + 4s + 5} P(s) \tag{2.14.15}$$

where

$$P(s) = \begin{bmatrix} 2 & (s+1)^2 \\ (s+1)^2 & -2 \end{bmatrix} \tag{2.14.16}$$

Using elementary operations, $P(s)$ will be first converted to Smith form.

Step I: Divide the first row of $P(s)$ by 1/2:

$$U_1(s)P(s) = \begin{bmatrix} 1 & 0.5(s+1)^2 \\ (s+1)^2 & -2 \end{bmatrix} \tag{2.14.17}$$

where

$$U_1(s) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.14.18}$$

Step II: Subtract $(s+1)^2$ times the first row from the second row:

$$U_2(s)U_1(s)P(s) = \begin{bmatrix} 1 & 0.5(s+1)^2 \\ 0 & -2 - 0.5(s+1)^4 \end{bmatrix} \tag{2.14.19}$$

where

$$U_2(s) = \begin{bmatrix} 1 & 0 \\ -(s+1)^2 & 1 \end{bmatrix} \tag{2.14.20}$$

Step III: Subtract $0.5(s+1)^2$ times the first column from the second column:

$$U_2(s)U_1(s)P(s)V_3(s) = \begin{bmatrix} 1 & 0 \\ 0 & -2-0.5(s+1)^4 \end{bmatrix} \tag{2.14.21}$$

where

$$V_3(s) = \begin{bmatrix} 1 & -0.5(s+1)^2 \\ 0 & 1 \end{bmatrix} \tag{2.14.22}$$

Step IV: Multiplying the second column by –2,

$$U_2(s)U_1(s)P(s)V_3(s)V_4(s) = \begin{bmatrix} 1 & 0 \\ 0 & 4+(s+1)^4 \end{bmatrix} \tag{2.14.23}$$

where

$$V_4(s) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \tag{2.14.24}$$

Now, it can be verified that Equation 2.14.23 has the Smith form (2.14.1), i.e.,

$$U(s)P(s)V(s) = \begin{bmatrix} 1 & 0 \\ 0 & 4+(s+1)^4 \end{bmatrix} \tag{2.14.25}$$

$$U(s) = U_2(s)U_1(s) = \begin{bmatrix} 0.5 & 0 \\ -0.5(s+1)^2 & 1 \end{bmatrix} \tag{2.14.26}$$

and

$$V(s) = V_3(s)V_4(s) = \begin{bmatrix} 1 & (s+1)^2 \\ 0 & -2 \end{bmatrix} \tag{2.14.27}$$

Therefore, from (2.14.15) and (2.14.25),

$$U^{-1}(s)G(s)V^{-1}(s) = \frac{1}{s^2+4s+5}\begin{bmatrix} 1 & 0 \\ 0 & 4+(s+1)^4 \end{bmatrix} \quad (2.14.28)$$

As $(s+1)^4 + 4 = (s^2+4s+5)(s^2+1)$,

$$U^{-1}(s)G(s)V^{-1}(s) = \begin{bmatrix} \dfrac{1}{s^2+4s+5} & 0 \\ 0 & \dfrac{s^2+1}{1} \end{bmatrix} \quad (2.14.29)$$

Equation 2.14.29 has the Smith–McMillan form. Therefore, poles are $-2 \pm j$, which are the roots of $s^2+4s+5=0$. Further, zeros are $\pm j$, which are the roots of $s^2+1=0$.

2.14.4 Poles and Zeros via an Irreducible MFD

Consider the following irreducible MFD of the multivariable transfer function $G(s)$:

$$G(s) = N(s)D^{-1}(s) \quad (2.14.30)$$

Poles of $G(s)$ are roots of $\det D(s) = 0$. Zeros of $G(s)$ are values of s for which $N(s)$ is not of full rank.

Example 2.16

For the transfer function matrix (2.12.6), the irreducible MFD is represented by

$$G(s) = \bar{N}(s)\bar{D}^{-1}(s) \quad (2.14.31)$$

where $\bar{N}(s)$ and $\bar{D}(s)$ are represented by (2.12.8) and (2.12.9), respectively. Now,

$$\det \bar{D}(s) = \frac{2}{3}(s+3)(s+4)(s+5) \quad (2.14.32)$$

Therefore, poles are located at –3, –4, and –5. There is no s for which the rank of $\bar{N}(s)$ is less than 2. Therefore, there is no zero.

2.15 STABILITY ANALYSIS

The stability of a linear and time-invariant system is not influenced by external inputs $\mathbf{u}(t)$. Therefore, $\mathbf{u}(t)$ will be set to zero, and the following initial-value problem is considered:

$$\dot{\mathbf{x}} = A\mathbf{x}(t) \; ; \quad \mathbf{x}(0) \neq 0 \tag{2.15.1}$$

The system (2.15.1) is described to be stable when

$$\mathbf{x}(t) \to 0 \;\; \text{as} \;\; t \to \infty \tag{2.15.2}$$

This property is also referred to as asymptotic stability. The condition (2.15.2) is satisfied when all the eigenvalues of A have negative (nonzero) real parts, i.e., they are located in the left half (excluding the imaginary axis) of the complex plane.

Define a positive definite (Appendix B) function $V(t)$ as

$$V(t) = \mathbf{x}^T(t)P\mathbf{x}(t) \quad \text{where} \quad P = P^T > 0 \tag{2.15.3}$$

Differentiating (2.15.3) with respect to time,

$$\dot{V}(t) = \dot{\mathbf{x}}^T(t)P\mathbf{x}(t) + \mathbf{x}^T(t)P\dot{\mathbf{x}}(t) \tag{2.15.4}$$

Substituting (2.15.1) into (2.15.4),

$$\dot{V}(t) = \mathbf{x}^T(t)[A^TP + PA]\mathbf{x}(t) \tag{2.15.5}$$

For a stable system, $\dot{V}(t)$ is a negative definite function. Therefore,

$$A^TP + PA = -Q \tag{2.15.6}$$

where $Q = Q^T > 0$.

Equation 2.15.6 is known as the Lyapunov equation, often written as

$$A^TP + PA + Q = 0 \tag{2.15.7}$$

An Important Property of the Lyapunov Equation

When $Q = Q^T > 0$, and all the eigenvalues of A have negative (nonzero) real parts, the Lyapunov equation 2.15.7 has a unique solution for P, satisfying $P = P^T > 0$ as follows (Kailath, 1980):

$$P = \int_0^\infty e^{A^Tt}Qe^{At}\,dt \tag{2.15.8}$$

EXERCISE PROBLEMS

P2.1 Consider the following transfer function:

$$G(s) = \frac{4s^3 + 25s^2 + 45s + 34}{s^3 + 6s^2 + 10s + 8}$$

Develop state space realizations using Methods I and II. Write state space equations and draw the block diagram for each method.

P2.2 Using the properties of the determinant, find the characteristic polynomial of the following matrix:

$$\begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 \\ -a_n & -a_{n-1} & . & . & . & -a_1 \end{bmatrix}$$

P2.3

a. If $\bar{A} = T^{-1}AT$, show that $e^{At} = Te^{\bar{A}t}T^{-1}$.

b. If there are n independent eigenvectors for the $n \times n$ matrix A, develop an algorithm to compute e^{At} using the result in part (a).

P2.4 Matlab Exercise

Consider the system shown in the Figure P.2.4 where

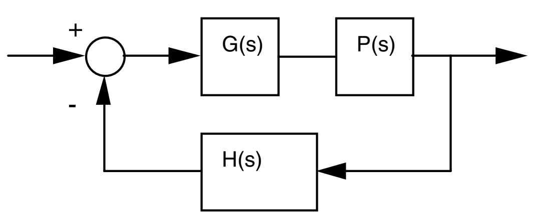

FIGURE P.2.4 A Feedback System.

$$G(s) = \frac{s+2}{s^2 + 6s + 8} \qquad P(s) = \frac{s^3 + 2s^2 + 9s + 10}{s^3 + s^2 + 10s + 20}$$

$$H(s) = \frac{s+1}{s+5}$$

a. Develop state space models for both open-loop and closed-loop systems.

b. Examine controllability and observability of these state space models.
c. Determine the stability of open- and closed-loop systems.

P2.5 Consider a single input/single output system with the following transfer function:

$$\frac{s+\alpha}{(s+1)(s+2)(s+3)}$$

a. Construct a state space realization $\{A, \mathbf{b}, \mathbf{c}\}$ that is controllable for all values of α. You should also show the corresponding simulation diagram.
b. Find the values of α for which the state space realization developed in part (a) is not observable.

P2.6 Consider a linear system described by the following state space equation:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -x_1 + u$$

$$y(t) = x_1(t) + x_2(t)$$

Given y(0) = 0, $\frac{dy}{dt}(0) = 1$, and u(0) = 0.

a. Determine $x_1(0)$ and $x_2(0)$.
b. Determine the response $x_1(t)$ and $x_2(t)$ via the matrix exponential when $u(t) = 1$ for $t > 0$.

P2.7 Consider a single input/single output system with the following transfer function:

$$\frac{s+2}{(s+1)(s+2)(s+3)}$$

a. Construct a state space realization {A, **b**, **c**} that is controllable. You should also show the corresponding simulation diagram.
b. Answer the following questions:
   i. Will it be possible to construct a realization that is both controllable and observable?
   ii. Will it be possible to construct a realization that is neither controllable nor observable?

P2.8 Consider the following system:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + \mathbf{b}u(t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find $u(t)$, $0 \le t \le 1$, such that $\mathbf{x}(1) = [1 \quad 0]^T$.

P2.9 Consider the state space realization using Method II. Show that this realization is controllable, provided there is no common factor between numerator and denominator of the transfer function.

P2.10 Consider the following state space realization:

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix}\mathbf{x}(t)$$

i. Find the transfer function of the system.
ii. Determine the observability and controllability of the state space realization.
iii. Let $y(0) = 1$, $\frac{dy}{dt}(0) = 0$, and u(0) =0. Find $\mathbf{x}(0)$.
iv. Find e^{At}.
v. Solve the state equations when $\mathbf{x}(0) = \mathbf{0}$ and $u(t) = 1$ for $t \ge 0$.

P2.11 Consider the following transfer function matrix:

$$H(s) = \begin{bmatrix} \dfrac{s+1}{s^2+s+1} & \dfrac{s+2}{s^3+2s^2+1} \end{bmatrix}$$

a. Develop the state space realization using Method I. Determine its controllability and observability.
b. Develop the state space realization using Method II. Determine its controllability and observability.

P2.12 Consider the problem of the control of a tire-tread extrusion line:

$$G(s) = \begin{bmatrix} \dfrac{0.071}{s+0.19} & \dfrac{0.007}{(s+0.23)^2} \\ \dfrac{0.24}{(s+0.23)^2} & \dfrac{0.027}{(s+0.3)^2} \end{bmatrix}$$

a. Find the controller-form state space realization using a suitable MFD.
b. Determine controllability and observability of your state space realization and discuss your results.
c. Find poles and zeros of the system.

P2.13 Consider a system with the following transfer function matrix:

$$G(s) = \begin{bmatrix} -\dfrac{s}{(s+1)} & \dfrac{1}{(s+1)} \\ \dfrac{(2s+1)}{s(s+1)} & \dfrac{1}{s+1} \end{bmatrix}$$

i. Find an irreducible right MFD.
ii. What is the order of a minimal realization?
iii. Find the poles and transmission zeros.

P2.14 Consider the linearized dynamics of a spark ignition engine (Abate et al., 1994):

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \frac{1}{d(s)} \begin{bmatrix} \alpha_1\alpha_4 & \alpha_6(s+\alpha_2) \\ \alpha_1(Js+\alpha_7-\alpha_5) & -\alpha_3\alpha_6 \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

where $y_1(t)$ = engine speed, $y_2(t)$ = relative air pressure of manifold, $u_1(t)$ = duty cycle of the throttle valve, and $u_2(t)$ = spark advance position. The polynomial $d(s)$ is defined as follows:

$$d(s) = Js^2 + (J\alpha_2 + \alpha_7 - \alpha_5)s + (\alpha_3\alpha_4 + \alpha_2\alpha_7 - \alpha_2\alpha_5)$$

Parameters J (mass-moment of inertia) and $\alpha_1 - \alpha_7$ are provided in Table 2.P.1 for three different operating conditions I, II, and III.

TABLE 2.P1
Parameters of a Spark Engine

	J	α_1	α_2	α_3	α_4	α_5	α_6	α_7
I	1	2.1608	0.1027	0.0357	0.5607	2.0183	4.4962	2.0283
II	1	3.4329	0.1627	0.1139	0.2539	1.7993	2.0247	1.8201
III	10	2.1608	0.1027	0.0357	0.5607	1.7993	4.4962	1.8201

a. Develop a minimum-order state space realization.
b. Find poles and zeros of the transfer function matrix for each operating condition.

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3 State Feedback Control and Optimization

First, the design of a full state feedback control system is presented for a single input/single output (SISO) system along with its impact on poles and zeros of the closed-loop system. Next, the full state feedback control system is presented for a multiinput/multioutput (MIMO) system. The necessary conditions for the optimal control are then derived and used to develop the linear quadratic (LQ) control theory and the minimum time control.

3.1 STATE VARIABLE FEEDBACK FOR A SINGLE INPUT SYSTEM

Consider the state space realization $\{A, \mathbf{b}, \mathbf{c}\}$; i.e.,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + \mathbf{b}u(t) \tag{3.1.1}$$

$$y(t) = \mathbf{c}\mathbf{x}(t) \tag{3.1.2}$$

The open-loop transfer function is given as

$$\frac{y(s)}{u(s)} = \mathbf{c}(sI - A)^{-1}\mathbf{b} = \frac{\mathbf{c}Adj(sI - A)\mathbf{b}}{\det(sI - A)} \tag{3.1.3}$$

The state variable feedback system is shown in Figure 3.1, in which

$$u(t) = v(t) - \mathbf{k}\mathbf{x}(t) \tag{3.1.4}$$

where $v(t)$ is the external input and $\mathbf{k}$ is the state feedback vector. Using (3.1.1) and (3.1.4),

$$\frac{d\mathbf{x}}{dt} = (A - \mathbf{b}\mathbf{k})\mathbf{x}(t) + \mathbf{b}v(t) \tag{3.1.5}$$

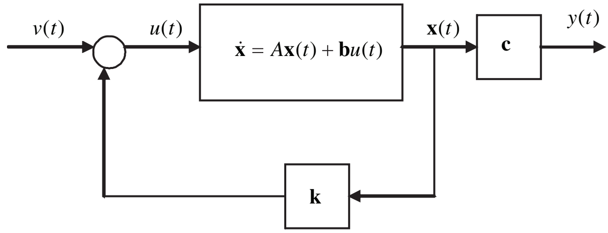

FIGURE 3.1 A state feedback control system.

Equation 3.1.5 represents the state dynamics of the closed-loop system. The transfer function of the closed-loop system is

$$\frac{y(s)}{v(s)} = \mathbf{c}(sI - A + \mathbf{bk})^{-1}\mathbf{b} = \frac{\mathbf{c}Adj(sI - A + \mathbf{bk})\mathbf{b}}{\det(sI - A + \mathbf{bk})} \tag{3.1.6}$$

It is desirable to study the effects of state feedback on poles and zeros of the closed-loop transfer function.

3.1.1 Effects of State Feedback on Poles of the Closed-Loop Transfer Function: Computation of State Feedback Gain Vector

It will be shown here that poles of the closed-loop transfer function can be located anywhere in the s-plane by the full state feedback, provided the state space realization is controllable. In other words, it is always possible to find the state feedback vector $\mathbf{k}$ corresponding to any desired locations of closed-loop poles, provided the state space realization is controllable.

Let

$$\frac{y(s)}{u(s)} = \frac{b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1}s + a_n} \tag{3.1.7}$$

Hence, the characteristic polynomial of the open-loop system is

$$a(s) = \det(sI - A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1}s + a_n \tag{3.1.8}$$

Let the desired characteristic polynomial of the closed-loop system be

$$\alpha(s) = \det(sI - A + \mathbf{bk}) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \ldots + \alpha_{n-1}s + \alpha_n \tag{3.1.9}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary real numbers.

Now,

$$\begin{aligned}\alpha(s) &= \det(sI - A + \mathbf{bk}) = \det\{(sI - A)(I + (sI - A)^{-1}\mathbf{bk})\} \\ &= a(s).\det(I + (sI - A)^{-1}\mathbf{bk})\end{aligned} \tag{3.1.10}$$

Using the identity $\det(I_n - PQ) = \det(I_m - QP)$, where P and Q are $n \times m$ and $m \times n$ matrices, respectively, Equation 3.1.10 can be written (Kailath, 1980) as

$$\alpha(s) = a(s)[1 + \mathbf{k}(sI - A)^{-1}\mathbf{b}] \tag{3.1.11}$$

or

$$\alpha(s) - a(s) = \mathbf{k}adj(sI - A)\mathbf{b} \tag{3.1.12}$$

Using Equation 2.3.4 and equating coefficients of $s^{n-1}, s^{n-2}, \ldots, s^0$ on both sides, we obtain

$$\begin{aligned}&\alpha_1 - a_1 = \mathbf{kb} \\ &\alpha_2 - a_2 = \mathbf{k}A\mathbf{b} + a_1\mathbf{kb} \\ &\alpha_3 - a_3 = \mathbf{k}A^2\mathbf{b} + a_1\mathbf{k}A\mathbf{b} + a_2\mathbf{kb} \\ &\vdots \\ &\alpha_n - a_n = \mathbf{k}A^{n-1}\mathbf{b} + a_1\mathbf{k}A^{n-2}\mathbf{b} + a_2\mathbf{k}A^{n-3}\mathbf{b} + \ldots + a_{n-1}\mathbf{kb}\end{aligned} \tag{3.1.13}$$

Representing the system of equations in matrix form,

$$\boldsymbol{\alpha} - \mathbf{a} = \mathbf{k}CU_t \tag{3.1.14}$$

where C is the controllability matrix,

$$\boldsymbol{\alpha} = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad . \quad . \quad \alpha_n] \tag{3.1.15}$$

$$\mathbf{a} = [a_1 \quad a_2 \quad a_3 \quad . \quad . \quad a_n] \tag{3.1.16}$$

and U_t is the upper triangular Toeplitz matrix defined as

$$U_t = \begin{bmatrix} 1 & a_1 & a_2 & . & . & a_{n-1} \\ 0 & 1 & a_1 & a_2 & . & a_{n-2} \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & 1 \end{bmatrix} \tag{3.1.17}$$

It can be seen that U_t is always nonsingular. Hence, for any choice of $\boldsymbol{\alpha}$, $\mathbf{k}$ can be found, provided C is nonsingular (or the system is controllable). The result is

$$\mathbf{k} = (\boldsymbol{\alpha} - \mathbf{a})U_t^{-1}C^{-1} \tag{3.1.18}$$

This is also known as the Bass–Gura formula for computing $\mathbf{k}$.

The fact that the eigenvalues can be arbitrarily relocated by state feedback is known as *modal controllability.* This can be understood by realizing that states of a system possess all the information about system dynamics. Hence, feeding back all the states is equivalent to using all the information in deciding the control input.

Another important point to note is that the order of the closed-loop system is the same as that of the open-loop system, which is not always true for the output feedback via a compensator.

Example 3.1: Balancing a Pointer

Consider the balancing of a pointer (Kailath, 1980) on your fingertip (Figure 3.2). If you try to do it yourself, you will find that the pointer will fall down when the fingertip does not move. Furthermore, the fingertip must move to balance the pointer. In fact, the acceleration of the fingertip serves as the control input. To develop a mathematical model, the following assumptions are made (Kailath, 1980):

1. The mass of the pointer is concentrated at the top end.
2. The angle ϕ is small.
3. The force F from the fingertip is applied only along the direction of the pointer.

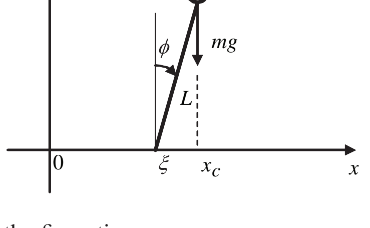

FIGURE 3.2 A pointer on the fingertip.

Applying Newton's second law of motion,

$$m\ddot{x}_c = F\sin\phi \approx F\phi \tag{3.1.19}$$

where x_c is the x-coordinate of the center of mass. For a small ϕ, the acceleration of the center of mass along the y-direction can be neglected. Therefore,

$$mg = F\cos\phi \approx F \tag{3.1.20}$$

From (3.1.19) and (3.1.20),

$$\ddot{x}_c = g\phi \tag{3.1.21}$$

Now,

$$x_c(t) = \xi(t) + L\phi(t) \tag{3.1.22}$$

where $\xi(t)$ is the x-coordinate of the fingertip. Substituting (3.1.22) into (3.1.21),

$$\ddot{\phi} = \frac{g}{L}\phi - u(t) \tag{3.1.23}$$

where

$$u(t) = \frac{\ddot{\xi}}{L} \tag{3.1.24}$$

The variable $u(t)$, which is proportional to the fingertip acceleration $\ddot{\xi}$, will be treated as the control input. The state variable equations are described as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \tag{3.1.25}$$

where

$$q_1 = \phi \quad \text{and} \quad q_2 = \dot{\phi} \tag{3.1.26}$$

Furthermore,

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{3.1.27}$$

Open-Loop Stability

$$\det(sI - A) = s^2 - \frac{g}{L} \tag{3.1.28}$$

Open-loop eigenvalues are $\sqrt{\frac{g}{L}}$ and $-\sqrt{\frac{g}{L}}$. Hence, the open-loop system is unstable which is exhibited by the fact that the pointer falls down if the fingertip does not have any acceleration, i.e., $u(t) = 0$. Furthermore, if L is smaller, the magnitude of the unstable eigenvalue $\sqrt{\frac{g}{L}}$ is larger. This matches with the fact that it is harder to balance a small pointer. Try it.

Controllability

The controllability matrix is

$$C = \begin{bmatrix} \mathbf{b} & A\mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{3.1.29}$$

The matrix C is nonsingular. Hence, the state space realization is controllable.

State Feedback Control

$$u(t) = -k_1 q_1 - k_2 q_2 = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{3.1.30}$$

$$A - b\mathbf{k} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} + k_1 & k_2 \end{bmatrix} \tag{3.1.31}$$

$$\det(sI - A + b\mathbf{k}) = \det \begin{bmatrix} s & -1 \\ -\left(\frac{g}{L} + k_1\right) & s - k_2 \end{bmatrix} = s^2 - k_2 s - \left(\frac{g}{L} + k_1\right) \tag{3.1.32}$$

Let the desired closed-loop poles be –1 and –2. In this case, the desired closed-loop characteristic equation will be

$$\det(sI - A + b\mathbf{k}) = (s+1)(s+2) = s^2 + 3s + 2 \tag{3.1.33}$$

Matching the coefficients of polynomials,

$$-k_2 = 3 \Rightarrow k_2 = -3 \tag{3.1.34}$$

$$-\left(\frac{g}{L} + k_1\right) = 2 \Rightarrow k_1 = -\frac{g}{L} - 2 \tag{3.1.35}$$

Comments

1. To determine state feedback vector, the Bass–Gura formula (3.1.18) can also be used. But for a low-order problem, it is more straightforward to directly match the coefficients of polynomials. The Bass–Gura formula is extremely useful to solve higher order problems via a computer software such as MATLAB®.
2. The control input is $u(t) = -k_1\phi - k_2\dot{\phi}$. Its implementation requires measurements of ϕ and $\dot{\phi}$, and real-time computation of $-k_1\phi - k_2\dot{\phi}$. When a human being tries to balance a pointer, one can say that the eyes are estimating the values of ϕ and $\dot{\phi}$, and the brain is deciding on a suitable control input and instructing the motor to move the fingertip with proper acceleration.

3.1.2 Effects of State Feedback on Zeros of the Closed-Loop Transfer Function

If the state space realization is controllable, it can be converted to the controller canonical form $\{A_c, \mathbf{b}_c, \mathbf{c}_c\}$ described in Chapter 2, Section 2.2. Under the state feedback, the system matrix of the closed-loop system is given as

$$A_c - \mathbf{b}_c\mathbf{k} = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ & & & & & \\ 0 & 0 & 0 & . & . & 1 \\ -a_n - k_1 & -a_{n-1} - k_2 & . & . & . & -a_1 - k_n \end{bmatrix} \tag{3.1.36}$$

Hence, the realization $\{A_c - \mathbf{b}_c\mathbf{k}, \mathbf{b}_c, \mathbf{c}_c\}$ remains in the controller canonical form. Therefore, the transfer function of the closed-loop system is given as

$$\mathbf{c}_c(sI - A_c + \mathbf{b}_c\mathbf{k})^{-1}\mathbf{b}_c = \frac{b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + (a_1 + k_n)s^{n-1} + \ldots + (a_n + k_1)} \tag{3.1.37}$$

This expression clearly indicates that the numerator polynomial of the transfer function remains unchanged. In other words, the state feedback has no influence on the zeros of the transfer function (Kailath, 1980).

Example 3.2: Bicycle Dynamics

Consider a simple model of a bicycle (Lowell and McKell, 1982; Astrom et al., 2005), in which the rider, wheels, front-fork assembly, and the rear frame are treated as a single rigid body as shown by the plane in Figure 3.3. The total mass is m and the location of the center of gravity (cg) is shown in the figure. Assume that the forward velocity v of the bicycle is a constant. The small angle θ is the perturbation from the bicycle's upright position. As the weight mg will further try to increase this angle, the rider turns the handlebar by a small angle α, so that the bicycle begins to travel in a circle with the instant radius r and the instant center of rotation O. This circular travel is represented by the angle ϕ about the vertical axis passing through the rear-wheel contact point. Therefore, the acceleration of the cg in the direction normal to the plane is

$$h\ddot{\theta} + b\ddot{\phi} + \frac{v^2}{r} \tag{3.1.38}$$

where v^2 / r is the centripetal acceleration, and b is the distance of cg from the vertical axis passing through the rear-wheel contact point. Applying Newton's second law,

$$m\left(h\ddot{\theta} + b\ddot{\phi} + \frac{v^2}{r}\right) = mg\theta \tag{3.1.39}$$

where $mg\theta$ is the component of the weight along the direction normal to the frame for small θ.

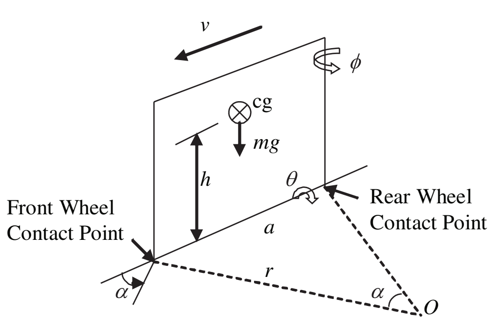

FIGURE 3.3 Fundamentals of bicycle dynamics.

From the definition of the instantaneous center for rotation:

$$r\dot{\phi} = v \quad \text{and} \quad r\alpha = a \tag{3.1.40a,b}$$

Substituting (3.1.40) into (3.1.39), the input/output equation is obtained:

$$\ddot{\theta} - \frac{g}{h}\theta = -\frac{bv}{ha}\dot{\alpha} - \frac{v^2}{ah}\alpha \tag{3.1.41}$$

where angles $\theta(t)$ and $\alpha(t)$ are the output and the input, respectively. Taking the Laplace transform of (3.1.41) with zero initial conditions:

$$\frac{\theta(s)}{\alpha(s)} = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2} \tag{3.1.42}$$

where

$$b_1 = -\frac{bv}{ha}\,; \quad b_2 = -\frac{v^2}{ha}\,; \quad a_1 = 0\,; \quad a_2 = -\frac{g}{h} \tag{3.1.43}$$

Now, a state space model can be constructed via either Method I or Method II in Chapter 2, Section 2.2.

There are two poles of the system:

$$-\sqrt{\frac{g}{h}} \quad \text{and} \quad -\sqrt{\frac{g}{h}} \tag{3.1.44}$$

And there is one zero:

$$-\frac{v}{b} \tag{3.1.45}$$

The open-loop system is unstable, and has a zero in the left half plane. The system can be stabilized by a simple feedback law:

$$\alpha = k_p\theta \tag{3.1.46}$$

Substituting (3.1.46) into (3.1.39), the closed-loop system is

$$\ddot{\theta}+\frac{bv}{ha}k_p\dot{\theta}+\left(\frac{v^2}{ah}k_p-\frac{g}{h}\right)\theta=0 \tag{3.1.47}$$

For stability,

$$k_p > \frac{ag}{v^2} \tag{3.1.48}$$

Note that the control law (3.1.46) has to be implemented by the rider using his or her eye, brain, or hand. Equation 3.1.48 indicates that the lower bound of k_p is smaller at a higher velocity. In fact, the damping is also higher at a higher velocity, Equation 3.1.47. Therefore, the rider finds it easier to stabilize the bicycle at a higher velocity.

State Feedback Control

Using Method I (Chapter 2, Section 2.2):

$$\dot{\mathbf{x}} = A\mathbf{x}(t)+\mathbf{b}u(t) \tag{3.1.49}$$

$$y(t) = \mathbf{c}\mathbf{x}(t) \tag{3.1.50}$$

where

$$A=\begin{bmatrix} 0 & 1 \\ g/h & 0 \end{bmatrix}; \quad \mathbf{b}=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{c}=\begin{bmatrix} b_1 & b_2 \end{bmatrix} \tag{3.1.51}$$

and

$$\det(sI-A+\mathbf{bk})=\det\begin{bmatrix} s & -1 \\ -g/h+k_1 & s+k_2 \end{bmatrix}=s^2+k_2s+(k_1-g/h) \tag{3.1.52}$$

Let the desired characteristic polynomial be

$$\det(sI-A+\mathbf{bk})=s^2+\alpha_1 s+\alpha_2 \tag{3.1.53}$$

Matching coefficients of polynomials in (3.1.52) and (3.1.53),

$$k_1=\alpha_2+g/h \tag{3.1.54}$$

and

$$k_2 = \alpha_1 \tag{3.1.55}$$

And, the closed-loop transfer function is

$$\mathbf{c}(sI - A + \mathbf{bk})^{-1}\mathbf{b} = \frac{b_1 s + b_2}{s^2 + \alpha_1 s + \alpha_2} \tag{3.1.56}$$

Note that the zero of the transfer function remains unchanged, whereas the poles have been located arbitrarily via state feedback control:

$$u(t) = u_r(t) - \mathbf{kx}(t) \tag{3.1.57}$$

where $u_r(t)$ is the reference input, which will be zero in this case. Compared to the state feedback law (3.1.57), the proportional controller (3.1.46) is simpler and hence easier to be implemented by a rider. This may explain why a person can ride a bicycle for hours, whereas he or she can balance a pointer on a finger only for a few minutes as full state feedback (Equation 3.1.30) is necessary for stability.

3.1.3 State Feedback Control for a Nonzero and Constant Output

Let the external input v(t) be a constant v_0. From Equation 3.1.5, the steady state value of the state vector $\mathbf{x}_s$ is given by

$$0 = (A - \mathbf{bk})\mathbf{x}_s + \mathbf{b}v_0 \tag{3.1.58}$$

or

$$\mathbf{x}_s = -(A - \mathbf{bk})^{-1}\mathbf{b}v_0 \tag{3.1.59}$$

Hence, the steady state output is given as

$$y_s = \mathbf{cx}_s = -\mathbf{c}(A - \mathbf{bk})^{-1}\mathbf{b}v_0 \tag{3.1.60}$$

If the desired value of the output is y_d, the corresponding command input v_0 is obtained by setting $y_s = y_d$ and solving (3.1.60). The result is

$$v_0 = -\frac{y_d}{\mathbf{c}(A - \mathbf{bk})^{-1}\mathbf{b}} \quad \text{provided} \quad \mathbf{c}(A - \mathbf{bk})^{-1}\mathbf{b} \neq 0 \tag{3.1.61}$$

Now, let us examine the conditions under which $\mathbf{c}(A-\mathbf{bk})^{-1}\mathbf{b} \neq 0$. Substituting s = 0 in Equation 3.1.6, it is clear that $\mathbf{c}(A-\mathbf{bk})^{-1}\mathbf{b} \neq 0$ if there is no zero of the closed-loop transfer function at s = 0. As locations of zeros remain unchanged under state feedback, the command input for the nonzero set point can be found, provided the open-loop transfer function does not have any zero located at $s = 0$ (Kailath, 1980).

Example 3.3: Spring-Mass System

For the spring-mass-damper system (Chapter 2, Figure 2.4), state equations are defined by (2.5.16) and (2.5.17).

Case I: Position Output

If the position of the mass, x, is measured by a sensor,

$$y(t) = \mathbf{cx}(t) \tag{3.1.62}$$

where

$$\mathbf{c} = [1 \quad 0] \tag{3.1.63}$$

Let the state feedback control law be

$$u(t) = v_0 - \mathbf{kx}(t) \tag{3.1.64}$$

It can be shown that

$$\mathbf{c}(A-\mathbf{bk})^{-1}\mathbf{b} = -\left(k_1 + \frac{\beta}{m}\right)^{-1} \tag{3.1.65}$$

From Equation 3.1.61, to achieve the desired set point y_d for the output,

$$v_0 = -\frac{y_d}{\mathbf{c}(A-\mathbf{bk})^{-1}\mathbf{b}} = y_d\left(k_1 + \frac{\beta}{m}\right) \tag{3.1.66}$$

If f_0 is the force corresponding to v_0,

$$v_0 = \frac{f_0}{m} \tag{3.1.67}$$

From (3.1.66) and (3.1.67),

$$f_0 = y_d(mk_1 + \beta) \tag{3.1.68}$$

From (3.1.64),

$$f(t) = f_0 - mk_1 x - mk_2 \dot{x} \tag{3.1.69}$$

Substituting (3.1.69) into the differential Equation 2.5.14,

$$m\ddot{x} + (\alpha + mk_2)\dot{x} + (\beta + mk_1)x = f_0 \tag{3.1.70}$$

Therefore, position and velocity feedback coefficients k_1 and k_2 add to the stiffness and damping coefficient, respectively. Furthermore, conditions (3.1.66) and (3.1.68) represent the static equilibrium condition.

Case II: Velocity Output

If the velocity of the mass, $\dot{x}$, is measured by a sensor,

$$y(t) = \mathbf{cx}(t) \tag{3.1.71}$$

where

$$\mathbf{c} = [0 \quad 1] \tag{3.1.72}$$

In this case,

$$\mathbf{c}(A - \mathbf{bk})^{-1}\mathbf{b} = 0 \tag{3.1.73}$$

And a nonzero set point y_d cannot be achieved. Mathematically, this is due to the presence of a zero at $s = 0$ as the transfer function is

$$\frac{y(s)}{u(s)} = \frac{ms}{ms^2 + \alpha s + \beta} \tag{3.1.74}$$

Physically, nonzero constant velocity is not possible when the applied force is a constant.

3.1.4 State Feedback Control under Constant Input Disturbances: Integral Action

Consider the following system:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b}u(t) + \mathbf{w} \tag{3.1.75}$$

$$y(t) = \mathbf{c}\mathbf{x}(t) \tag{3.1.76}$$

where **w** is a constant disturbance vector of unknown magnitudes. The objective is to develop a state feedback control algorithm such that $\lim_{t\to\infty} y(t) = 0.$ The reader can verify that the algorithm (3.1.4) will not be able to achieve this goal. The required algorithm is developed by defining a new variable q(t) as follows:

$$\frac{dq}{dt} = y(t) = \mathbf{c}\mathbf{x}(t); \qquad q(0) = 0 \tag{3.1.77}$$

Defining a new state vector,

$$\mathbf{p}(t) = \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} \tag{3.1.78}$$

Using (3.1.75) to (3.1.77),

$$\frac{d\mathbf{p}}{dt} = A_a\mathbf{p}(t) + \mathbf{b}_a u(t) + \begin{bmatrix} \mathbf{w} \\ 0 \end{bmatrix} \tag{3.1.79}$$

where

$$A_a = \begin{bmatrix} A & 0 \\ \mathbf{c} & 0 \end{bmatrix}; \quad \mathbf{b}_a = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \tag{3.1.80}$$

Lemma

The augmented system $\{A_a, \mathbf{b}_a\}$ is controllable provided that the original system $\{A, \mathbf{b}\}$ is controllable and

$$rank\begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & 0 \end{bmatrix} = n + 1 \tag{3.1.81}$$

Proof

The controllability matrix (Gopal, 1984) for the system $\{A_a, \mathbf{b}_a\}$ is

$$C_a = \begin{bmatrix} \mathbf{b} & A\mathbf{b} & A^2\mathbf{b} & . & . & A^n\mathbf{b} \\ 0 & \mathbf{cb} & \mathbf{c}A\mathbf{b} & . & . & \mathbf{c}A^{n-1}\mathbf{b} \end{bmatrix} \tag{3.1.82}$$

$$= \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c} & 0 \end{bmatrix} \begin{bmatrix} 0 & C \\ 1 & 0 \end{bmatrix} \tag{3.1.83}$$

The matrix

$$\begin{bmatrix} 0 & C \\ 1 & 0 \end{bmatrix} \tag{3.1.84}$$

is nonsingular if the original system $\{A, \mathbf{b}\}$ is controllable; i.e., the matrix C is nonsingular. Therefore, C_a is nonsingular if the condition (3.1.81) is satisfied. Now, the state feedback control for the augmented system is

$$u(t) = -\mathbf{kx}(t) - k_q q(t) \tag{3.1.85}$$

Substituting (3.1.85) into (3.1.79),

$$\begin{bmatrix} \frac{d\mathbf{x}}{dt} \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} A - \mathbf{bk} & -\mathbf{b}k_q \\ \mathbf{c} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} \mathbf{w} \\ 0 \end{bmatrix} \tag{3.1.86}$$

Assuming that C_a is nonsingular, it is always possible to find $\mathbf{k}$ and k_q such that eigenvalues of system (3.1.86) are in the left half of the s-plane. Because $\mathbf{w}$ is a constant vector, steady state values of $\mathbf{x}(t)$ and q(t) will be constants for a stable closed-loop system. Hence, in steady state, $\frac{dq}{dt} = 0$. From Equation 3.1.77, $\lim_{t \to \infty} y(t) = 0$. Finally, the control law (3.1.85) can be represented as

$$u(t) = -\mathbf{kx}(t) - k_q \int_0^t y(t)dt \tag{3.1.87}$$

Hence, by feeding back the integral of the output in addition to states, we can have the output go to zero in the presence of a constant disturbance vector with unknown magnitudes (Kailath, 1980).

Example 3.4: Integral Output Feedback

For the spring-mass-damper system (Chapter 2, Figure 2.4), state equations are defined by (2.5.16) and (2.5.17). In addition, consider a constant but unknown disturbance force acting on the mass.

Case I: Position Output

If the position of the mass, x, is measured by a sensor,

$$y(t) = \mathbf{cx}(t) \tag{3.1.88}$$

where

$$\mathbf{c} = [1 \quad 0] \tag{3.1.89}$$

In this case,

$$rank\begin{bmatrix} A_a & \mathbf{b} \\ \mathbf{c} & 0 \end{bmatrix} = rank\begin{bmatrix} 0 & 1 & 0 \\ -\alpha/m & -\beta/m & 1 \\ 1 & 0 & 0 \end{bmatrix} = 3 \tag{3.1.90}$$

Therefore, the condition (3.1.81) is satisfied and the augmented system is controllable.

For the control law (3.1.52), define

$$\mathbf{k}_a = [\mathbf{k} \quad k_q] = [k_1 \quad k_2 \quad k_q] \tag{3.1.91}$$

Then, the closed-loop characteristic polynomial is

$$\det(sI - (A_a - \mathbf{b}_a\mathbf{k}_a)) = s^3 + \left(\frac{\alpha}{m} + k_2\right)s^2 + \left(\frac{\beta}{m} + k_1\right)s + k_q \tag{3.1.92}$$

Let the desired characteristic polynomial be

$$\det(sI - (A_a - \mathbf{b}_a\mathbf{k}_a)) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 \tag{3.1.93}$$

Matching the coefficients of polynomials,

$$k_1 = \alpha_2 - \frac{\beta}{m}; \quad k_2 = \alpha_1 - \frac{\alpha}{m}; \quad \text{and} \quad k_q = \alpha_3 \tag{3.1.94}$$

Case II: Velocity Output

If the velocity of the mass, $\dot{x}$, is measured by a sensor,

$$y(t) = \mathbf{cx}(t) \tag{3.1.95}$$

where

$$\mathbf{c} = [0 \quad 1] \tag{3.1.96}$$

In this case,

$$rank\begin{bmatrix} A_a & \mathbf{b} \\ \mathbf{c} & 0 \end{bmatrix} = rank\begin{bmatrix} 0 & 1 & 0 \\ -\alpha/m & -\beta/m & 1 \\ 0 & 1 & 0 \end{bmatrix} = 2 \tag{3.1.97}$$

Therefore, the condition (3.1.48) is not satisfied, and the augmented system is not controllable.

3.2 COMPUTATION OF STATE FEEDBACK GAIN MATRIX FOR A MULTIINPUT SYSTEM

The state feedback control law is

$$\mathbf{u}(s) = \mathbf{r}(s) - K\mathbf{x}(s) \tag{3.2.1}$$

where $\mathbf{r}(s)$ and K are reference input vector and state feedback gain matrix, respectively.

Using (2.13.8) and (2.13.18),

$$\mathbf{u}(s) = \mathbf{r}(s) - K\Psi(s)\boldsymbol{\xi}(s) \tag{3.2.2}$$

Substituting (3.2.2) into (2.13.6),

$$\left[D_{hc}S(s) + (D_{lc} + K)\Psi(s)\right]\boldsymbol{\xi}(s) = \mathbf{r}(s) \tag{3.2.3}$$

Equation 3.2.3 yields

$$\boldsymbol{\xi}(s) = D_k^{-1}(s)\mathbf{r}(s) \tag{3.2.4}$$

where

$$D_k(s) = D_{hc}S(s) + (D_{lc} + K)\Psi(s) = D(s) + K\Psi(s) \tag{3.2.5}$$

Substituting (3.2.4) into (2.13.5),

$$\mathbf{y}(s) = G_K(s)\mathbf{r}(s) \tag{3.2.6}$$

where

$$G_K(s) = N(s)D_K^{-1}(s) \tag{3.2.7}$$

$G_K(s)$ is the closed-loop transfer function with full state feedback. Examining the structures of (3.2.5) and (3.2.7), the following points (Kailath, 1980) should be noted:

- State feedback does not alter the numerator polynomial *N(s)*.
- State feedback does not alter D_{hc}. Therefore, the state feedback control cannot change the column degrees of *D(s)*.

Let the characteristic polynomial of the closed-loop system be described by the following monic polynomial:

$$\alpha(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1}s + \alpha_n \tag{3.2.8}$$

Assume that the column degrees (Chapter 2, Section 2.8) of *D(s)* are arranged as

$$k_1 \le k_2 \le \ldots \le k_m; \quad \sum_{i=1}^{m} k_i = n \tag{3.2.9}$$

Rewrite the characteristic polynomial (3.2.8) as follows:

$$\alpha(s) = s^n + \alpha_1(s)s^{n-k_1} + \alpha_2(s)s^{n-k_1-k_2} + \ldots + \alpha_m(s) \tag{3.2.10}$$

where $\alpha_i(s)$ is a polynomial of degree less than k_i. Then, it can be verified that

$$\det \begin{bmatrix} s^{k_1} + \alpha_1(s) & \alpha_2(s) & . & 0 & \alpha_m(s) \\ -1 & s^{k_2} & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & s^{k_{m-1}} & 0 \\ 0 & 0 & . & -1 & s^{k_m} \end{bmatrix} = \alpha(s) \tag{3.2.11}$$

From (3.2.7), the closed-loop characteristic polynomial is given by $\det(D_K(s))$. Because $\alpha(s)$ is a monic polynomial,

$$\det(sI - A_c + B_c K) = \det(D_{hc}^{-1} D_K) = \alpha(s) \tag{3.2.12}$$

From (3.2.5),

$$\det(S(s) + D_{hc}^{-1}(D_{lc} + K)\Psi(s)) = \alpha(s) \tag{3.2.13}$$

Comparing (3.2.11) and (3.2.13),

$$\begin{bmatrix} s^{k_1} + \alpha_1(s) & \alpha_2(s) & . & 0 & \alpha_m(s) \\ -1 & s^{k_2} & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & s^{k_{m-1}} & 0 \\ 0 & 0 & . & -1 & s^{k_m} \end{bmatrix} = S(s) + D_{hc}^{-1}(D_{lc} + K)\Psi(s) \tag{3.2.14}$$

Using the definition of $S(s)$, Equation 3.2.14 yields

$$K\Psi(s) = D_{hc} \begin{bmatrix} \alpha_1(s) & \alpha_2(s) & . & 0 & \alpha_m(s) \\ -1 & 0 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & . & -1 & 0 \end{bmatrix} - D_{lc}\Psi(s) \tag{3.2.15}$$

Factoring out $\Psi(s)$ on the right-hand side yields the state feedback gain matrix K. Equation 3.2.11 is not a unique way to express the desired characteristic polynomial $\alpha(s)$. As shown in Example 3.5, there are other matrices with their determinants equal to $\alpha(s)$. Therefore, the state feedback gain matrix K is not unique to obtain the desired closed-loop characteristic polynomial $\alpha(s)$ for a multiinput system (Kailath, 1980).

Example 3.5

For $\bar{D}_H(s)$ in Equation 2.13.32, $k_1 = 2$ and $k_2 = 1$. And,

$$D_{hc} = \begin{bmatrix} 2/3 & -2/3 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D_{lc} = \begin{bmatrix} 16/3 & 10 & -2 \\ 0 & 0 & 4 \end{bmatrix} \tag{3.2.16}$$

Let the desired characteristic polynomial be

$$\alpha(s) = s^3 + 30s^2 + 300s + 1000 \tag{3.2.17}$$

Because $k_1 = 2$ and $k_2 = 3$, this polynomial can be rearranged to be

$$\alpha(s) = s^3 + \alpha_1(s)s + \alpha_2(s) \tag{3.2.18}$$

where

$$\alpha_1(s) = 30s + 300 \quad \text{and} \quad \alpha_2(s) = 1000 \tag{3.2.19}$$

From (3.2.15),

$$(K_c + D_{lc})\Psi(s) = D_{hc}\begin{bmatrix} \alpha_1(s) & \alpha_2(s) \\ -1 & 0 \end{bmatrix} \tag{3.2.20}$$

where

$$\Psi(s) = \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.2.21}$$

From (3.2.20),

$$(K_c + D_{lc})\Psi(s) = D_{hc}\begin{bmatrix} 30 & 300 & 1000 \\ 0 & -1 & 0 \end{bmatrix}\Psi(s) \tag{3.2.22}$$

Equating coefficients of $\Psi(s)$ on both sides,

$$K_c + D_{lc} = D_{hc}\begin{bmatrix} 30 & 300 & 1000 \\ 0 & -1 & 0 \end{bmatrix} \tag{3.2.23}$$

Therefore,

$$K_c = \begin{bmatrix} 44/3 & 572/3 & 2006/3 \\ 0 & -1 & -4 \end{bmatrix} \tag{3.2.24}$$

Alternatively,

$$(K_c + D_{lc})\Psi(s) = D_{hc}\begin{bmatrix} \alpha_1(s) & \alpha_2(s)/10 \\ -10 & 0 \end{bmatrix} \tag{3.2.25}$$

Using (3.2.19),

$$(K_c + D_{lc})\Psi(s) = D_{hc}\begin{bmatrix} 30 & 300 & 100 \\ 0 & -10 & 0 \end{bmatrix}\Psi(s) \tag{3.2.26}$$

Equating coefficients of $\Psi(s)$ on both sides,

$$K_c + D_{lc} = D_{hc}\begin{bmatrix} 30 & 300 & 100 \\ 0 & -10 & 0 \end{bmatrix} \tag{3.2.27}$$

Therefore,

$$K_c = \begin{bmatrix} 44/3 & 50/3 & 206/3 \\ 0 & -10 & -4 \end{bmatrix} \tag{3.2.28}$$

There exists another K_c matrix to achieve the desired characteristic polynomial (3.2.18).

3.3 STATE FEEDBACK GAIN MATRIX FOR A MULTIINPUT SYSTEM FOR DESIRED EIGENVALUES AND EIGENVECTORS

The closed-loop system is described by

$$G_K(s) = N(s)D_K^{-1}(s) \tag{3.3.1}$$

where

$$D_k(s) = D_{hc}S(s) + (D_{lc} + K)\Psi(s) = D(s) + K\Psi(s) \tag{3.3.2}$$

and

$$D(s) = D_{hc}S(s) + D_{lc}\Psi(s) \tag{3.3.3}$$

Let the desired eigenvalues of the closed-loop system be μ_i; $i = 1, 2, ..., n$. In this case,

$$\det D_k(\mu_i) = 0; i = 1, 2, ..., n \tag{3.3.4}$$

It can be shown that eigenvectors $\mathbf{f}_i$ associated with the system matrix of the controller form realization (Chapter 2, Section 2.13) of the transfer matrix $G_K(s)$ can be expressed as

$$\mathbf{f}_i = \Psi(\mu_i)\mathbf{p}_i \tag{3.3.5}$$

where the vector $\mathbf{p}_i$ satisfies

$$D_K(\mu_i)\mathbf{p}_i = 0 \tag{3.3.6}$$

Using (3.3.2) and (3.3.6),

$$D(\mu_i)\mathbf{p}_i + K\Psi(\mu_i)\mathbf{p}_i = 0 \tag{3.3.7}$$

Substituting (3.3.5) into (3.3.7),

$$K\mathbf{f}_i = -\mathbf{g}_i; \quad i = 1, 2, \ldots, n \tag{3.3.8}$$

where

$$\mathbf{g}_i = D(\mu_i)\mathbf{p}_i \tag{3.3.9}$$

Equation 3.3.8 can be put in the following form (Kailath, 1980):

$$K\begin{bmatrix}\mathbf{f}_1 & \mathbf{f}_2 & . & . & \mathbf{f}_n\end{bmatrix} = -\begin{bmatrix}\mathbf{g}_1 & \mathbf{g}_2 & . & . & \mathbf{g}_n\end{bmatrix} \tag{3.3.10}$$

Assuming that eigenvectors $\{\mathbf{f}_i\}$ are linearly independent,

$$K = -\begin{bmatrix}\mathbf{g}_1 & \mathbf{g}_2 & . & . & \mathbf{g}_n\end{bmatrix}\begin{bmatrix}\mathbf{f}_1 & \mathbf{f}_2 & . & . & \mathbf{f}_n\end{bmatrix}^{-1} \tag{3.3.11}$$

Linear independence of eigenvectors $\mathbf{f}_i$ is guaranteed when eigenvalues μ_i are distinct. Although it is possible to get a unique solution for K, even when some of the eigenvalues are repeated, it will be assumed that eigenvalues μ_i are distinct for the following discussion.

To use Equation 3.3.11, $\mathbf{g}_i$ and $\mathbf{f}_i$ are obtained (Kailath, 1980) as follows:

1. Choose unrepeated closed-loop eigenvalues μ_i.
2. Choose the desired closed-loop eigenvectors $\mathbf{f}_i$ to satisfy the following constraints:
   a. With reference to Equation 3.3.5, $\mathbf{f}_i$ must belong to the range space of $\Psi(\mu_i)$.
   b. All the eigenvectors $\mathbf{f}_i$ are linearly independent.
   c. If $\mathbf{f}_i$ corresponds to a complex eigenvalue μ_i, the complex conjugate of $\mathbf{f}_i$ must correspond to the eigenvalue which is the complex conjugate of μ_i.
3. Use Equation 3.3.5 to solve for $\mathbf{p}_i$. Premultiply both sides of Equation 3.3.5 by $\Psi^T(\mu_i)$:

$$\Psi^T(\mu_i)\Psi(\mu_i)\mathbf{p}_i = \Psi^T(\mu_i)\mathbf{f}_i \tag{3.3.12}$$

Because the matrix $\Psi(\mu_i)$ is of full rank, the square matrix $\Psi^T(\mu_i)\Psi(\mu_i)$ is nonsingular. Therefore,

$$\mathbf{p}_i = (\Psi^T(\mu_i)\Psi(\mu_i))^{-1}\Psi^T(\mu_i)\mathbf{f}_i \tag{3.3.13}$$

4. From Equation 3.3.9, $\mathbf{g}_i$ is computed.

Example 3.6: Electronics Navigation or Gyro Box (Schultz and Inman, 1994)

An electronics navigation or gyro box is mounted on passive spring-damper isolators located at the bottom four corners (Figure 3.4). There are isolators along x and z directions at each bottom four corner. There are also three actuators providing active control forces u_1, u_2, and u_3 as shown in Figure 3.4. Free body diagram of the system is shown in Figure 3.5.

Applying Newton's law, the system of differential equations of motion is written as

$$M\ddot{\mathbf{q}} + E\dot{\mathbf{q}} + K_s\mathbf{q}(t) = \mathbf{v}(t) \tag{3.3.14}$$

$$\mathbf{v}(t) = B_f\mathbf{u}(t) \tag{3.3.15}$$

where the mass matrix M, the damping matrix E, the stiffness matrix K_s, and the input force matrix B_f are expressed as:

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_{yy} \end{bmatrix} \quad E = \begin{bmatrix} 4c_z & 0 & 0 \\ 0 & 4c_x & -4\ell_z c_x \\ 0 & -4\ell_z c_x & 4(c_z\ell_x^2 + c_x\ell_z^2) \end{bmatrix} \tag{3.3.16a,b}$$

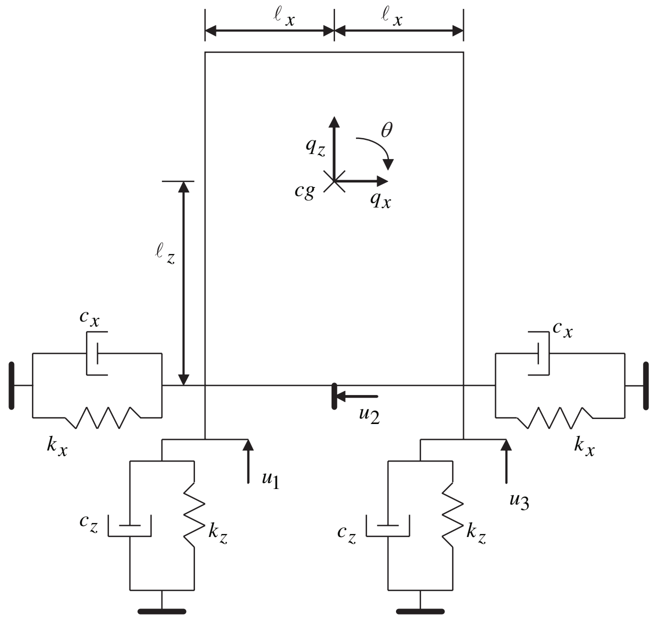

FIGURE 3.4 Electronics navigation or gyro box.

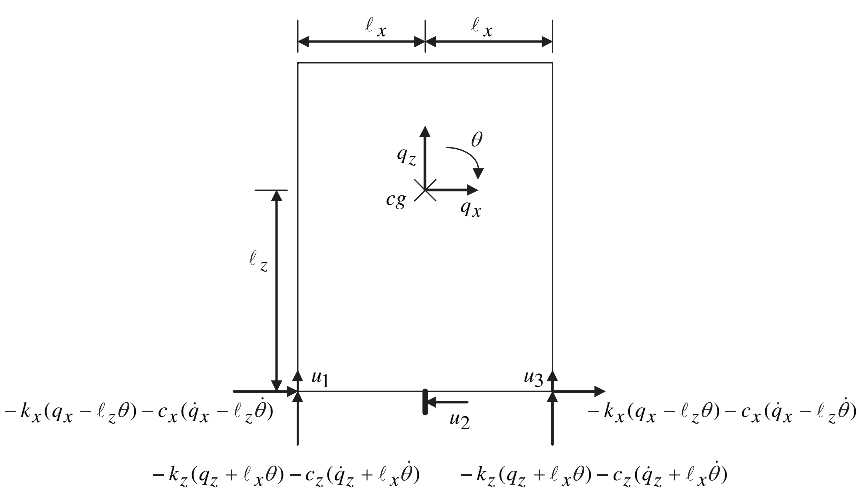

FIGURE 3.5 Free body diagram of gyro box.

$$K_s = \begin{bmatrix} 4k_z & 0 & 0 \\ 0 & 4k_x & -4\ell_z k_x \\ 0 & -4\ell_z k_x & 4(k_z\ell_x^2 + k_x\ell_z^2) \end{bmatrix} \quad B_f = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ \ell_x & \ell_z & -\ell_x \end{bmatrix} \tag{3.3.17a,b}$$

For a mechanical system, it is usual to define states as

$$\mathbf{p}_1 = \mathbf{q} \quad \text{and} \quad \mathbf{p}_2 = \dot{\mathbf{q}} \tag{3.3.18a,b}$$

Then,

$$\dot{\mathbf{p}}_1 = \mathbf{p}_2 \tag{3.3.19}$$

and from (3.3.14),

$$\dot{\mathbf{p}}_2 = -M^{-1}K_s\mathbf{p}_1 - M^{-1}E\mathbf{p}_2 + M^{-1}\mathbf{v}(t) \tag{3.3.20}$$

Putting (3.3.19) and (3.3.20) in the matrix form,

$$\dot{\mathbf{p}} = A_p\mathbf{p}(t) + B_p\mathbf{v}(t) \tag{3.3.21}$$

where

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}; \quad A_p = \begin{bmatrix} 0 & I \\ -M^{-1}K_s & -M^{-1}E \end{bmatrix}; \quad B_p = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tag{3.3.22a,b,c}$$

Expressing (3.3.14) in the matrix fraction description (MFD) form,

$$\mathbf{q}(s) = N(s)D^{-1}(s)\mathbf{v}(s) \tag{3.3.23}$$

where

$$N(s) = I \quad \text{and} \quad D(s) = (Ms^2 + Es + K_s) \tag{3.3.24a,b}$$

Here,

$$D(s) = D_{hc}\begin{bmatrix} s^2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s^2 \end{bmatrix} + D_{\ell c}\Psi(s) \tag{3.3.25}$$

where

$$D_{hc} = M \tag{3.3.26}$$

$$D_{\ell c} = \begin{bmatrix} 4c_z & 4k_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 4c_x & 4k_x & -4\ell_z c_x & -4\ell_z k_x \\ 0 & 0 & -4\ell_z c_x & -4\ell_z k_x & 4(c_z\ell_x^2 + c_x\ell_z^2) & 4(k_z\ell_x^2 + k_x\ell_z^2) \end{bmatrix} \tag{3.3.27}$$

and

$$\Psi(s) = \begin{bmatrix} s & 0 & 0 \\ 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \\ 0 & 0 & 1 \end{bmatrix} \tag{3.3.28}$$

Let the desired characteristic polynomial for the closed-loop system be

$$\alpha(s) = s^6 + \beta_1 s^5 + \beta_2 s^4 + \beta_3 s^3 + \beta_4 s^2 + \beta_5 s + \beta_6 \tag{3.3.29}$$

Here, $k_1 = k_2 = k_3 = 2$. Therefore, Equation 3.3.29 should be expressed as

$$\alpha(s) = s^6 + s^4\alpha_1(s) + s^2\alpha_2(s) + \alpha_3(s) \tag{3.3.30}$$

where

$$\alpha_1(s) = \beta_1 s + \beta_2, \quad \alpha_2(s) = \beta_3 s + \beta_4, \quad \text{and} \quad \alpha_3(s) = \beta_5 s + \beta_6 \tag{3.3.31}$$

It can be easily verified that

$$\alpha(s) = \det\begin{bmatrix} s^2 + \alpha_1(s) & \dfrac{\alpha_2(s)}{\gamma} & \dfrac{\alpha_3(s)}{\gamma\lambda} \\ -\gamma & s^2 & 0 \\ 0 & -\lambda & s^2 \end{bmatrix} \tag{3.3.32}$$

where γ and λ are any arbitrary real numbers. Factoring out $\Psi(s)$ on both sides of Equation 3.2.15,

$$K = D_{hc}\begin{bmatrix} \beta_1 & \beta_2 & \beta_3/\gamma & \beta_4/\gamma & \beta_5/(\gamma\lambda) & \beta_6/(\gamma\lambda) \\ 0 & -\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 \end{bmatrix} - D_{\ell c} \tag{3.3.33}$$

This result clearly indicates the gain matrix K is not unique because γ and λ can be chosen arbitrarily. Furthermore, this gain matrix is associated with the controller form state space realization (Chapter 2, Section 2.13), which is developed below.

Following definitions (2.13.16) and (2.13.17),

$$A_c^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_c^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.3.34}$$

Therefore,

$$A_c = A_c^0 - B_c^0 D_{hc}^{-1} D_{lc} = \begin{bmatrix} \frac{4c_z}{m} & \frac{4k_z}{m} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4c_x}{m} & \frac{4k_x}{m} & \frac{-4\ell_z c_x}{m} & \frac{-4\ell_z k_x}{m} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{-4\ell_z c_x}{I_{yy}} & \frac{-4\ell_z k_x}{I_{yy}} & \frac{4(c_z\ell_x^2 + c_x\ell_z^2)}{I_{yy}} & \frac{4(k_z\ell_x^2 + k_x\ell_z^2)}{I_{yy}} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.3.35}$$

$$B_c = B_c^0 D_{hc}^{-1} = \begin{bmatrix} 1/m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/I_{yy} \\ 0 & 0 & 0 \end{bmatrix} \tag{3.3.36}$$

The state equations are

$$\dot{\mathbf{x}} = A_c\mathbf{x}(t) + B_c\mathbf{u}(t) \tag{3.3.37}$$

with the control law

$$\mathbf{u}(t) = -K\mathbf{x}(t) \tag{3.3.38}$$

where K is given by Equation 3.3.33. The control law is not directly implementable because sensors provide the states $\mathbf{p}$(t), not $\mathbf{x}$(t). Therefore, to implement this controller, it is necessary to find the similarity transformation that will convert the state space realization (3.3.21) to the state space realization (3.3.37). This is where the similarity transformation (2.13.46) can be used for parameter values as follows.

$$\ell_z = 0.11 \text{ m}, \;\; k_z = 17500 \text{ N/m}, \;\; c_z = 0.002k_z$$

$$\ell_x = 0.08 \text{ m}, \;\; k_x = 8750 \text{ N/m}, \;\; c_x = 0.002k_x \tag{3.3.39}$$

$$m = 10 \text{ kg}, \;\; I_{yy} = 0.0487 \text{ kg-m-m}$$

Then

$$A_c = \begin{bmatrix} -14 & 700 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & -350 & 0.77 & 385 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 158.11 & 79055.44 & -35.79 & -17895.277 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Define

$$\mathbf{p}(t) = T\mathbf{z}(t) \tag{3.3.40}$$

where the similarity transformation matrix T is defined by Equation 2.13.46. Substituting (3.3.40) into (3.3.21),

$$\dot{\mathbf{z}} = A_{cc}\mathbf{z}(t) + B_{cc}\mathbf{v}(t) \tag{3.3.41}$$

where

$$A_{cc} = T^{-1}A_pT =$$

$$\begin{bmatrix} -14 & 700 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & -350 & 158.11 & 79055.44 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.77 & 385 & -35.79 & -17895.277 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.3.42}$$

and

$$B_{cc} = T^{-1}B_p = B_c^0 \tag{3.3.43}$$

It is seen that structures of A_{cc} and B_{cc} are similar to those of A_c and B_c, respectively. Using (2.13.21) and (2.3.22), new $D_{hc} = D_{hcc}$ and new $D_{\ell c} = D_{\ell cc}$ are obtained as follows:

$$D_{hcc} = I_6 \tag{3.3.44}$$

and

$$D_{\ell cc} = \begin{bmatrix} 14 & -700 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 350 & -158.11 & -79055.44 \\ 0 & 0 & -0.77 & -385 & 35.79 & 17895.277 \end{bmatrix} \tag{3.3.45}$$

Now, formulae (3.2.15) and (3.3.11) should be used with these new $D_{hc} = D_{hcc}$ and new $D_{\ell c} = D_{\ell cc}$. A Matlab code 3.1 is attached to do the following:

1. Locate the eigenvalues of the closed-loop system, μ_i, such that the damping factor of each vibratory mode is five times the corresponding value of the open-loop system, and undamped natural frequencies remain unchanged.
2. Locate the eigenvalues of the closed-loop system as described in part (1), and eigenvectors as:

$$\mathbf{f}_1 = \begin{bmatrix} \mu_1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{f}_2 = \begin{bmatrix} \mu_2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{f}_3 = \begin{bmatrix} 0 \\ 0 \\ \mu_3 \\ 1 \\ 0 \\ 0 \end{bmatrix};$$

$$\mathbf{f}_4 = \begin{bmatrix} 0 \\ 0 \\ \mu_4 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{f}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mu_5 \\ 1 \end{bmatrix}; \quad \mathbf{f}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mu_6 \\ 1 \end{bmatrix} \tag{3.3.46}$$

It should be noted the choice of eigenvectors (3.3.46) satisfies the required constraints.

Matlab Program 3.1: Full State Feedback Control of Electronics Navigation or Gyro Box

```
%
clear all
close all
%
lz=0.11;
m=10;
kz=17500;
Iyy=0.0487;
lx=0.08;
kx=8750;
%
```

```
M=diag([m m Iyy]);
Ks=[4*kz 0 0;0 4*kx -4*lz*kx;0, -4*lz*kx
4*((kx*lz*lz)+(kz*lx*lx))];
E=0.002*Ks;
Bf=[1 0 1;0 -1 0;lx lz -lz];
%
Ac0=zeros(6,6);
Ac0(2,1)=1.;
Ac0(4,3)=1;
Ac0(6,5)=1;
%
Bc0=zeros(6,3);
Bc0(1,1)=1;
Bc0(3,2)=1;
Bc0(5,3)=1;
%
Dhc=M;
Dlc=[E(1,1) Ks(1,1) 0 0 0 0;0 0 E(2,2) Ks(2,2) E(2,3)
Ks(2,3);0 0 E(3,2) Ks(3,2) E(3,3) Ks(3,3)];
Ac=Ac0-Bc0*inv(Dhc)*Dlc;
Bc=Bc0*inv(Dhc);
%
Ap=[0*eye(3) eye(3);-inv(M)*Ks -inv(M)*E];
eigop=eig(Ap);
Bp=[0*eye(3);inv(M)];
CI=[Bp Ap*Bp];
%Soloution of Eq. (2.13.39)
Coeff=inv(CI)*Ap*Ap*Bp;
Beta=Coeff(4:6,:);
%
%Similarity Transformation Matrix T, Eq. (2.13.46)
```

```
for i=1:3
 T(:,2*i-1)=Bp(:,i);
 T(:,2*i)=Ap*Bp(:,i)-Bp*Beta(:,i);
end
%
Acc=inv(T)*Ap*T;
Bcc=inv(T)*Bp;
%
%New Dhc=Dhcc and New Dlc=Dlcc
%
Accd=Acc-Ac0;
Dhcc=eye(3);
Dlcc=-[Accd(1,:);Accd(3,:);Accd(5,:)];
%
%Find Closed-Loop Eigenvalues
%
im=sqrt(-1);
for i=1:6
 rp=real(eigop(i));
 if (imag(eigop(i))<0.)imm=-im;
 end
 if (imag(eigop(i))>0.)imm=im;
 end
%Open Loop Frequencies and Damping Factors
 ogg(i)=abs(eigop(i));
 zeta(i)=-rp/ogg(i);
%Closed Loop Damping Factor=5* Open Loop Damping
Factor
 zetacl(i)=5*zeta(i);
```

```
%Desired Closed Loop Eigenvalues

 mu(i)=-zetacl(i)*ogg(i)+imm*ogg(i)*sqrt(1.-
zetacl(i)^2);

end

%

% Desired Closed-Loop Characteristic Polynomial

%

chcl=poly(mu);

%

%Non-unique State Feedback Gain Matrix KU for
%Specified Eigenvalues

%gamma and lambda can be chosen arbitrarily.

%

gamma=10;

lambda=1000;

%

ve1=[chcl(2) chcl(3)];

ve2=[chcl(4) chcl(5)]/gamma;

ve3=[chcl(6) chcl(7)]/(gamma*lambda);

%

mave=[ve1 ve2 ve3;0 -gamma 0 0 0 0;0 0 0 -lambda 0 0];

%

KU=Dhcc*mave-Dlcc;

%

% Unique State Feedback Gain Matrix KK for Specified
%Eigenvalues and Eigenvectors

%Choose eigenvectors appropriately

ff(:,1)=[mu(1) 1 0 0 0 0].';

ff(:,2)=[mu(2) 1 0 0 0 0].';

ff(:,3)=[0 0 mu(3) 1 0 0].';
```

```
ff(:,4)=[0 0 mu(4) 1 0 0].';
ff(:,5)=[0 0 0 0 mu(5) 1].';
ff(:,6)=[0 0 0 0 mu(6) 1].';
%
for i=1:6
 psi=[mu(i) 1 0 0 0 0;0 0 mu(i) 1 0 0;0 0 0 0 mu(i)
1].';
 DD=Dhcc*(mu(i)^2)+Dlcc*psi;
 pp=inv(psi.'*psi)*psi.'*ff(:,i);
%Equation (3.3.9)
 gg(:,i)=DD*pp;
end
%
%Solution of Eq. (3.3.11)
%
KK=-gg*inv(ff);
```

3.4 FUNDAMENTALS OF OPTIMAL CONTROL THEORY

Consider a general dynamic system of order n,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)); \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.4.1}$$

where $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are n-dimensional state and m-dimensional input vectors, respectively. The objective is to determine $\mathbf{u}(t)$; $0 \le t \le t_f$, such that the following objective function (Ray, 1981) is minimized or maximized:

$$I(\mathbf{u}(t)) = G(\mathbf{x}(t_f)) + \int_0^{t_f} F(\mathbf{x}, \mathbf{u})dt \tag{3.4.2}$$

Note that state equations serve as constraints for the optimization of I. In addition, constraints on the input of the following types will be considered:

$$u_{i*} \le u_i \le u_i^* \tag{3.4.3}$$

3.4.1 Necessary Conditions for Optimality

Let $\mathbf{u}_o(t)$ be a candidate for the optimal input vector, and let the corresponding state vector be $\mathbf{x}_o(t)$, i.e.,

$$\frac{d\mathbf{x}_0(t)}{dt} = \mathbf{f}(\mathbf{x}_o(t), \mathbf{u}_o(t)) \tag{3.4.4}$$

In order to see whether $\mathbf{u}_o(t)$ is indeed an optimal solution, this candidate optimal input is perturbed (Ray, 1981) by a small amount $\delta\mathbf{u}(t)$; i.e.,

$$\mathbf{u}(t) = \mathbf{u}_o(t) + \delta\mathbf{u}(t) \tag{3.4.5}$$

The change in the value of the objective function can be written as

$$\delta I = I(\mathbf{u}_o(t)+\delta\mathbf{u}(t)) - I(\mathbf{u}_o(t))$$

$$= \left(\frac{\partial G}{\partial \mathbf{x}}\right)\delta\mathbf{x}(t_f) + \int_0^{t_f} \left[\left(\frac{\partial F}{\partial \mathbf{x}}\right)\delta\mathbf{x} + \left(\frac{\partial F}{\partial \mathbf{u}}\right)\delta\mathbf{u}\right]dt + \left[F(t_f) + \left(\frac{\partial G}{\partial \mathbf{x}}\right)\mathbf{f}(t_f)\right]\delta t_f \tag{3.4.6}$$

If the solution of (3.4.1) with $\mathbf{u}(t)$ given by (3.4.5) is $\mathbf{x}_o(t) + \delta\mathbf{x}(t)$,

$$\frac{d(\mathbf{x}_o(t)+\delta\mathbf{x}(t))}{dt} = \mathbf{f}(\mathbf{x}_o(t)+\delta\mathbf{x}(t),\ \mathbf{u}_o(t)+\delta\mathbf{u}(t)) \tag{3.4.7}$$

Linearizing (3.4.7),

$$\frac{d(\delta\mathbf{x})}{dt} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\delta\mathbf{x}(t) + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)\delta\mathbf{u}(t) \tag{3.4.8}$$

Multiplying Equation 3.4.8 by $\boldsymbol{\lambda}^T(t)$ and integrating from 0 to t_f,

$$\int_0^{t_f} \boldsymbol{\lambda}^T(t)\frac{d(\delta\mathbf{x})}{dt}dt - \int_0^{t_f} \boldsymbol{\lambda}^T(t)\left[\left(\frac{\partial f}{\partial \mathbf{x}}\right)\delta\mathbf{x}(t) + \left(\frac{\partial f}{\partial \mathbf{u}}\right)\delta\mathbf{u}(t)\right]dt = 0 \tag{3.4.9}$$

where $\boldsymbol{\lambda}(t)$ is an n-dimensional vector. Adding (3.4.9) to (3.4.6) and evaluating the first integral in (3.4.9) by parts (Ray, 1981),

$$\delta I = \left[F(t_f) + \left(\frac{\partial G}{\partial \mathbf{x}}\right)\mathbf{f}(t_f) \right]\delta t_f + \boldsymbol{\lambda}^T(0)\delta\mathbf{x}(0) + \left[\left(\frac{\partial G}{\partial \mathbf{x}}\right) - \boldsymbol{\lambda}^T(t_f)\right]\delta\mathbf{x}(t_f)$$
$$+ \int_0^{t_f} \left[\left(\frac{\partial H}{\partial \mathbf{x}}\right)\delta\mathbf{x} + \left(\frac{\partial H}{\partial \mathbf{u}}\right)\delta\mathbf{u}(t) + \frac{d\boldsymbol{\lambda}^T}{dt}\delta\mathbf{x}\right]dt \tag{3.4.10}$$

where the function H is known as *Hamiltonian*, defined as follows:

$$H = F(\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}^T(t)\mathbf{f}(\mathbf{x},\mathbf{u}) \tag{3.4.11}$$

Because $\boldsymbol{\lambda}(t)$ is arbitrary, it is chosen to satisfy

$$\frac{d\boldsymbol{\lambda}^T}{dt} = -\left(\frac{\partial H}{\partial \mathbf{x}}\right) \tag{3.4.12}$$

Terms outside the integral in (3.4.10) are known as boundary conditions terms, which are removed for the specified problem. For example, if t_f and $\mathbf{x}(0)$ are specified, $\delta t_f = 0$ and $\delta\mathbf{x}(0) = 0$, and the third outside term in (3.4.10) vanishes under the following condition:

$$\boldsymbol{\lambda}^T(t_f) = \left(\frac{\partial G}{\partial \mathbf{x}}\right)_{t=t_f} \tag{3.4.13}$$

Hence, the Equation 3.4.10 can be written as

$$\delta I = \int_0^{t_f} \left(\frac{\partial H}{\partial \mathbf{u}}\right)\delta\mathbf{u}\,dt \tag{3.4.14}$$

Minimization of *I*

If $\mathbf{u}_o(t)$ is an optimal solution,

$$\delta I \geq 0 \quad \text{for any perturbation } \delta\mathbf{u}(t) \tag{3.4.15}$$

Case I: No Constraint on Input

For condition (3.4.15) to be true,

$$\left(\frac{\partial H}{\partial \mathbf{u}}\right) = 0 \quad \text{for every } t \tag{3.4.16}$$

This is the necessary condition for optimality.

Case II: With Constraints (3.4.3) on Inputs
In view of the condition (3.4.15),

$$\text{If } u_{io} = u_i^*, \quad \left(\frac{\partial H}{\partial u_i}\right) \le 0 \tag{3.4.17}$$

$$\text{If } u_{io} = u_{i*}, \quad \left(\frac{\partial H}{\partial u_i}\right) \ge 0 \tag{3.4.18}$$

Maximization of *I*

If $\mathbf{u}_o(t)$ is an optimal solution,

$$\delta I \le 0 \qquad \text{for any perturbation } \boldsymbol{\delta}\mathbf{u}(t) \tag{3.4.19}$$

Case I: No Constraint on Input
For condition (3.4.19) to be true,

$$\left(\frac{\partial H}{\partial \mathbf{u}}\right) = 0 \;\; \text{for every } t \tag{3.4.20}$$

This is the necessary condition for optimality and is identical to Equation 3.4.16.

Case II: With Constraints (3.4.3) on Inputs
In view of the condition (3.4.19),

$$\text{If } u_{io} = u_i^*, \quad \left(\frac{\partial H}{\partial u_i}\right) \ge 0 \tag{3.4.21}$$

$$\text{If } u_{io} = u_{i*}, \quad \left(\frac{\partial H}{\partial u_i}\right) \le 0 \tag{3.4.22}$$

3.4.2 Properties of Hamiltonian for an Autonomous System

For an autonomous system, the function **f** is not an explicit function of time. Therefore, from Equation 3.4.11,

$$\frac{dH}{dt} = \frac{\partial H}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial H}{\partial \mathbf{u}}\frac{d\mathbf{u}}{dt} + \frac{\partial H}{\partial \boldsymbol{\lambda}}\frac{d\boldsymbol{\lambda}}{dt} \tag{3.4.23}$$

But

$$\frac{\partial H}{\partial \boldsymbol{\lambda}} = \mathbf{f}^T \tag{3.4.24}$$

Substituting (3.4.1) and (3.4.12) into (3.4.23),

$$\frac{dH}{dt} = \frac{\partial H}{\partial \mathbf{u}} \frac{d\mathbf{u}}{dt} \tag{3.4.25}$$

Without any constraints on inputs, $\frac{\partial H}{\partial \mathbf{u}} = 0$. When there are constraints, inputs can be either maximum or minimum constant values according to Equation 3.14.17 and Equation 3.4.18 or Equation 3.14.21 and Equation 3.4.22, respectively. In this case, $\frac{d\mathbf{u}}{dt} = 0$. Therefore, for optimal inputs,

$$\frac{dH}{dt} = 0 \tag{3.4.26}$$

In other words, Hamiltonian H is a constant along an optimal trajectory for an autonomous system.

Special Case

Final time t_f not specified; i.e., $\delta t_f \neq 0$. In this case, to remove the corresponding boundary condition term in Equation 3.4.10,

$$F(t_f) + \left(\frac{\partial G}{\partial \mathbf{x}} \right) \mathbf{f}(t_f) = 0 \tag{3.4.27}$$

When final conditions on states are not specified, the condition (3.4.13) holds, and Equation 3.4.27 reduces to

$$H(t_f) = F(t_f) + \boldsymbol{\lambda}^T(t_f)\mathbf{f}(t_f) = 0 \tag{3.4.28}$$

The condition (3.4.28) is valid even when some or all states are specified at the final time. For example, consider that all states are specified at the final time. In this case, with the first-order term in the Taylor series expansion (Ray, 1981),

$$\mathbf{x}_o(t_f) = \mathbf{x}(t_f + \delta t_f) = \mathbf{x}(t_f) + \dot{\mathbf{x}}(t_f)\delta t_f \tag{3.4.29}$$

or

$$\delta\mathbf{x}(t_f) = \mathbf{x}(t_f) - \mathbf{x}_o(t_f) = -\mathbf{f}(t_f)\delta t_f \tag{3.4.30}$$

It is interesting to note that $\delta\mathbf{x}(t_f) \neq 0$. Substituting Equation 3.4.30 into (3.4.10) and using (3.4.12),

$$\delta I = [F(t_f) + \boldsymbol{\lambda}^T(t_f)\mathbf{f}(t_f)]\delta t_f + \boldsymbol{\lambda}^T(0)\delta\mathbf{x}(0) + + \int_0^{t_f} \left(\frac{\partial H}{\partial \mathbf{u}}\right)\delta\mathbf{u}(t)dt \tag{3.4.31}$$

Therefore, the condition (3.4.28) is again needed to remove boundary condition terms.

In summary, $H(t_f) = 0$ when t_f is not specified. Because H is a constant, it is concluded that $H(t) = 0$ along an optimal trajectory for an autonomous system when t_f is not specified.

Example 3.7: Travel over Maximum Distance in Specified Time without Constraint on Final Velocity

Consider the simple mass m, which is subjected to force $f(t)$. The differential equation of motion is

$$\ddot{x} = u \tag{3.4.32}$$

where

$$u(t) = \frac{f(t)}{m} \tag{3.4.33}$$

Find the optimal control input $u(t)$, such that the vehicle covers the maximum distance (Biegler, 1982) in a fixed time t_f, subject to the following constraints:

$$a \leq u(t) \leq b \tag{3.4.34}$$

Without any loss of generality take $x(0) = 0$. Assume that the vehicle starts from rest; i.e.,

$$\dot{x}(0) = 0 \tag{3.4.35}$$

Solution

Define the following state variables:

$$x_1 = x \quad \text{and} \quad x_2 = \dot{x} \tag{3.4.36}$$

Hence, state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix} = \mathbf{f} \tag{3.4.37}$$

The objective in this problem is to maximize

$$I = x_1(t_f) \tag{3.4.38}$$

Therefore,

$$G = x_1(t) \quad \text{and} \quad F = 0 \tag{3.4.39}$$

The Hamiltonian H for this problem is

$$H = F + \boldsymbol{\lambda}^T \mathbf{f} = [\lambda_1 \quad \lambda_2] \begin{bmatrix} x_2 \\ u \end{bmatrix} = \lambda_1 x_2 + \lambda_2 u \tag{3.4.40}$$

Adjoint equations are

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1} = 0 \tag{3.4.41}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2} = -\lambda_1 \tag{3.4.42}$$

Then, conditions on state variables are as follows:

$$x_1(0) = 0 \Rightarrow \delta x_1(0) = 0 \tag{3.4.43a}$$

$$x_2(0) = 0 \Rightarrow \delta x_2(0) = 0 \tag{3.4.43b}$$

$$x_1(t_f) unspecified \Rightarrow \delta x_1(t_f) \neq 0 \tag{3.4.43c}$$

$$x_2(t_f) unspecified \Rightarrow \delta x_2(t_f) \neq 0 \tag{3.4.44d}$$

Also, because the final time t_f is fixed,

$$\delta t_f = 0 \tag{3.4.45}$$

For the boundary condition terms to be zero,

$$\boldsymbol{\lambda}^T(t_f) = \left(\frac{\partial G}{\partial \mathbf{x}}\right)_{t=t_f} \tag{3.4.46}$$

Therefore,

$$\lambda_1(t_f) = \left(\frac{\partial G}{\partial x_1}\right)_{t=t_f} = 1 \tag{3.4.47a}$$

$$\lambda_2(t_f) = \left(\frac{\partial G}{\partial x_2}\right)_{t=t_f} = 0 \tag{3.4.47b}$$

Solutions of adjoint equations are

$$\lambda_1(t) = c \quad \text{and} \quad \lambda_2(t) = -ct + d \tag{3.4.48a,b}$$

where c and d are constants. To satisfy the final conditions,

$$c = 1 \quad \text{and} \quad d = t_f \tag{3.4.49a,b}$$

In other words,

$$\lambda_1(t) = 1 \quad \text{and} \quad \lambda_2(t) = -t + t_f \tag{3.4.50a,b}$$

Optimality condition:

$$\frac{\partial H}{\partial u} = \lambda_2 \tag{3.4.51}$$

$\frac{\partial H}{\partial u} = \lambda_2$ is plotted in Figure 3.6. $\frac{\partial H}{\partial u}$ is not equal to zero for a finite time interval. This implies that the optimal u cannot take any intermediate value between a and b. Furthermore, $\frac{\partial H}{\partial u} > 0$ for $0 \le t < t_f$. Equation 3.4.21 implies that the optimal control input is

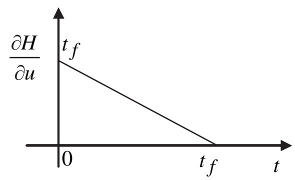

FIGURE 3.6 $\frac{\partial H}{\partial u} = \lambda_2$ vs. time t.

$$u = b \quad \text{for} \quad 0 \le t \le t_f \tag{3.4.52}$$

The optimal strategy is the maximum acceleration, which is employed by a rider out of common sense, when the objective is to cover the maximum distance in a fixed time, without any constraint on the vehicle velocity at the final time. Now, the optimal state trajectory can be easily obtained by solving the state equations with given initial conditions:

$$x_2(t) = bt \quad \text{and} \quad x_1(t) = \frac{1}{2}bt^2 \tag{3.4.53}$$

Therefore, the maximum value of the objective function I is $\frac{1}{2}bt_f^2$. Along the optimal trajectory,

$$H = bt + (t_f - t)b = bt_f \tag{3.4.54}$$

As expected, the Hamiltonian H is a constant along the optimal trajectory.

Example 3.8: Travel over Maximum Distance in Specified Time with Constraint That Final Velocity Is Equal to Zero.

Consider the system, which is same as that in Example 3.7. Find the optimal control input $u(t)$ such that the vehicle covers the maximum distance in a fixed time t_f and ends at rest (Biegler, 1982) subject to the following constraints:

$$a \le u(t) \le b \tag{3.4.55}$$

Here, it is also given that $a < 0$ and $b > 0$.

Solution

Define the following state variables:

$$x_1 = x \quad \text{and} \quad x_2 = \dot{x} \tag{3.4.56}$$

Hence, state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix} = \mathbf{f} \tag{3.4.57}$$

The objective in this problem is to maximize

$$I = x_1(t_f) \tag{3.4.58}$$

Therefore,

$$G = x_1(t) \quad \text{and} \quad F = 0 \tag{3.4.59a,b}$$

The Hamiltonian H for this problem is

$$H = F + \boldsymbol{\lambda}^T \mathbf{f} = [\lambda_1 \quad \lambda_2] \begin{bmatrix} x_2 \\ u \end{bmatrix} = \lambda_1 x_2 + \lambda_2 u \tag{3.4.60}$$

Adjoint equations are

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1} = 0 \tag{3.4.61}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2} = -\lambda_1 \tag{3.4.62}$$

Then, conditions on state variables are as follows:

$$x_1(0) = 0 \Rightarrow \delta x_1(0) = 0 \tag{3.4.63a}$$

$$x_2(0) = 0 \Rightarrow \delta x_2(0) = 0 \tag{3.4.63b}$$

$$x_1(t_f) \mathit{unspecified} \Rightarrow \delta x_1(t_f) \neq 0 \tag{3.4.63c}$$

$$x_2(t_f) = 0 \Rightarrow \delta x_2(t_f) = 0 \tag{3.4.63d}$$

Also, because the final time t_f is fixed,

$$\delta t_f = 0 \tag{3.4.64}$$

Now, consider the following boundary condition term:

$$\left[\left(\frac{\partial G}{\partial \mathbf{x}}\right) - \boldsymbol{\lambda}^T(t_f)\right]\delta\mathbf{x}(t_f) = \left[\left(\frac{\partial G}{\partial x_1}\right) - \lambda_1(t_f)\right]\delta x_1(t_f) + \left[\left(\frac{\partial G}{\partial x_2}\right) - \lambda_2(t_f)\right]\delta x_2(t_f) \tag{3.4.65}$$

Because $\delta x_2(t_f) = 0$,

$$\left[\left(\frac{\partial G}{\partial \mathbf{x}}\right) - \lambda^T(t_f)\right]\delta\mathbf{x}(t_f) = \left[\left(\frac{\partial G}{\partial x_1}\right) - \lambda_1(t_f)\right]\delta x_1(t_f) \tag{3.4.66}$$

Hence, for all boundary condition terms to be zero,

$$\lambda_1(t_f) = \left(\frac{\partial G}{\partial x_1}\right)_{t=t_f} = 1 \tag{3.4.67}$$

Solutions of adjoint equations are

$$\lambda_1(t) = c \quad \text{and} \quad \lambda_2(t) = -ct + d \tag{3.4.68}$$

where c and d are constants. To satisfy the final condition,

$$c = 1 \tag{3.4.69}$$

In other words,

$$\lambda_1(t) = 1 \quad \text{and} \quad \lambda_2(t) = -t + d \tag{3.4.70a,b}$$

Optimality condition:

$$\frac{\partial H}{\partial u} = \lambda_2 = -t + d \tag{3.4.71}$$

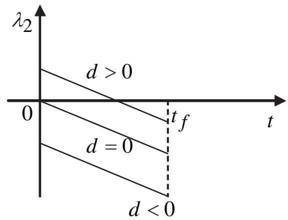

FIGURE 3.7 $\frac{\partial H}{\partial u} = \lambda_2$ vs. time t.

The value of d is not known. Any of these three possibilities exists: $d < 0$, $d = 0$, or $d > 0$. $\frac{\partial H}{\partial u} = \lambda_2$ is plotted in Figure 3.7 for all these possibilities. The first thing to note is that $\frac{\partial H}{\partial u}$ is not equal to zero for a finite time interval. This implies that the optimal $u(t)$ cannot take any intermediate value between a and b. Secondly, $d < 0$ and $d = 0$ are ruled out because they will lead to $u(t) = a < 0$ (Equation 3.4.22). Hence, $d > 0$ and the optimal control input is

$$u(t) = \begin{cases} b & for \quad t \le t_s \\ a & for \quad t > t_s \end{cases} \tag{3.4.72}$$

where t_s is the switching instant, which is, when $\lambda_2(t)$ changes sign in Figure 3.7, $d > 0$.

Solving the second state equation,

$$x_2(t_s) = bt_s \quad \text{and} \quad x_2(t_f) = bt_s + a(t_f - t_s) \tag{3.4.73}$$

For $x_2(t_f) = 0$,

$$t_s = \frac{-a}{b-a} t_f \tag{3.4.74}$$

3.5 LINEAR QUADRATIC REGULATOR (LQR) PROBLEM

Consider the linear system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + B\mathbf{u}(t); \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.5.1}$$

The objective is to drive the state vector $\mathbf{x}(t)$ to the origin of the state space (zero state vector) from any nonzero initial values of states. If a state feedback control law is used, $\mathbf{x}(t)$ will quickly die out provided closed-loop poles are located far inside the left half of the s-plane. However, elements of the feedback gain vector can be large in magnitudes and the control cost can be high. On the other hand, if closed-loop poles are located close to the open-loop poles, there will not be much increase in the rate of decay of $\mathbf{x}(t)$ and a relatively small amount of control action will be required. Hence, the location of closed-loop poles is a trade-off between the rate of decay of $\mathbf{x}(t)$ and the magnitude of control input. To make this trade-off, the following objective function is chosen:

$$I = \frac{1}{2}\mathbf{x}^T(t_f)S_f\mathbf{x}(t_f) + \frac{1}{2}\int_0^{t_f}[\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{u}^T(t)R\mathbf{u}(t)]dt \tag{3.5.2}$$

where the final time t_f is fixed. Without any loss of generality, matrices S_f, Q, and R are chosen to be symmetric (Appendix B). In addition, S_f and Q are chosen to be positive semidefinite, and R to be positive definite. Symbolically, these are expressed as

$$S_f = S_f^T \geq 0\ , \quad Q = Q^T \geq 0\ , \quad \text{and} \quad R = R^T > 0$$

Problem

Find $\mathbf{u}(t)$; $0 \leq t \leq t_f$, such that the objective function (3.5.2) is minimized.

3.5.1 Solution (Open-Loop Optimal Control)

For this optimal control problem, the Hamiltonian (3.4.11) is

$$H = \frac{1}{2}(\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{u}^T(t)R\mathbf{u}(t)) + \boldsymbol{\lambda}^T(t)(A\mathbf{x}(t) + B\mathbf{u}(t)) \tag{3.5.3}$$

The necessary condition (3.4.16) for optimality yields

$$\frac{\partial H}{\partial \mathbf{u}} = R\mathbf{u}(t) + B^T\boldsymbol{\lambda}(t) = 0$$

or

$$\mathbf{u}(t) = -R^{-1}B^T\boldsymbol{\lambda}(t) \tag{3.5.4}$$

The dynamics of $\lambda(t)$ is given by Equation 3.4.12 with final conditions (3.4.13). Hence,

$$\frac{d\boldsymbol{\lambda}}{dt} = -Q\mathbf{x}(t) - A^T\boldsymbol{\lambda}(t); \qquad \boldsymbol{\lambda}(t_f) = S_f\mathbf{x}(t_f) \tag{3.5.5}$$

Equation 3.5.1 and Equation 3.5.5 represent a two-point boundary value problem (TPBVP) which can be solved to find $\boldsymbol{\lambda}(t)$ and $\mathbf{x}(t)$. Putting (3.5.1) and (3.5.5) in the matrix form,

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\lambda} \end{bmatrix} = M \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} \tag{3.5.6}$$

where

$$M = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \tag{3.5.7}$$

Solving (3.5.6),

$$\begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} \mathbf{x}(0) \\ \boldsymbol{\lambda}(0) \end{bmatrix} \tag{3.5.8}$$

To determine $\boldsymbol{\lambda}(0)$, the matrix e^{Mt} is partitioned as follows:

$$e^{Mt} = \begin{bmatrix} \mathrm{E}_{11}(t) & \mathrm{E}_{11}(t) \\ \mathrm{E}_{21}(t) & \mathrm{E}_{22}(t) \end{bmatrix} \tag{3.5.9}$$

From (3.5.8) and (3.5.9),

$$\mathbf{x}(t) = \mathrm{E}_{11}(t)\mathbf{x}(0) + \mathrm{E}_{12}(t)\lambda(0) \tag{3.5.10}$$

$$\lambda(t) = \mathrm{E}_{21}(t)\mathbf{x}(0) + \mathrm{E}_{22}(t)\lambda(0) \tag{3.5.11}$$

Imposing the condition $\boldsymbol{\lambda}(t_f) = S_f\mathbf{x}(t_f)$,

$$\mathrm{E}_{21}(t_f)\mathbf{x}(0) + \mathrm{E}_{22}(t_f)\boldsymbol{\lambda}(0) = S_f[\mathrm{E}_{11}(t_f)\mathbf{x}(0) + \mathrm{E}_{12}(t_f)\boldsymbol{\lambda}(0)] \tag{3.5.12}$$

From (3.5.12),

$$\boldsymbol{\lambda}(0) = [E_{22}(t_f) - S_f E_{12}(t_f)]^{-1}[S_f E_{11}(t_f) - E_{21}(t_f)]\mathbf{x}(0) \tag{3.5.13}$$

Substituting this $\boldsymbol{\lambda}(0)$ into (3.5.8), $\boldsymbol{\lambda}(t)$ is obtained. Then, the optimal $\mathbf{u}(t)$ is found from (3.5.4). However, this TPBVP must be solved again if initial conditions change. Furthermore, the control inputs are implementable in an open-loop fashion only, as they are not in the forms of functions of states.

3.5.2 Solution (Closed-Loop Optimal Control)

Use the following transformation (Ray, 1981):

$$\boldsymbol{\lambda}(t) = S(t)\mathbf{x}(t) \tag{3.5.14}$$

where $S(t)$ is a symmetric $n \times n$ matrix. Substituting (3.5.14) into (3.5.5),

$$\frac{dS}{dt}\mathbf{x}(t) + S(t)\frac{d\mathbf{x}}{dt} = -Q\mathbf{x}(t) - A^T S(t)\mathbf{x}(t) \tag{3.5.15}$$

Using (3.5.1),

$$\left(\frac{dS}{dt} + SA - SBR^{-1}B^T S + Q + A^T S\right)\mathbf{x}(t) = 0 \tag{3.5.16}$$

Because Equation 3.5.16 is true for all $\mathbf{x}(t)$,

$$\frac{dS}{dt} + SA - SBR^{-1}B^T S + Q + A^T S = 0 \tag{3.5.17}$$

Equation 3.4.13 and Equation 3.5.14 yield

$$S(t_f) = S_f \tag{3.5.17b}$$

Equation 3.5.17 is known as the *Riccati equation*. This nonlinear differential equation can be numerically solved backward in time to determine $S(t)$. From (3.5.4) and (3.5.14),

$$\mathbf{u}(t) = -K(t)\mathbf{x}(t) \tag{3.5.18}$$

where

$$K(t) = R^{-1}B^T S(t) \tag{3.5.19}$$

The structure of (3.5.18) indicates that the $K(t)$ is the optimal state feedback gain matrix. Because the solution of $S(t)$ does not depend on system states, this gain is optimal for all initial conditions on states.

3.5.3 Cross Term in the Objective Function

Consider a more general form of the quadratic objective function (Anderson and Moore, 1990):

$$I = \frac{1}{2}\mathbf{x}^T(t_f)S_f\mathbf{x}(t_f) + \frac{1}{2}\int_0^{t_f}[\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{u}^T(t)R\mathbf{u}(t) + 2\mathbf{x}^T(t)N\mathbf{u}(t)]dt \tag{3.5.20}$$

It can be seen that

$$\mathbf{x}^T(t)Q\mathbf{x}(t) + 2\mathbf{x}^T(t)N\mathbf{u}(t) + \mathbf{u}^T(t)R\mathbf{u}(t) = \mathbf{x}^T(t)Q_m\mathbf{x}(t) + \mathbf{v}^T(t)R\mathbf{v}(t) \tag{3.5.21}$$

where

$$Q_m = Q - NR^{-1}N^T \tag{3.5.22}$$

and

$$\mathbf{v}(t) = \mathbf{u}(t) + R^{-1}N^T\mathbf{x}(t) \tag{3.5.23}$$

Equation 3.5.21 can be proved by simply multiplying out the terms on the right-hand side. Hence, Equation 3.5.20 can be rewritten as

$$I = \frac{1}{2}\mathbf{x}^T(t_f)S_f\mathbf{x}(t_f) + \frac{1}{2}\int_0^{t_f}[\mathbf{x}^T(t)Q_m\mathbf{x}(t) + \mathbf{v}^T(t)R\mathbf{v}(t)]dt \tag{3.5.24}$$

and the plant equation (3.5.1) is modified with Equation 3.5.23:

$$\dot{\mathbf{x}} = A_m\mathbf{x}(t) + B\mathbf{v}(t) \tag{3.5.25}$$

where

$$A_m = A - BR^{-1}N^T \tag{3.5.26}$$

Assuming that $Q_m \geq 0$, Equation 3.5.24 and Equation 3.5.25 constitute a standard LQ problem for which the optimal state feedback control law is

$$\mathbf{v}(t) = -R^{-1}B^T S_m(t)\mathbf{x}(t) \tag{3.5.27}$$

where

$$\dot{S}_m = -S_m A_m - A_m^T S_m + S_m BR^{-1}B^T S_m - Q_m; \quad S_m(t_f) = S_f \tag{3.5.28}$$

From (3.5.23) and (3.5.27),

$$\mathbf{u}(t) = -K_m(t)\mathbf{x}(t) \tag{3.5.29}$$

where the optimal state feedback gain matrix is given by

$$K_m(t) = R^{-1}(B^T S_m(t) + N^T) \tag{3.5.30}$$

Example 3.9: Minimum Energy Control of a DC Motor

Consider the position controller shown in Figure 3.8. The actuator is an armature-controlled DC motor (Kuo, 1995). The torque produced by the motor $T_m(t)$ is proportional to the armature current $i_a(t)$; i.e.,

$$T_m(t) = k_i i_a(t) \tag{3.5.31}$$

Let the back emf developed across the armature be $e_b(t)$, i.e.,

$$e_b(t) = k_b \dot{\theta}_m(t) \tag{3.5.32}$$

where k_b is the back emf constant and $\theta_m(t)$ is the angular position of the rotor. Applying Kirchoff's law to the armature,

$$u(t) = R_a i_a(t) + e_b(t) \tag{3.5.33}$$

where $u(t)$ is the input voltage and $R_a(t)$ is the armature resistance. The inductance of the armature windings has been neglected.

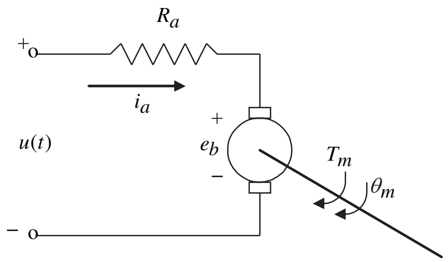

FIGURE 3.8 An armature-controlled DC motor.

Applying Newton's second law,

$$J_m\ddot{\theta}_m + B_m\dot{\theta}_m = T_m(t) \tag{3.5.34}$$

where J_m and B_m are mass moment of inertia and equivalent viscous damping, respectively. From (3.5.31) to (3.5.33),

$$T_m(t) = \frac{k_i}{R_a}u(t) - \frac{k_i k_b}{R_a}\dot{\theta}_m \tag{3.5.35}$$

From (3.5.34) and (3.5.35),

$$\ddot{\theta}_m + \alpha\dot{\theta}_m = \beta u(t) \tag{3.5.36}$$

where

$$\alpha = \frac{B_m}{J_m} + \frac{k_i k_b}{R_a J_m} \tag{3.5.37}$$

and

$$\beta = \frac{k_i}{R_a J_m} \tag{3.5.38}$$

From (3.5.36), state equations are

$$\dot{\mathbf{x}} = A\mathbf{x}(t) + bu(t) \tag{3.5.39}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} \theta_m(t) \\ \dot{\theta}_m(t) \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \tag{3.5.40}$$

Let us define an optimal control problem: Find the input u(t), $0 \le t \le t_f$, such that the energy consumed by the DC motor is minimized over a fixed final time t_f, with the following initial and final conditions on the states:

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}(t_f) = \begin{bmatrix} \theta_f \\ 0 \end{bmatrix} \tag{3.5.41}$$

The energy consumed by the DC motor over a fixed final time t_f is given by the following integral:

$$I_d = \int_{t=0}^{t_f} T_m(t) d\theta_m = \int_0^{t_f} T_m \dot{\theta}_m dt \tag{3.5.42}$$

Using (3.5.35), (3.5.40), and (3.5.42)

$$I_d = \int_0^{t_f} \frac{k_i}{R_a}(ux_2 - k_b x_2^2) dt \tag{3.5.43}$$

To satisfy the requirements for the existence of solution to the LQ control, the objective function (3.5.43) is modified to be

$$I = \int_0^{t_f} \frac{k_i}{R_a}(ux_2 - k_b x_2^2) + \gamma x_2^2 + \rho u^2 dt \; ; \quad \gamma > 0 \quad \text{and} \quad \rho > 0 \tag{3.5.44}$$

Hence, with respect to the definition (3.5.20),

$$I = \frac{1}{2}\int_0^{t_f} \mathbf{x}^T Q \mathbf{x} + 2\mathbf{x}^T N \mathbf{u} + \mathbf{u}^T R \mathbf{u} dt \tag{3.5.45}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 2(\rho - \frac{k_i k_b}{R_a}) \end{bmatrix}; \quad N = \begin{bmatrix} 0 \\ \frac{k_i}{R_a} \end{bmatrix}; \quad R = 2\rho \tag{3.5.46}$$

and

$$Q_m = \begin{bmatrix} 0 & 0 \\ 0 & (2\gamma - \frac{2k_i k_b}{R_a} - \frac{k_i^2}{2R_a^2\rho}) \end{bmatrix} \tag{3.5.47}$$

For $Q_m \geq 0$, ρ and γ must be chosen to satisfy

$$2\gamma - \frac{2k_i k_b}{R_a} - \frac{k_i^2}{2R_a^2\rho} \geq 0 \tag{3.5.48}$$

In order to have final conditions on states to be zero, the following transformation of the state vector is introduced:

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}(t_f) \tag{3.5.49}$$

Then,

$$\tilde{\mathbf{x}}(0) = -\mathbf{x}(t_f) \quad \text{and} \quad \tilde{\mathbf{x}}(t_f) = 0 \tag{3.5.50a,b}$$

Substituting (3.5.49) into (3.5.39), and noting that $A\mathbf{x}(t_f) = 0$,

$$\dot{\tilde{\mathbf{x}}} = A\tilde{\mathbf{x}}(t) + \mathbf{b}u(t) \tag{3.5.51}$$

Because the objective function does not contain x_1 and $x_2 = \tilde{x}_2$, Equation 3.5.45 can be rewritten as

$$I = \frac{1}{2}\int_0^{t_f} \tilde{\mathbf{x}}^T Q\tilde{\mathbf{x}} + 2\tilde{\mathbf{x}}^T N\mathbf{u} + \mathbf{u}^T R\mathbf{u}dt \tag{3.5.52}$$

In the LQ control, there is no constraint on the final conditions. Therefore, the objective function is modified to contain the final state vector:

$$I = \frac{1}{2}\tilde{\mathbf{x}}^T(t_f)S_f\tilde{\mathbf{x}}(t_f) + \frac{1}{2}\int_0^{t_f} \tilde{\mathbf{x}}^T Q\tilde{\mathbf{x}} + 2\tilde{\mathbf{x}}^T N\mathbf{u} + \mathbf{u}^T Rudt \tag{3.5.53}$$

where

$$S_f = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}; \quad a > 0 \text{ and } b > 0 \tag{3.5.54}$$

Relaxing constraints on final states, the LQ problem can be solved to minimize (3.5.53) for the linear system (3.5.51) with nonzero $\tilde{\mathbf{x}}(0)$. By choosing a and b to be large numbers, $\tilde{\mathbf{x}}(t_f)$ can be forced to be close to zero.

3.5.4 Important Case: Infinite Final Time

When the final time $t_f \to \infty$, the optimal gain $K_m(t)$ or $S_m(t)$ turns out to be a constant (Kwakernaak and Sivan, 1972). The system of differential equations (3.5.28) reduces to a system of algebraic equations:

$$S_m A_m - S_m B R^{-1} B^T S_m + Q_m + A_m^T S_m = 0 \tag{3.5.55}$$

This is known as the algebraic Riccati equation (ARE) and is probably one of the most commonly used equations in modern control theory. Since the matrix S_m is a constant, the feedback gain matrix also turns out to be a constant as follows:

$$K_o = R^{-1}(B^T S_m + N^T) \tag{3.5.56}$$

Hence from (3.5.1) and (3.5.56), the closed-loop system dynamics is represented by

$$\frac{d\mathbf{x}}{dt} = (A - BK_o)\mathbf{x}(t) \tag{3.5.57}$$

The eigenvalues of the matrix $A - BK_o$ are the optimal closed-loop poles. In the next section, a method is described to determine these optimal closed-loop poles first via root locus plots. Then, the optimal feedback gain can be obtained via pole placement techniques.

Because $Q_m \geq 0$, there exists a matrix H known as the square root of Q_m, such that

$$Q_m = \mathrm{H}\mathrm{H}^T \tag{3.5.58}$$

If (A, H) is observable, S_m is positive definite and is the only solution of ARE, Equation 3.5.55, with this property. Furthermore, the optimal closed-loop system (3.5.57) is asymptotically stable (Anderson and Moore, 1990; Kailath, 1980).

Example 3.10: A First-Order System

Consider a first-order system

$$\dot{x} = x + u \tag{3.5.59}$$

and the objective function

$$I = \frac{5}{2}x^2(t_f) + \frac{1}{2}\int_0^{t_f} 2x^2 + 3u^2 dt \tag{3.5.60}$$

Here,

$$A = 1,\ \mathbf{b} = 1,\ Q = 2,\ R = 3,\ \ S_f = 5 \tag{3.5.61}$$

Therefore, the Riccati equation becomes

$$\dot{S} = -2S + \frac{S^2}{3} - 2\ ; \quad S(t_f) = 5 \tag{3.5.62}$$

Nonlinear differential equation (3.5.62) has to be numerically solved backward in time to determine $S(t)$ for $0 \le t \le t_f$. Matlab routine ODE23 or ODE45 can be used for this purpose. Then, the optimal control law will be

$$u(t) = -\frac{S(t)}{3}x(t)\ ; \quad 0 \le t \le t_f \tag{3.5.63}$$

If $t_f \to \infty$, ARE becomes

$$0 = -2S + \frac{S^2}{3} - 2 \tag{3.5.64}$$

Solving (3.5.64),

$$S = 3 \pm \sqrt{15} \tag{3.5.65}$$

In order to have a stable closed-loop system, a "+" sign must be chosen. The optimal state feedback law is

$$u(t) = -Kx(t) \tag{3.5.66}$$

where

$$K = 1 + \frac{\sqrt{15}}{3} \tag{3.5.67}$$

Substituting (3.5.66) into (3.5.59), the closed-loop system dynamics is

$$\dot{x} = -\frac{\sqrt{15}}{3}x \tag{3.5.68}$$

which is stable.

Example 3.11: Active Suspension with Optimal Linear State Feedback (Thomson, 1976)

Differential equations of motion for the linear vehicle model shown in Figure 3.9 are

$$m_2\ddot{x}_2 = u \tag{3.5.69}$$

$$m_1\ddot{x}_1 + \lambda_1(x_1 - x_0) = -u \tag{3.5.70}$$

Defining $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$,

$$\dot{x}_1 = x_3 \tag{3.5.71}$$

$$\dot{x}_2 = x_4 \tag{3.5.72}$$

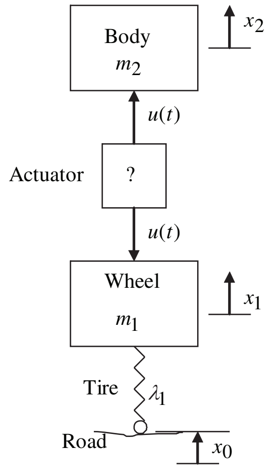

FIGURE 3.9 A quarter car model.

$$\dot{x}_3 = -\frac{\lambda_1}{m_1}(x_1 - x_0) - \frac{u(t)}{m_1} \tag{3.5.73}$$

$$\dot{x}_4 = \frac{u(t)}{m_2} \tag{3.5.74}$$

Assume that the road profile is a step function. In this case, x_0 is a constant, and a new set of state variables is defined as

$$\hat{x}_1 = x_1 - x_0 \tag{3.5.75}$$

$$\hat{x}_2 = x_2 - x_0 \tag{3.5.76}$$

$$\hat{x}_3 = \dot{\hat{x}}_1 = \dot{x}_1 \tag{3.5.77}$$

$$\hat{x}_4 = \dot{\hat{x}}_2 = \dot{x}_2 \tag{3.5.75}$$

and the new set of state equations can be written as

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda_1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{m_1} \\ \frac{1}{m_2} \end{bmatrix} u(t) \tag{3.5.76}$$

The performance index is chosen as

$$I = \frac{1}{2}\int_0^\infty [\rho u^2 + q_1(x_0 - x_1)^2 + q_2(x_1 - x_2)^2]dt \tag{3.5.77}$$

Note the following:

$$(x_0 - x_1)\text{: tire dynamic deflection} \tag{3.5.78}$$

$$(x_1 - x_2)\text{: relative wheel travel} \tag{3.5.79}$$

Moreover, the actuator force $u(t)$, which is proportional to the vertical acceleration of the body, is also a measure of the ride discomfort. Hence, the minimization of the objective function I will result in a trade-off between the minimization of the actuator input or ride discomfort and minimization of the weighted sum of tire dynamic deflection and relative wheel travel. Weighting parameters are ρ, q_1, and q_2. Because

$$\begin{aligned} q_1(x_0 - x_1)^2 + q_2(x_1 - x_2)^2 &= q_1\hat{x}_1^2 + q_2(\hat{x}_1 - \hat{x}_2)^2 \\ &= (q_1 + q_2)\hat{x}_1^2 + q_2\hat{x}_2^2 - 2q_2\hat{x}_1\hat{x}_2 \end{aligned} \tag{3.5.80}$$

the objective function I can be expressed as

$$I = \frac{1}{2}\int_0^\infty [\rho u^2 + \hat{\mathbf{x}}^T Q\hat{\mathbf{x}}]dt \tag{3.5.81}$$

where

$$Q = \begin{bmatrix} q_1 + q_2 & -q_2 & 0 & 0 \\ -q_2 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} \tag{3.5.82a,b}$$

The optimal control law is

$$u(t) = -\mathbf{k}\hat{\mathbf{x}} = -\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}\hat{\mathbf{x}} = -k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4 \tag{3.5.83}$$

where

$$\mathbf{k} = \rho^{-1}\mathbf{b}^T S \tag{3.5.84}$$

and

$$SA + A^T S - S\mathbf{b}\rho^{-1}\mathbf{b}^T S + Q = 0 \tag{3.5.85}$$

Physical realization of the control law:

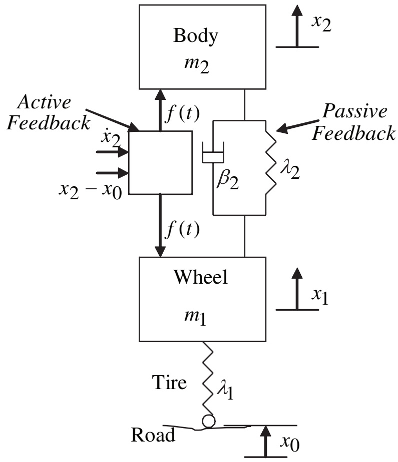

FIGURE 3.10 Active or passive vehicle suspension system.

$$\begin{aligned} u(t) &= -k_1(\hat{x}_1 - \hat{x}_2) - k_3(\hat{x}_3 - \hat{x}_4) - (k_1 + k_2)\hat{x}_2 - (k_3 + k_4)\hat{x}_4 \\ &= u_p + f(t) \end{aligned} \tag{3.5.86}$$

where

$$u_p(t) = \lambda_2(x_1 - x_2) + \beta_2(\dot{x}_1 - \dot{x}_2) \tag{3.5.87}$$

and

$$f(t) = -(k_1 + k_2)(x_2 - x_0) - (k_3 + k_4)\dot{x}_2 \tag{3.5.88}$$

The part of the control input $u_p(t)$ has been applied by a spring and a damper that provide passive feedback of $(x_1 - x_2)$ and $(\dot{x}_1 - \dot{x}_2)$ (Figure 3.10). The remaining part $f(t)$ is applied by active feedback via sensors to measure $(x_2 - x_0)$ and $\dot{x}_2$, and an actuator to apply the force $f(t)$. In the study by Thompson (1976), the relative distance $(x_2 - x_0)$ is measured by an ultrasonic transmitter/receiver, and the velocity $\dot{x}_2$ is obtained by integrating the signal from an accelerometer. The actuator is shown to be an electrohydraulic system.

3.6 SOLUTION OF LQR PROBLEM VIA ROOT LOCUS PLOT: SISO CASE

Let the quadratic objective function be

$$I = \frac{1}{2}\int_0^{\infty} y^2(t) + ru^2(t)dt \tag{3.6.1}$$

where $u(t)$ and $y(t)$ are input and output of a SISO linear system. Note that

$$y^2(t) = y^T(t)y(t) = \mathbf{x}^T(t)\mathbf{c}^T\mathbf{c}\mathbf{x}(t) \tag{3.6.2}$$

Referring to (3.5.2),

$$Q = \mathbf{c}^T\mathbf{c} \quad \text{and} \quad R = r \tag{3.6.3}$$

Hence, from Equation 3.5.1, Equation 3.5.4, and Equation 3.5.5,

$$\begin{bmatrix} \dfrac{d\mathbf{x}(t)}{dt} \\ \dfrac{d\boldsymbol{\lambda}(t)}{dt} \end{bmatrix} = M\begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{bmatrix} \tag{3.6.4}$$

where

$$M = \begin{bmatrix} A & -r^{-1}\mathbf{b}\mathbf{b}^T \\ -\mathbf{c}^T\mathbf{c} & -A^T \end{bmatrix} \tag{3.6.5}$$

The matrix M is known as the Hamiltonian matrix. It can be shown (Kwakernaak and Sivan, 1972) that

$$(-1)^n\Delta(s) = a_1(s)a_1(-s) + r^{-1}a_2(s)a_2(-s) \tag{3.6.6}$$

where

$$\Delta(s) = \det(sI - M) \tag{3.6.7}$$

and $a_1(s)$ and $a_2(s)$ are denominator and numerator polynomials of the open-loop transfer function, i.e.,

$$\mathbf{c}(sI - A)^{-1}\mathbf{b} = \frac{a_2(s)}{a_1(s)} \tag{3.6.8}$$

It will be assumed that there are no common factors between $a_2(s)$ and $a_1(s)$. Equation 3.6.6 establishes the fact that if s^* is a root of $\Delta(s)=0$, $-s^*$ would also be the root. Hence, eigenvalues of M are symmetric with respect to the imaginary axis of the s-plane. Recall that eigenvalues of any real matrix are always symmetric with respect to the real axis. Hence, eigenvalues of M are symmetric with respect to both real and imaginary axes. Furthermore,

$$\begin{aligned}(-1)^n \Delta(j\omega) &= a_1(j\omega)a_1(-j\omega)+r^{-1}a_2(j\omega)a_2(-j\omega)\\ &= \left|a_1(j\omega)\right|^2+r^{-1}\left|a_2(j\omega)\right|^2\end{aligned} \tag{3.6.9}$$

Note that $\left|a_i(j\omega)\right|$, $i=1$ and 2, will be zero only when $s=0$ is a root of $a_i(s)=0$. Because it has been assumed that there are no common factors between a_1(s) and a_2(s), $|a_1(j\omega)|$ and $|a_2(j\omega)|$ cannot simultaneously be zero. As a result, $\left|\Delta(j\omega)\right|>0$; i.e., none of the eigenvalues is on the imaginary axis of the s-plane.

Fact

The optimal poles are the stable eigenvalues of M (Kailath, 1980).

The Symmetric Root Locus

From (3.6.6),

$$a_1(s)a_1(-s)+r^{-1}a_2(s)a_2(-s)=0 \tag{3.6.10}$$

Hence,

$$r^{-1}\frac{a_2(s)a_2(-s)}{a_1(s)a_1(-s)}=-1 \tag{3.6.11}$$

Let

$$a_2(s)=\prod_{i=1}^{m}(s-z_i) \tag{3.6.12}$$

$$a_1(s)=\prod_{i=1}^{n}(s-p_i) \tag{3.6.13}$$

$$r^{-1}(-1)^{m-n}\frac{\prod_{i=1}^{m}(s-z_i)(s+z_i)}{\prod_{i=1}^{n}(s-p_i)(s+p_i)}=-1 \tag{3.6.14}$$

Equation 3.6.14 is in the standard root locus form (Kuo, 1995). When $n-m$ is even, the 180° root locus plot will be constructed. If $n-m$ is odd, the 0° root locus will be constructed.

Case I: High Cost of Control ($r \to \infty$)

When the cost of control action is high, it is desired to use a small value of input $u(t)$. This can be achieved by selecting a very large value of r. When $r \to \infty$, Equation 3.6.6 yields

$$\Delta(s)=a_1(s)a_1(-s)=0 \tag{3.6.15}$$

Because the optimal poles are the stable roots of $\Delta(s)=0$, we have the following two situations:

1. Stable open-loop system: If all the eigenvalues of the matrix A are in the left half of the s-plane, optimal closed-loop poles are the same as open-loop poles.
2. Unstable open-loop system: Optimal closed-loop poles are: (a) stable open-loop poles and (b) reflections of unstable open-loop poles about the imaginary axis.

Case II: Low Cost of Control ($r \to 0$)

When $r \to 0$, Equation 3.6.10 yields

$$a_2(s)a_2(-s)=0 \tag{3.6.16}$$

Hence, m optimal closed-loop poles are the stable roots of $a_2(s)a_2(-s)=0$. They are either open-loop left-half zeros or the reflections of open-loop right-half zeros about the imaginary axis. The remaining $(n-m)$ optimal closed-loop poles are located near infinity. To find their locations, Equation 3.6.10 is written as follows for large s by ignoring lower powers of s:

$$s^n(-s)^n+r^{-1}s^m(-s)^m(b_0)^2=0 \tag{3.6.17}$$

where

$$a_2(s) = b_0 s^m + \ldots + b_m \tag{3.6.18}$$

From (3.6.17),

$$s^{2(n-m)} = (-1)^{n-m+1} b_0^2 r^{-1} \tag{3.6.19}$$

or

$$s = [(-1)^{(n-m+1)}]^{\frac{1}{2(n-m)}} (b_0^2 r^{-1})^{\frac{1}{2(n-m)}} \tag{3.6.20}$$

n − m + 1 odd

$$s = (-1)^{\frac{1}{2(n-m)}} (b_0^2 r^{-1})^{\frac{1}{2(n-m)}} \tag{3.6.21}$$

Recall that

$$-1 = e^{j\pi(2\ell+1)}, \text{ where } \ell \text{ is an integer}$$

Hence

$$(-1)^{\frac{1}{2(n-m)}} = e^{\frac{j\pi(2\ell+1)}{2(n-m)}}; \quad \ell = 0, 1, 2, \ldots, 2(n-m)-1 \tag{3.6.22}$$

Substitution of (3.6.22) into (3.6.21) yields $2(n - m)$ roots of Equation 3.6.19.

n − m + 1 even

$$s = (1)^{\frac{1}{2(n-m)}} (b_0^2 r^{-1})^{\frac{1}{2(n-m)}} \tag{3.6.23}$$

Recall that

$$1 = e^{j\pi 2\ell}, \text{ where } \ell \text{ is an integer} \tag{3.6.24}$$

Hence

$$(1)^{\frac{1}{2(n-m)}} = e^{\frac{j\pi 2\ell}{2(n-m)}}; \quad \ell = 0, 1, 2, \ldots\ldots\ldots, 2(n-m)-1 \tag{3.6.25}$$

Substitution of (3.6.25) into (3.6.23) yields $2(n-m)$ roots of Equation 3.6.19.

To summarize, the $2(n - m)$ roots of equation (3.6.19) lie on a circle of radius $(b_0^2 r^{-1})^{\frac{1}{2(n-m)}}$ in a pattern described by (3.6.21) and (3.6.25). This pattern is known as Butterworth configuration (Kailath, 1980).

Example 3.12: Noncolocated Sensor and Actuator (Bryson, 1979)

Consider the two-degree-of-freedom spring-mass system shown in Figure 3.11, where the force $u_0(t)$ is applied on the left mass and the position of the right mass $y(t)$ is the output. This model has been extensively used to simulate noncolocated sensor and actuator in the structural vibration control.

The governing differential equations of motion are

$$m_0\ddot{x} + k(x-y) = u_0(t) \tag{3.6.26}$$

$$m_0\ddot{y} + k(y-x) = 0 \tag{3.6.27}$$

Nondimensional time t' is defined as

$$t' = t\sqrt{\frac{k}{m_0}} \tag{3.6.28}$$

Therefore,

$$\frac{d(.)}{dt'} = \sqrt{\frac{m_0}{k}}\frac{d(.)}{dt} \quad \text{and} \quad \frac{d^2(.)}{dt'^2} = \frac{m_0}{k}\frac{d^2(.)}{dt^2} \tag{3.6.29a,b}$$

Hence, Equation 3.6.26 and Equation 3.6.27 can be written as

$$x'' + (x-y) = u(t') \quad \text{where} \quad u = \frac{u_0}{k} \tag{3.6.30}$$

$$y'' + (y-x) = 0 \tag{3.6.31}$$

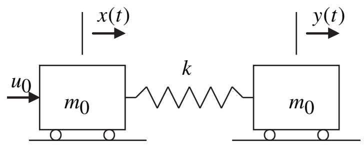

FIGURE 3.11 A two-degree-of-freedom system.

Taking the Laplace transform of (3.6.30) and (3.6.31) with zero initial conditions,

$$s^2 x(s) + (x(s) - y(s)) = u(s) \tag{3.6.32}$$

$$s^2 y(s) + (y(s) - x(s)) = 0 \tag{3.6.33}$$

Some simple algebra yields the SISO transfer function:

$$\frac{y(s)}{u(s)} = \frac{1}{s^2(s^2+2)} = \frac{a_2(s)}{a_1(s)} \tag{3.6.34}$$

Hence, the root locus equation is

$$r^{-1}\frac{a_2(s)a_2(-s)}{a_1(s)a_1(-s)} = -1 \tag{3.6.35}$$

or

$$r^{-1}\frac{1}{s^4(s^2+2)^2} = -1 \tag{3.6.36}$$

The 180° root locus is shown in Figure 3.12. There are eight branches and all of them end at infinity. Angles of asymptotes are

$$(2l+1)\frac{\pi}{8}; \quad l = 0,1,2,3,4,5,6,7 \tag{3.6.37}$$

All these asymptotes intersect the real axis at the origin of complex plane.

3.7 LINEAR QUADRATIC TRAJECTORY CONTROL

It is desired to find an optimal control law in such a way as to cause the output $\mathbf{y}(t)$ to track or follow a desired trajectory $\boldsymbol{\eta}(t)$. Hence, the objective function (3.5.2) is modified (Sage and White, 1977) to be

$$I = \frac{1}{2}\mathbf{z}^T(t_f)S_f\mathbf{z}(t_f) + \frac{1}{2}\int_0^{t_f}[\mathbf{z}^T(t)Q\mathbf{z}(t) + \mathbf{u}^T(t)R\mathbf{u}(t)]dt \tag{3.7.1}$$

where $\mathbf{z}(t)$ is the trajectory error defined as

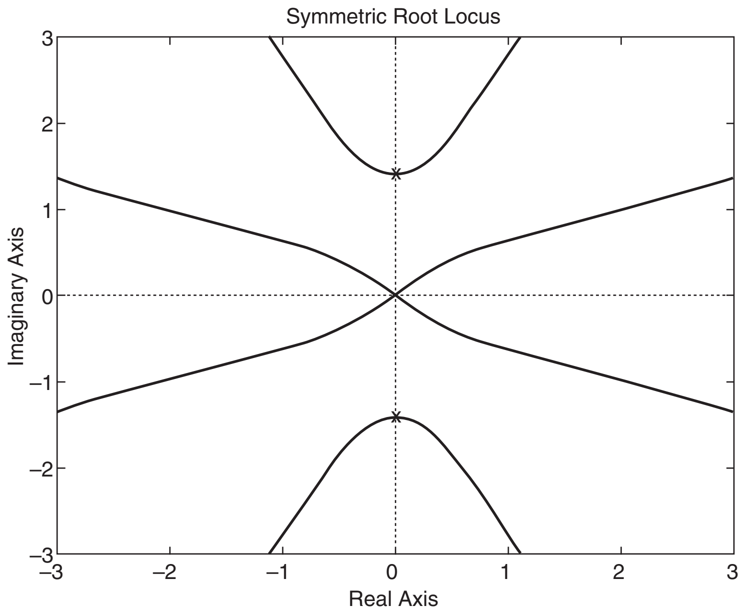

FIGURE 3.12 A symmetric root locus plot.

$$\mathbf{z}(t) = \boldsymbol{\eta}(t) - \mathbf{y}(t) \tag{3.7.2}$$

The state space equation (3.5.1) is modified (Sage and White, 1977) to include a deterministic external input or the plant *noise* vector $w(t)$:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + B\mathbf{u}(t) + \mathbf{w}(t); \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.7.3}$$

From Equation 3.4.11, the Hamiltonian H is defined as

$$H = \frac{1}{2}[\mathbf{z}^T(t)Q\mathbf{z}(t) + \mathbf{u}^T(t)R\mathbf{u}(t)] + \boldsymbol{\lambda}^T(t)[A\mathbf{x}(t) + B\mathbf{u}(t) + \mathbf{w}(t)] \tag{3.7.4}$$

The optimality condition (3.4.16) yields

$$\mathbf{u}(t) = -R^{-1}B^T\boldsymbol{\lambda}(t) \tag{3.7.5}$$

Equation 3.4.12 yields

$$\frac{d\lambda}{dt} = -\{C^TQ[C\mathbf{x}(t) - \boldsymbol{\eta}(t)] + A^T\boldsymbol{\lambda}(t)\} \tag{3.7.6}$$

with the terminal condition

$$\boldsymbol{\lambda}(t_f) = C^T S(t_f)[C\mathbf{x}(t_f) - \boldsymbol{\eta}(t_f)] \tag{3.7.7}$$

In order to determine the closed-loop control law, the transformation (3.5.14) is modified (Sage and White, 1977) to be

$$\boldsymbol{\lambda}(t) = S(t)\mathbf{x}(t) - \boldsymbol{\xi}(t) \tag{3.7.8}$$

where $\boldsymbol{\xi}(t)$ is to be determined. Differentiating (3.7.8),

$$\frac{d\boldsymbol{\lambda}}{dt} = \frac{dS}{dt}\mathbf{x}(t) + S(t)\frac{d\mathbf{x}}{dt} - \frac{d\boldsymbol{\xi}}{dt} \tag{3.7.9}$$

Using (3.7.3), (3.7.5), and (3.7.6),

$$\begin{aligned}&\left(\frac{dS}{dt} + SA - SBR^{-1}B^T S + C^T QC + A^T S\right)\mathbf{x}(t)\\ &+\left(-\frac{d\boldsymbol{\xi}}{dt} + SBR^{-1}B^T\boldsymbol{\xi} + S\mathbf{w}(t) - C^T Q\boldsymbol{\eta}(t) - A^T\boldsymbol{\xi}(t)\right) = 0\end{aligned} \tag{3.7.10}$$

Using (3.7.8), the terminal condition (3.7.7) can be written as

$$S(t_f)\mathbf{x}(t_f) - \boldsymbol{\xi}(t_f) = C^T S(t_f)C\mathbf{x}(t_f) - C^T S(t_f)\boldsymbol{\eta}(t_f) \tag{3.7.11}$$

The solution of (3.7.10) can be obtained by solving it as two separate problems:

$$\frac{dS}{dt} + SA - SBR^{-1}B^T S + C^T QC + A^T S = 0$$

with

$$S(t_f) = C^T S(t_f)C \tag{3.7.12}$$

and

$$-\frac{d\boldsymbol{\xi}}{dt} + SBR^{-1}B^T\boldsymbol{\xi} + S\mathbf{w}(t) - C^T Q\boldsymbol{\eta}(t) - A^T\boldsymbol{\xi}(t) = 0$$

with

$$\boldsymbol{\xi}(t_f) = C^T S(t_f)\boldsymbol{\eta}(t_f) \tag{3.7.13}$$

Lastly, from (3.7.5) and (3.7.8), the control law is

$$\mathbf{u}(t) = -R^{-1}B^T[S(t)\mathbf{x}(t) - \boldsymbol{\xi}(t)] = -K(t)\mathbf{x}(t) + R^{-1}B^T\boldsymbol{\xi}(t) \tag{3.7.14}$$

The state feedback gain matrix $K(t)$ is the same as that given by (3.5.19). Hence, the solution of the linear quadratic trajectory control problem is composed of two parts: (1) a linear regulator part and (2) a correction term containing $\boldsymbol{\xi}(t)$. The computation of $\boldsymbol{\xi}(t)$ requires the solution of (3.7.13) backward in time. Hence, it is required that $\mathbf{w}(t)$ and $\boldsymbol{\eta}(t)$ are exactly known *a priori* for all time t. From the disturbance rejection point of view, the control law can be described to be noncausal.

Example 3.13: Optimal Control of Sun Tracking Solar Concentrators (Hughes, 1979)

A solar collector consists of a concentrator and a receiver. As a concentrator, point focusing parabolic dishes have been used. It reflects the sun's energy towards its focal point and the receiver accepts the concentrated energy for further conversions. The axis of the paraboloid must be pointed at the sun in order to produce the required flux densities at the receiver aperture. Whenever there is a pointing error, energy is lost; hence, this energy loss is minimized by an appropriate control technique.

A linear model of a single axis of the concentrator, which is driven by an electric motor, is shown in Figure 3.13, where

$\theta_0(t)$: Collector's line of sight (LOS)
$\theta_i(t)$: Sun's position
$u(t)$: Command input to the motor
ξ: Damping ratio
K_s: System gain
ω_n: Natural frequency

The state space model of the system can be written as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b}u(t) \tag{3.7.15}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) \tag{3.7.16}$$

where

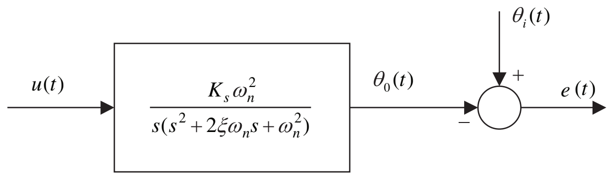

FIGURE 3.13 A model for sun tracking solar concentrator.

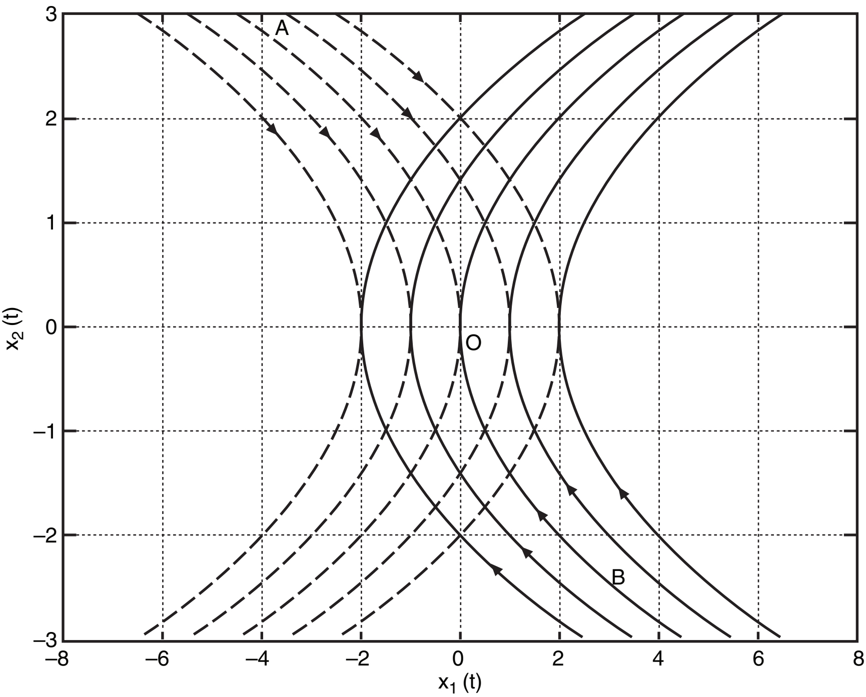

FIGURE 3.14 State space trajectories (___ $u = 1$, ---- $u = 1$).

$$\mathbf{x} = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \\ \ddot{\theta}_0 \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_n^2 & -2\xi\omega_n \end{bmatrix};$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ K_s\omega_n^2 \end{bmatrix}; \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.7.17}$$

Let $\boldsymbol{\eta}(t)$ be a vector of the sun's position, velocity, and acceleration, i.e.,

$$\boldsymbol{\eta}(t) = \begin{bmatrix} \theta_i \\ \dot{\theta}_i \\ \ddot{\theta}_i \end{bmatrix} \tag{3.7.18}$$

To minimize the energy loss, the output vector $\mathbf{y}(t)$ should follow the sun's trajectory $\boldsymbol{\eta}(t)$ as closely as possible. Hence, a linear servomechanism or LQ tracking problem is solved. The objective function is

$$J = \frac{1}{2}\int_0^{t_f - t_0} \mathbf{z}^T Q\mathbf{z} + ru^2\, dt \tag{3.7.19}$$

where

$$\mathbf{z}(t) = \boldsymbol{\eta}(t) - \mathbf{y}(t) \tag{3.7.20}$$

t_0 is the time of sunrise, and t_f is the time of sunset. Using (3.7.14), the control law is given as

$$u(t) = -K(t)\mathbf{x}(t) + r^{-1}b^T\boldsymbol{\xi}(t) \tag{3.7.21}$$

The variable $\boldsymbol{\xi}(t)$ is obtained by solving Equation 3.7.13 with $\mathbf{w}(t) = 0$. Because the desired trajectory $\boldsymbol{\eta}(t)$ changes every day, the variable $\boldsymbol{\xi}(t)$ has to be computed every day.

3.8 FREQUENCY-SHAPED LQ CONTROL

Let the quadratic objective function be

$$I = \int_0^{\infty} \mathbf{x}(t)^T Q\mathbf{x}(t) + \mathbf{u}(t)^T R\mathbf{u}(t)dt \tag{3.8.1}$$

Using Parseval's Theorem (Appendix B),

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{x}^T(-j\omega)Q\mathbf{x}(j\omega) + \mathbf{u}^T(-j\omega)R\mathbf{u}(j\omega)d\omega \tag{3.8.2}$$

Modify the objective function by making Q and R functions of the frequency ω (Gupta, 1980):

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{x}^T(-j\omega)Q(\omega)\mathbf{x}(j\omega) + \mathbf{u}^T(-j\omega)R(\omega)\mathbf{u}(j\omega)d\omega \tag{3.8.3}$$

Assume that $Q(\omega)$ and $R(\omega)$ are rational functions of ω^2. Factoring these matrices,

$$Q(\omega) = N_q^T(-j\omega)N_q(j\omega) \tag{3.8.4}$$

$$R(\omega) = N_r^T(-j\omega)N_r(j\omega) \tag{3.8.5}$$

Let the rank of the matrix $Q(\omega)$ be p. Then, $N_q(j\omega)$ will be an $p \times n$ matrix. For the existence of the solution of an LQ problem, the matrix $R(\omega)$ has to be of full rank. Therefore, $N_r(j\omega)$ will be an $m \times m$ matrix

Define

$$\mathbf{x}^T(-j\omega)Q(\omega)\mathbf{x}(j\omega) = \mathbf{z}^T(-j\omega)\mathbf{z}(j\omega) \tag{3.8.6}$$

where

$$N_q(j\omega)\mathbf{x}(j\omega) = \mathbf{z}(j\omega) \tag{3.8.7}$$

Similarly,

$$\mathbf{u}^T(-j\omega)R(\omega)\mathbf{u}(j\omega) = \boldsymbol{\chi}^T(-j\omega)\boldsymbol{\chi}(j\omega) \tag{3.8.8}$$

where

$$N_r(j\omega)\mathbf{u}(j\omega) = \boldsymbol{\chi}(j\omega) \tag{3.8.9}$$

The objective function (3.8.3) can be expressed as

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{z}^T(-j\omega)\mathbf{z}(j\omega) + \boldsymbol{\chi}^T(-j\omega)\boldsymbol{\chi}(j\omega)d\omega \tag{3.8.10}$$

Using Parseval's Theorem (Appendix B),

$$I = \int_{0}^{\infty} \mathbf{z}^T(t)\mathbf{z}(t) + \boldsymbol{\chi}^T(t)\boldsymbol{\chi}(t)dt \tag{3.8.11}$$

From (3.8.7), $\mathbf{z}(t)$ is the output of the linear system with the transfer function matrix $N_q(s)$ and input $\mathbf{x}(t)$; i.e.,

$$\mathbf{z}(s) = N_q(s)\mathbf{x}(s) \tag{3.8.12}$$

Similarly, from (3.8.9), $\boldsymbol{\chi}(t)$ is the output of the linear system with the transfer function matrix $N_r(s)$ and input $\mathbf{u}(t)$; i.e.,

$$\boldsymbol{\chi}(s) = N_r(s)\mathbf{u}(s) \tag{3.8.13}$$

Let the state space model of the MIMO system (3.8.12) be

$$\dot{\boldsymbol{\xi}} = A_q\boldsymbol{\xi} + B_q\mathbf{x}(t) \tag{3.8.14}$$

$$\mathbf{z}(t) = C_q\boldsymbol{\xi} + D_q\mathbf{x}(t) \tag{3.8.15}$$

Similarly, let the state space model of the MIMO system (3.8.13) be

$$\dot{\mathbf{v}} = A_r\mathbf{v} + B_r\mathbf{u}(t) \tag{3.8.16}$$

$$\boldsymbol{\chi}(t) = C_r\mathbf{v} + D_r\mathbf{u}(t) \tag{3.8.17}$$

Define the augmented state vector $\mathbf{x}_a(t)$:

$$\mathbf{x}_a(t) = \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\xi} \\ \mathbf{v} \end{bmatrix} \tag{3.8.18}$$

Let the augmented state space model be

$$\dot{\mathbf{x}}_a(t) = A_a\mathbf{x}_a(t) + B_a\mathbf{u}(t) \tag{3.8.19}$$

where

$$A_a = \begin{bmatrix} A & 0 & 0 \\ B_q & A_q & 0 \\ 0 & 0 & A_r \end{bmatrix} \quad \text{and} \quad B_a = \begin{bmatrix} B \\ 0 \\ B_r \end{bmatrix} \tag{3.8.20}$$

Now, it can be shown that

$$\mathbf{z}^T\mathbf{z}+\boldsymbol{\chi}^T\boldsymbol{\chi}=\mathbf{x}_a^TQ_f\mathbf{x}_a+2\mathbf{x}_a^TN_f\mathbf{u}+\mathbf{u}^TR_f\mathbf{u} \tag{3.8.21}$$

where

$$Q_f=\begin{bmatrix} D_q^TD_q & D_q^TC_q & 0 \\ C_q^TD_q & C_q^TC_q & 0 \\ 0 & 0 & C_r^TC_r \end{bmatrix},\quad N_f=\begin{bmatrix} 0 \\ 0 \\ C_r^TD_r \end{bmatrix},\quad \text{and}\quad R_f=D_r^TD_r \tag{3.8.22}$$

Therefore, the objective function (3.8.11) is

$$I=\int_0^\infty \mathbf{x}_a^TQ_f\mathbf{x}_a+2\mathbf{x}_a^TN_f\mathbf{u}+\mathbf{u}^TR_f\mathbf{u}dt \tag{3.8.23}$$

Using (3.5.29), the optimal control law will be

$$\mathbf{u}(t)=-K_a\mathbf{x}_a(t) \tag{3.8.24}$$

where

$$K_a=R_f^{-1}(PB_a+N_f)^T \tag{3.8.25}$$

and

$$P(A_a-B_aR_f^{-1}N_f^T)+(A_a-B_aR_f^{-1}N_f^T)^TP-PB_aR_f^{-1}B_a^TP+Q_f-N_fR_f^{-1}N_f^T=0 \tag{3.8.26}$$

Representing K_a as

$$K_a=\begin{bmatrix} K_x & K_\xi & K_\nu \end{bmatrix} \tag{3.8.27}$$

$$\mathbf{u}(t)=-K_x\mathbf{x}(t)-K_\xi\boldsymbol{\xi}(t)-K_\nu\mathbf{v}(t) \tag{3.8.28}$$

Example 3.14: A Simple Harmonic Oscillator

Consider a second-order system:

$$\ddot{y}+\omega_n^2y=u(t) \tag{3.8.29}$$

for which the state equations are

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u(t) \tag{3.8.30}$$

$$y(t) = x_1(t) \tag{3.8.31}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \tag{3.8.32}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.8.33a,b}$$

Standard LQ Control

Let the objective function be

$$I = \int_0^\infty (y^2 + u^2)dt \tag{3.8.34}$$

The optimal state feedback control law is

$$u(t) = -\mathbf{k}\mathbf{x}(t) \tag{3.8.35}$$

where $\mathbf{k}$ is the state feedback gain vector.

For $\omega_n = 10$ rad/sec,

$$\mathbf{k} = \begin{bmatrix} 0.005 & 0.1 \end{bmatrix} \tag{3.8.36}$$

and the eigenvalues of the optimal closed-loop system are $-0.05 \pm 10j$. Therefore, if the disturbance to the system is a sinusoidal function with the frequency equal to 10 rad/sec, its effects will be extremely large.

Frequency-Shaped LQ Control

Let the objective function be

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} [|y(j\omega)|^2 \, |Q(j\omega)|^2 + |u(j\omega)|^2]d\omega \tag{3.8.37}$$

where

$$Q(s) = \frac{\omega_n^2 s}{s^2 + 2\varsigma\omega_n s + \omega_n^2} \tag{3.8.38}$$

For a small value of the damping ratio ς, $|Q(j\omega_n)|$ will be extremely large, and as a result the response of the optimal closed-loop system is expected to be insensitive to external disturbance at the frequency $\omega = \omega_n$. Now, let

$$Q(s)y(s) = z(s) \tag{3.8.39}$$

The state space realization of the system (3.8.39) can be written as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\varsigma\omega_n \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} y(t) \tag{3.8.40}$$

and

$$z(t) = \xi_2(t) \tag{3.8.41}$$

Combining (3.8.30) and (3.8.40),

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_n^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_n^2 & 0 & -\omega_n^2 & -2\varsigma\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t) \tag{3.8.42}$$

From (3.8.39), the objective function (3.8.37) can be written as

$$I = \int_0^\infty (z^2 + u^2)dt \tag{3.8.43}$$

Equation 3.8.42 and Equation 3.8.43 constitute a standard LQ control problem. The optimal state feedback control law is

$$u(t) = -\mathbf{k}_a[x_1 \quad x_2 \quad \xi_1 \quad \xi_2]^T \tag{3.8.44}$$

where $\mathbf{k}_a$ is the optimal state feedback gain vector. For $\omega_n = 10$ rad/sec and $\varsigma = 0.1$,

$$\mathbf{k}_a = [4.1717 \quad 2.8885 \quad -4.1717 \quad 0.0178] \tag{3.8.45}$$

And the eigenvalues of the optimal closed-loop system are $-1.217 \pm 10.9791j$ and $-1.2272 \pm 8.96921j$. Therefore, if the disturbance to the system is a sinusoidal function with frequency 10 rad/sec, its effect will be small.

3.9 MINIMUM-TIME CONTROL OF A LINEAR TIME-INVARIANT SYSTEM

Consider the linear system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + B\mathbf{u}(t); \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.9.1}$$

It is desired to apply a control $\mathbf{u}(t)$, such that the system reaches the origin of the state space, in a minimum time when inputs must satisfy the following constraints:

$$|u_i(t)| \leq \alpha_i ; \quad i = 1, 2, \ldots, m \tag{3.9.2}$$

If the final time is denoted by t_f, the objective is to minimize the following objective function:

$$I = t_f = \int_0^{t_f} 1 dt \tag{3.9.3}$$

Therefore, the Hamiltonian H is

$$H = 1 + \boldsymbol{\lambda}^T (A\mathbf{x} + B\mathbf{u}) \tag{3.9.4}$$

The adjoint equations are

$$\frac{d\boldsymbol{\lambda}}{dt} = -\frac{\partial H}{\partial \mathbf{x}} = -A^T \boldsymbol{\lambda} \tag{3.9.5}$$

The solution of (3.9.5) is

$$\boldsymbol{\lambda}(t) = e^{-A^T t} \boldsymbol{\lambda}(0) \tag{3.9.6}$$

Substituting (3.9.6) into (3.9.4),

$$H = 1 + \boldsymbol{\lambda}^T (0) e^{-At} A\mathbf{x} + \boldsymbol{\lambda}^T (0) e^{-At} B\mathbf{u} \tag{3.9.7}$$

Therefore,

$$\frac{\partial H}{\partial u_i} = p_i \tag{3.9.8}$$

where

$$p_i = \boldsymbol{\lambda}^T(0)e^{-At}\mathbf{b}_i \tag{3.9.9}$$

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & . & . & \mathbf{b}_m \end{bmatrix} \tag{3.9.10}$$

$$\mathbf{u}^T = \begin{bmatrix} u_1 & u_2 & . & . & u_m \end{bmatrix} \tag{3.9.11}$$

Using (3.4.17) and (3.4.18), the optimal control inputs are

$$u_{0i}(t) = \begin{cases} +\alpha_i & if \quad p_i < 0 \\ -\alpha_i & if \quad p_i > 0 \end{cases} \tag{3.9.12}$$

Normality of Linear Systems

When $p_i(t) = 0$ over a finite interval $[t_1 \quad t_2]$, it is called a singular control case. This also implies that higher derivatives of $p_i(t)$, with respect to time, are zero over this finite interval. Therefore, for $t \in [t_1, t_2]$,

$$\begin{aligned} & p_i = \boldsymbol{\lambda}^T(0)e^{-At}\mathbf{b}_i = 0 \\ & -\frac{dp_i}{dt} = \boldsymbol{\lambda}^T(0)e^{-At}A\mathbf{b}_i = 0 \\ & \frac{d^2 p_i}{dt^2} = \boldsymbol{\lambda}^T(0)e^{-At}A^2\mathbf{b}_i = 0 \\ & \vdots \\ & (-1)^{n-1}\frac{d^{n-1}p_i}{dt^{n-1}} = \boldsymbol{\lambda}^T(0)e^{-At}A^{n-1}\mathbf{b}_i = 0 \end{aligned} \tag{3.9.13}$$

Equation 3.9.13 can be expressed as

$$\boldsymbol{\lambda}^T(0)e^{-At}C_i = 0 \tag{3.9.14}$$

where

$$C_i = [\mathbf{b}_i \quad A\mathbf{b}_i \quad A^2\mathbf{b}_i \quad . \quad . \quad A^{n-1}\mathbf{b}_i] \tag{3.9.15}$$

is the controllability matrix with respect to u_i.

It should be noted that $\boldsymbol{\lambda}(0) \neq 0$ because $H = 1$ otherwise; i.e., $H \neq 0$ and the solution will not be optimal (Section 3.4.2). Therefore, for Equation 3.9.14 to be true,

$$rank(C_i) < n \tag{3.9.16}$$

This analysis implies that there are no finite intervals on which $p_i(t) = 0$, provided C_i is not singular for any i, i = 1, 2, ... m. A system for which C_i is not singular for any i is called *normal* (Gopal, 1984).

Existence and Uniqueness Theorems on Minimum-Time Control (Gopal, 1984)

1. If the linear time-invariant system is controllable, and if all the eigenvalues of A have nonpositive real parts, a time-optimal control exists that transfers any initial state to the origin of the state space in a minimum time.
2. If the linear time-invariant system is normal, and if the time-optimal control exists, it is unique.
3. If eigenvalues of the matrix A are real and a unique time-optimal exists, each control component can switch at the most (n–1) times, where n is the dimension of the state space.

Example 3.15: Time-Optimal Control of a Rigid Body or a Double Integrator System

Consider the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + Bu(t) \tag{3.9.17}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.9.18}$$

and

$$|u| \leq 1 \tag{3.9.19}$$

The objective is to find the optimal control $u(t)$, such that the system reaches the origin of the state space in a minimum time from an initial state $\mathbf{x}(0)$.

First, note that eigenvalues of A are 0 and 0, which are real and nonpositive. Furthermore,

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.9.20}$$

Therefore, the system is controllable and normal as well. Application of the existence and uniqueness theorem indicates that a unique optimal control exists with at the most one switching.

$$\text{Hamiltonian } H = 1 + \lambda_1 x_2 + \lambda_2 u \tag{3.9.21}$$

$$\frac{\partial H}{\partial u} = \lambda_2 \tag{3.9.22}$$

Hence, the optimal control input is given by

$$u(t) = \begin{cases} -1 & if \quad \lambda_2 > 0 \\ +1 & if \quad \lambda_2 < 0 \end{cases} \tag{3.9.23}$$

Equation 3.9.23 can be expressed as

$$u(t) = -\operatorname{sgn}(\lambda_2(t)) \tag{3.9.24}$$

Adjoint equations are

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1} = 0 \tag{3.9.25}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2} = -\lambda_1 \tag{3.9.26}$$

The solution of (3.9.25) is

$$\lambda_1(t) = \lambda_1(0) \tag{3.9.27}$$

Substituting (3.9.27) into (3.9.26),

$$\lambda_2(t) = -\lambda_1(0)t + \lambda_2(0) \tag{3.9.28}$$

Because states are specified at initial and final time, there are no constraints on adjoint variables. Hence, state equations must be used to determine adjoint variables.

If $u = 1$, the solution to (3.9.17) is

$$x_2(t) = t + x_2(0) \tag{3.9.29}$$

$$x_1(t) = \frac{t^2}{2} + x_2(0)t + x_1(0) \tag{3.9.30}$$

Eliminating t between (3.9.29) and (3.9.30),

$$x_1 = \frac{x_2^2}{2} + x_1(0) - \frac{(x_2(0))^2}{2} \tag{3.9.31}$$

Equation 3.9.31 describes state space trajectories (which are parabolas) when $u = 1$. These trajectories are shown as solid curves in Figure 3.14.

If $u = -1$, solution to (3.9.17) is

$$x_2(t) = -t + x_2(0) \tag{3.9.32}$$

$$x_1(t) = -\frac{t^2}{2} + x_2(0)t + x_1(0) \tag{3.9.33}$$

Eliminating t between (3.9.32) and (3.9.33),

$$x_1 = -\frac{x_2^2}{2} + x_1(0) + \frac{(x_2(0))^2}{2} \tag{3.9.34}$$

Equation 3.9.34 describes state space trajectories (which are parabolas) when $u = -1$. These trajectories are shown as dashed curves in Figure 3.14.

Because it is known that the optimal control input can only have one switching at the most, only four cases exist:

Case I:

$$u = 1, \quad 0 \le t \le t_f \tag{3.9.35}$$

Initial states must be such that the system is on segment BO in Figure 3.14.

Case II:

$$u = -1\,,\ \ 0 \le t \le t_f \tag{3.9.36}$$

Initial states must be such that the system is on segment AO in Figure 3.14.

Case III:

$$u = \begin{cases} +1 & 0 \le t \le t_s \\ -1 & t_s < t \le t_f \end{cases} \tag{3.9.37}$$

Initial states must be such that the system will follow one of solid parabolas, which intersect the segment AO. At the switching instant t_s, the system will reach the segment AO and go to the origin of the state space along the segment AO.

Case IV:

$$u = \begin{cases} -1 & 0 \le t \le t_s \\ +1 & t_s < t \le t_f \end{cases} \tag{3.9.38}$$

Initial states must be such that the system will follow one of dashed parabolas, which intersect the segment BO. At the switching instant t_s, the system will reach the segment BO and go to the origin of the state space along the segment BO.

The switching curve is composed of segments AO and BO (Figure 3.14). For segment AO,

$$x_1(0) + \frac{(x_2(0))^2}{2} = 0 \tag{3.9.39}$$

Equation 3.9.39 implies the following equation for the curve AO:

$$x_1(t) = -\frac{1}{2}(x_2(t))^2 \tag{3.9.40}$$

For segment BO,

$$x_1(0) - \frac{(x_2(0))^2}{2} = 0 \tag{3.9.41}$$

Equation 3.9.41 implies the following equation for the curve BO:

$$x_1(t) = \frac{1}{2}(x_2(t))^2 \tag{3.9.42}$$

Combining (3.9.40) and (3.9.42), the equation of the switching curve AOB can be expressed as

$$x_1(t) = -\frac{1}{2}x_2(t)\left|x_2(t)\right| \tag{3.9.43}$$

On the basis of (3.9.43), a switching function (Gopal, 1984) is defined as follows:

$$s(\mathbf{x}(t)) = x_1(t) + \frac{1}{2}x_2(t)\left|x_2(t)\right| \tag{3.9.44}$$

If $s > 0$, $\mathbf{x}(t)$ lies above the curve AOB. In this case, $u = -1$ because dotted parabolas are directed toward the switching curve.
If $s < 0$, $\mathbf{x}(t)$ lies below the curve AOB. In this case, $u = 1$ because solid parabolas are directed toward the switching curve.
If $s = 0$ and $x_2 > 0$, $\mathbf{x}(t)$ lies on the curve AO. In this case, $u = -1$.
If $s = 0$ and $x_2 < 0$, $\mathbf{x}(t)$ lies on the curve BO. In this case, $u = 1$.

In summary, the time-optimal control in feedback form is

$$u(t) = \begin{cases} -1 & when \quad s(\mathbf{x}(t)) > 0 \\ +1 & when \quad s(\mathbf{x}(t)) = 0 \quad and \quad x_2(t) < 0 \\ +1 & when \quad s(\mathbf{x}(t)) < 0 \\ -1 & when \quad s(\mathbf{x}(t)) = 0 \quad and \quad x_2(t) > 0 \end{cases} \tag{3.9.45}$$

The structure of the optimal feedback control system is shown in Figure 3.15.

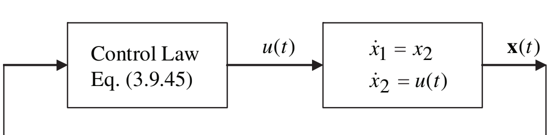

FIGURE 3.15 Implementation of bang-bang control via state feedback.

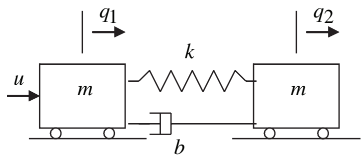

FIGURE 3.16 A system with rigid and flexible modes.

Example 3.16: Minimum-Time Control of a System with Rigid and Flexible Modes

The differential equations of motion of the mechanical system shown in Figure 3.16 are

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\begin{bmatrix} b & -b \\ -b & b \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}+\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}=\begin{bmatrix} u(t) \\ 0 \end{bmatrix} \tag{3.9.46}$$

Natural frequencies are found to be

$$\omega_0^2 = 0 \tag{3.9.47}$$

and

$$\omega_1^2 = \frac{2k}{m} \tag{3.9.48}$$

Modal vectors are as follows:

$$\begin{bmatrix} 1 & 1 \end{bmatrix}^T \text{ corresponding to } \omega_0^2 = 0 \text{ (Rigid mode)} \tag{3.9.49}$$

and

$$\begin{bmatrix} 1 & -1 \end{bmatrix}^T \text{ corresponding to } \omega_1^2 = 2k/m \text{ (Flexible mode)} \tag{3.9.50}$$

Now, the displacement vector can be represented as a linear combination of these modal vectors:

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \Phi \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{3.9.51}$$

where

$$\Phi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3.9.52}$$

Substituting (3.9.51) into (3.9.46), and premultiplying both sides by Φ^T,

$$\ddot{r}_1 = \frac{1}{2m} u \tag{3.9.53}$$

and

$$\ddot{r}_2 + 2\varsigma\omega_1\dot{r}_2 + \omega_1^2 r_2 = \frac{1}{2m} u \tag{3.9.54}$$

where $\varsigma = 2b/(2m\omega_1)$. Defining state variables as

$$x_1 = r_1 \text{ , } x_2 = \dot{r}_1 \text{ , } x_3 = r_2 \text{ , and } x_4 = \dot{r}_2 \tag{3.9.55}$$

State equations are

$$\dot{\mathbf{x}} = A\mathbf{x}(t) + \mathbf{b}u(t) \tag{3.9.56}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_1^2 & -2\varsigma\omega_1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0.5m \\ 0 \\ 0.5m \end{bmatrix} \tag{3.9.57}$$

Let the initial and final states be

$$x(0) = \begin{bmatrix} -L & 0 & 0 & 0 \end{bmatrix}^T \quad \text{and} \quad \mathbf{x}(t_f) = 0 \qquad \text{(3.9.58a and b)}$$

and

$$|u(t)| \leq \alpha \tag{3.9.59}$$

The minimum time control will be bang-bang and unique. However, because all eigenvalues are not real, the maximum number of switching is not necessarily (n–1). Let switching instants be t_i, i = 1, 2, …, k, and initially $u(t) = \alpha$ as the position of the system has to be increased. Solving state equations with this input and initial condition (3.9.58a), and then imposing the final condition (3.9.58b), the following nonlinear equations (Pao and Singhose,1998; Singhose and Pao, 1997) are obtained:

$$(-1)^k t_f + 2\sum_{i=1}^{k} (-1)^{i-1} t_i \tag{3.9.60}$$

$$(-1)^{k+1} t_f^2 + 2\sum_{i=1}^{k} (-1)^i (t_i)^2 = \frac{4mL}{\alpha} \tag{3.9.61}$$

$$1 + (-1)^{k+1} e^{\varsigma\omega_1 t_f} \cos(\omega_d t_f) + 2\sum_{i=1}^{k} (-1)^i e^{\varsigma\omega_1 t_i} \cos(\omega_d t_i) = 0 \tag{3.9.62}$$

$$(-1)^{k+1} e^{\varsigma\omega_1 t_f} \sin(\omega_d t_f) + 2\sum_{i=1}^{k} (-1)^i e^{\varsigma\omega_1 t_i} \sin(\omega_d t_i) = 0 \tag{3.9.63}$$

where $\omega_d = \omega_n\sqrt{1-\varsigma^2}$. There are many solutions of t_i, i = 1, 2, …, k, that satisfy (3.9.60) to (3.9.63). However, only one of them will satisfy Equation 3.9.12.

Example 3.17: Input Shaping

Consider the prototype second-order system (Kuo, 1995):

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u(t) \tag{3.9.64}$$

where ξ, ω_n, and u(t) are the damping ratio, undamped natural frequency, and the reference input, respectively. The reference input is often a unit step function for which the response of an underdamped system is

$$y(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \chi e^{-\xi\omega_n t}\sin\omega_d t; \quad t \geq 0 \tag{3.9.65}$$

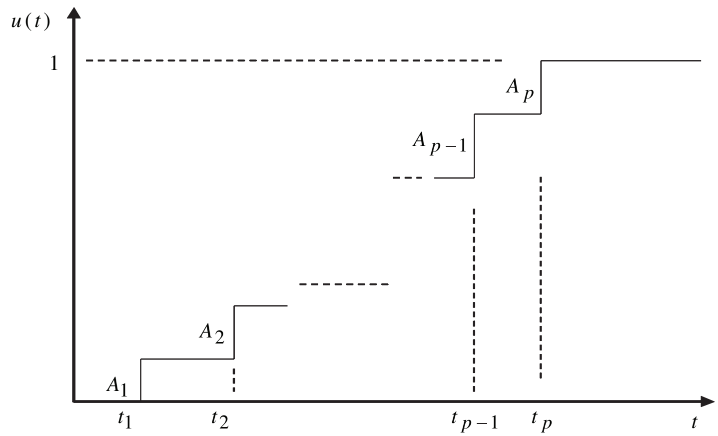

FIGURE 3.17 A staircase input.

where

$$\chi = \frac{\xi}{\sqrt{1-\xi^2}} \quad \text{and} \quad \omega_d = \omega_n\sqrt{1-\xi^2} \tag{3.9.66a,b}$$

The value of the steady state error is guaranteed to be zero. However, there is a large amount of oscillation, and the settling time can be large for a small damping level.

The concept of input shaping is to modify the command input so that that the system reaches the final position with zero oscillation. For the staircase input shown in Figure 3.17, the response can be written as

$$y(t) = \sum_{i=1}^{p} A_i[1 - e^{-\xi\omega_n(t-t_i)}(\cos\omega_d(t-t_i) + \chi\sin\omega_d(t-t_i))]u_s(t-t_i) \tag{3.9.67}$$

where $u_s(t-t_i)$ is a unit step function applied at $t = t_i$.

For $t > t_p$,

$$y(t) = g(t) + \sum_{i=1}^{p} A_i \tag{3.9.68}$$

where

$$g(t) = -e^{-\xi\omega_n t}[\alpha \sin \omega_d t + \beta \cos \omega_d t] \tag{3.9.69}$$

$$\alpha = \sum_{i=1}^{p} e^{\xi\omega_n t_i} A_i (\sin \omega_d t_i + \chi \cos \omega_d t_i) \tag{3.9.70}$$

$$\beta = \sum_{i=1}^{p} e^{\xi\omega_n t_i} A_i (\cos \omega_d t_i - \chi \sin \omega_d t_i) \tag{3.9.71}$$

The conditions for $g(t)$ to be zero for $t > t_p$ are

$$\alpha = \sum_{i=1}^{p} e^{\xi\omega_n t_i} A_i (\sin \omega_d t_i + \chi \cos \omega_d t_i) = 0 \tag{3.9.72}$$

$$\beta = \sum_{i=1}^{p} e^{\xi\omega_n t_i} A_i (\cos \omega_d t_i - \chi \sin \omega_d t_i) = 0 \tag{3.9.73}$$

Equation 3.9.72 and Equation 3.9.73 are equivalent to the following well-known conditions (Singer and Seering, 1990):

$$\sum_{i=1}^{p} e^{\xi\omega_n t_i} A_i \sin \omega_d t_i = 0 \tag{3.9.74}$$

$$\sum_{i=1}^{p} e^{\xi\omega_n t_i} A_i \cos \omega_d t_i = 0 \tag{3.9.75}$$

Consider the case of $p = 2$. Without any loss of generality, assume that

$$t_1 = 0 \quad \text{and} \quad A_1 + A_2 = 1 \tag{3.9.76a and b}$$

In this case, Equation 3.9.74 and Equation 3.9.75 can be solved for two unknowns. The results are

$$t_2 = \frac{\pi}{\omega_d}, \quad A_1 = \frac{1}{1+K}, \text{ and } K = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \tag{3.9.77a, b, and c}$$

Therefore, from (3.9.68),

$$y(t) = 1 \quad \text{for } t > t_2 \tag{3.9.78}$$

In other words, the system reaches the desired value without any oscillation for $t > t_2$, when the input $u(t)$ is as follows:

$$u(t) = \begin{cases} (1+K)^{-1}; & 0 < t \leq \pi/\omega_d \\ 1; & t > \pi/\omega_d \end{cases} \tag{3.9.79}$$

Time Optimality of the Shaped Input

Define

$$x_1 = y - 1, \quad x_2 = \dot{y}, \quad v = u - 1 \tag{3.9.80}$$

The state equations are

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}v(t) \tag{3.9.81}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \tag{3.9.82}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} \tag{3.9.83a,b}$$

Now, initial and final states are

$$\mathbf{x}(0) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T \quad \text{and} \quad \mathbf{x}(t_f) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \tag{3.9.84a,b}$$

where t_f is the final time. Another constraint for a positive shaper is that the input $u(t)$ should be between 0 and 1. This implies the following constraint on $v(t)$:

$$-1 \leq v(t) \leq 0 \tag{3.9.85}$$

For minimum-time control, the Hamiltonian H is defined as follows:

$$H = 1 + \lambda_1 x_2 + \lambda_2(-\omega_n^2 x_1 - 2\xi\omega_n x_2 + \omega_n^2 v) \tag{3.9.86}$$

where adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$ satisfy the following equations:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = \omega_n^2 \lambda_2 \tag{3.9.87}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 + 2\xi\omega_n \lambda_2 \tag{3.9.88}$$

and

$$\frac{\partial H}{\partial v} = \omega_n^2 \lambda_2 \tag{3.9.89}$$

Therefore, the minimum-time control will depend on the sign of $\lambda_2(t)$. Differentiating (3.9.88) and using (3.9.87),

$$\ddot{\lambda}_2 - 2\xi\omega_n \dot{\lambda}_2 + \omega_n^2 \lambda_2 = 0 \tag{3.9.90}$$

Solving (3.9.90),

$$\lambda_2(t) = e^{\xi\omega_n t}\left[\frac{\dot{\lambda}_2(0) - \xi\omega_n \lambda_2(0)}{\omega_d}\sin\omega_d t + \lambda_2(0)\cos\omega_d t\right] \tag{3.9.91}$$

From (3.9.88),

$$\dot{\lambda}_2(0) = -\lambda_1(0) + 2\xi\omega_n \lambda_2(0) \tag{3.9.92}$$

Substituting (3.9.92) into (3.9.91),

$$\lambda_2(t) = e^{\xi\omega_n t}\left[\frac{-\lambda_1(0) + \xi\omega_n \lambda_2(0)}{\omega_d}\sin\omega_d t + \lambda_2(0)\cos\omega_d t\right] \tag{3.9.93}$$

Because both $\mathbf{x}(0)$ and $\mathbf{x}(t_f)$ are specified, $\lambda_1(0)$ and $\lambda_2(0)$ are not specified, and have to be determined so that the resulting input leads to satisfaction of initial and final conditions of states. To achieve the control input (3.9.79) that leads to satisfaction of initial and final conditions of states,

$$\lambda_2(0) = 0 \quad \text{and} \quad \lambda_1(0) < 0 \tag{3.9.94}$$

Then, from (3.9.93),

$$\lambda_2(t) = -\lambda_1(0)e^{\xi\omega_n t} \sin \omega_d t > 0 \quad \text{for } 0 < t < \tfrac{\pi}{\omega_d} \tag{3.9.95}$$

and

$$\lambda_2(t) = -\lambda_1(0)e^{\xi\omega_n t} \sin \omega_d t < 0 \quad \text{for } \tfrac{\pi}{\omega_d} < t < \tfrac{2\pi}{\omega_d} \tag{3.9.96}$$

The control input (3.9.79) is minimum-time and bang-bang if the constraint (3.9.85) is modified as follows:

$$-1 + (1 + K)^{-1} \le v(t) \le 0 \tag{3.9.97}$$

EXERCISE PROBLEMS

P3.1 Consider a linear system described by the following transfer function:

$$\frac{y(s)}{u(s)} = \frac{10}{s(s+1)}$$

Design a state feedback controller such that the eigenvalues of the closed-loop system are at –2 and –3.

P3.2 Consider a linear system described by the following state space equation:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -x_2 + u(t)$$

$$y(t) = x_1(t) + x_2(t)$$

i. Find the open-loop eigenvalues.

ii. Find the state feedback gain vector such that both eigenvalues for the closed-loop system are located at –2.

iii. Let $u(t) = -k_1 x_1(t) - k_2 x_2(t)$, and k_1 and k_2 be as calculated in part ii. Furthermore, $y(0) = 0$ and $\frac{dy}{dt}(0) = 1$.

a. Determine $x_1(0)$ and $x_2(0)$

b. Determine the response $x_1(t)$ and $x_2(t)$ via the matrix exponential.

P3.3 Consider a linear system described by the transfer function

$$\frac{y(s)}{u(s)} = \frac{10(s+3)}{s(s+1)(s+2)}$$

a. Develop a state space realization and construct the block diagram.

b. Find the state feedback gain vector such that eigenvalues of the closed-loop system are located at –2, –3, and –4.

c. Examine the controllability and observability of the closed-loop system.

P3.4 Consider the single-link flexible robot manipulator (Figure P3.4) which is driven at one end by a DC motor. The input torque is represented by $u(t)$. Assume that the response of the system can be adequately represented by considering only two modes of vibration. Parameters for the manipulator are given as follows:

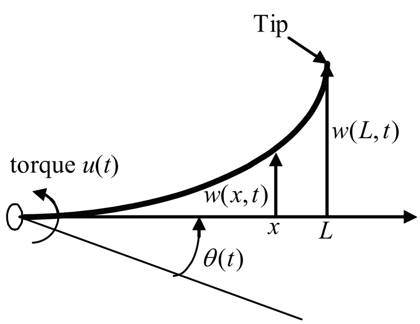

FIGURE P3.4 A single-link flexible manipulator.

$$EI = 2.04146 \text{ N-m}^2,\ L = 1.05 \text{ m}$$

$$m = 0.4252 \text{ kg},\ I_\alpha = 0.15628 \text{ kg-m}^2$$

where $E =$ Young's modulus of elasticity, $I =$ area moment of inertia of the beam cross-section, $L =$ length of the beam, $\rho =$ mass of the beam per unit length, $m =$ mass of the beam, and $I_\alpha =$ mass moment of inertia of the beam about the torque axis = $mL^2/3$.

The governing partial differential equation (PDE) of motion is

$$EI\frac{\partial^4 y}{\partial x^4}+\rho\frac{\partial^2 y}{\partial t^2}=0 \qquad \text{where } y(x,t)=w(x,t)+x\theta(t)$$

with the following boundary conditions:

$$EI\frac{\partial^2 y}{\partial x^2}\Big|_{x=0}+u(t)=0$$

$$w(0,t)=0$$

$$EI\frac{\partial^2 y}{\partial x^2}\Big|_{x=L}=0$$

$$EI\frac{\partial^3 y}{\partial x^3}\Big|_{x=L}=0$$

The solution of this PDE can be expressed as

$$y(x,t)=\sum_{i=0}^{\infty}\psi_i(x)q_i(t)$$

where $i=0$ corresponds to the rigid body mode with

$$\psi_0(x)=x \quad \text{and} \quad q_0(t)=\theta(t)$$

$\psi_i(x)$ with $i\geq 1$ are mode shapes of a pinned-free beam, which are expressed as

$$\psi_i(x)=\sin\alpha_i x+\frac{\sin\alpha_i L}{\sinh\alpha_i L}\sinh\alpha_i x$$

where α_i is the solution of the following transcendental equation:

$$\tan\alpha_i L=\tanh\alpha_i L$$

Natural frequencies ω_i of the pinned-free beam are related to α_i as follows:

$$\omega_i^2 = \frac{EI\alpha_i^4}{\rho}$$

Finally, it can be shown that

$$\frac{d^2 q_0}{dt^2} = \frac{u(t)}{I_\alpha}$$

$$\frac{d^2 q_i}{dt^2} + \omega_i^2 q_i = \frac{\psi_i'(0)}{\rho a_i} u(t); \quad i = 1, 2, \ldots, \infty$$

Design a full state feedback controller such that all the closed-loop poles are located at $-\sigma$, $-\sigma$, $-\sigma \pm j\omega_1$, and $-\sigma \pm j\omega_2$. Choose σ such that the damping ratio in each vibratory mode is higher than 0.1.

P3.5 Consider a linear system for which the transfer function is given as

$$\frac{y(s)}{u(s)} = \frac{s}{(s-1)(s+1)}$$

i. Construct a suitable state space realization and find the state feedback gain vector such that the closed-loop poles are located at –2 and –3.
ii. Consider the control law $u(t) = v(t) - k_1 x_1(t) - k_2 x_2(t)$, where $x_1(t)$ and $x_2(t)$ are states. Can a nonzero value of the steady state output be obtained? Explain your answer.
iii. Let $u(t) = v(t) - k_1 x_1(t) - k_2 x_2(t)$. Determine the zeros of the closed-loop transfer function.

P3.6 Consider the following system:

$$\frac{dx}{dt} = -x(t) + u(t) + w$$

where w is a scalar constant disturbance of *unknown* magnitude and $u(t)$ is the control input.

a. Let $u(t) = -k\, x(t)$. Find an expression for the steady state value of x and show that the steady state value of x cannot be zero for a finite value of k.
b. Develop a suitable state feedback control law such that $x(t) \to x_d \neq 0$ as $t \to \infty$ for finite values of state feedback gains. Justify your answer.

P3.7 Consider a system with the following transfer function matrix:

$$G(s) = [0 \quad 1]\begin{bmatrix} -\dfrac{s-2}{2} & \dfrac{s}{2} \\ s & 0 \end{bmatrix}^{-1}$$

i. Find a state space realization in controller form.
ii. Determine the state feedback gain matrix corresponding to the controller form state space realization such that the closed-loop eigenvalues are located at –1 and –2 and the closed-loop eigenvectors are $[1 \quad 0]^T$ and $[0 \quad 1]^T$.

P3.8 Consider the flexible tetrahedral truss structure (Appendix F) with collocated sensors and actuators. The controller is to be designed on the basis of the first four modes of vibration.

Design a full state feedback controller such that all the closed-loop poles are located at $-0.1\omega_i \pm j\omega_i$; $i = 1, 2, 3$, and 4. Find at least two state feedback gain matrices.

P3.9 Find the open-loop optimal control inputs $u_1(t)$ (and $u_2(t)$) for the system

$$\frac{dx_1}{dt} = x_2 + u_1$$

$$\frac{dx_2}{dt} = u_2(t)$$

which minimizes

$$I = \frac{1}{2}\int_0^2 [u_1^2(t) + u_2^2(t)]dt$$

Given: $x_1(0) = x_2(0) = 1$ and $x_1(2) = 0$.

P3.10 The system

$$\frac{dx}{dt} = -x(t) + u(t)$$

is to be transferred from $x(0) = 5$ to $x(4) = 0$ such that

$$I = \frac{1}{2}\int_0^4 u^2(t)dt$$

is minimized. Find the optimal control input $u(t)$.

P3.11 Consider the Example 3.9. It is desired to move the load shaft with inertia 1 kg-m^2 by 30° in 2 sec and minimize the work done by the DC motor. Furthermore, this load shaft must start from rest and end at rest. The parameters for the DC motor are as follows: k_b = 0.04297 V-sec/rad, R_a = 1.025 Ω, and k_i = 0.04297 N-m/A.

i. Set up a suitable objective function such that the optimal controller can be implemented in a closed-loop fashion. Find the optimal feedback gain vector.
ii. Find the open-loop optimal input voltage for the DC motor such that the constraints on final states are exactly satisfied. Compare the optimal control input to that for item i.

P3.12 Consider the following system:

$$\frac{dx}{dt} = x(t) + u(t)$$

It is desired to develop an optimal state feedback law such that the following objective function is minimized:

$$I = \frac{1}{2}\int_0^{t_f} [x^2(t) + u^2(t)]dt$$

a. For a finite time t_f, determine the equation for finding the optimal state feedback gain.
b. If $t_f \to \infty$, what is the optimal state feedback gain?

P3.13 Consider the second-order system

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -x_1 - 2x_2 + u(t)$$

and the performance index

$$I = 20x_1^2(5) + \int_0^5 x_1^2(t) + 2x_2^2(t) + x_1(t)x_2(t) + u^2(t)dt$$

a. Set up the Riccati differential equations with proper boundary conditions.

b. Solve the Riccati differential equation using the Matlab routine ode23 or ode45, and find the time-varying state feedback gain vector and the closed-loop response. Also, find the optimal value of I.

P3.14 Consider a plant consisting of a DC motor, the shaft of which has the angular velocity $\omega(t)$. The DC motor is driven by the input voltage $u(t)$. The dynamics of the deviation in the angular speed from its constant nominal value $\bar{\omega}$ is given by

$$\frac{d\omega}{dt} = -0.5(\omega(t) - \bar{\omega}) + (u(t) - \bar{u})$$

where $\bar{u}$ is the voltage corresponding to $\bar{\omega}$. A feedback control system is to be designed such that the following objective function is minimized:

$$I = \int_0^\infty [(\omega(t) - \bar{\omega})^2 + r(u(t) - \bar{u})^2]dt \ ; \quad r > 0$$

i. Find the optimal state feedback gain by solving the algebraic Riccati equation.
ii. Using the gain found in item i, find the response of the closed-loop system for a specified $\omega(0) - \bar{\omega}$.
iii. Assuming that $\omega(0) - \bar{\omega} = 1$ unit, determine the value of r such that the magnitude of optimal $u(t) - \bar{u}$ never exceeds 1 unit.

P3.15 An unstable spring-mass-damper is described by the following differential equation:

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} + x = u(t)$$

where $u(t)$ is the control input. It is desired to design a state feedback control system such that

$$I = \int_0^\infty x^2 + ru^2 dt$$

is minimized. Assuming $r \to \infty$, find the optimal location of poles and the corresponding state feedback gain vector.

P3.16 A simple spring-mass-damper system is shown in Figure P3.16a. Because of some external disturbances, the initial displacement $x(0)$ and velocity $\frac{dx}{dt}(0)$ are not equal to zero. It is desired to introduce additional spring (stiffness = k_1) and viscous damper (damping coefficient = c_1) such that the following objective function is minimized:

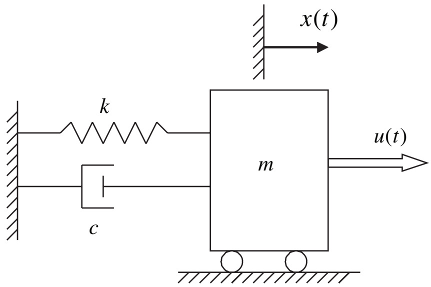

FIGURE P3.16A A spring-mass system.

$$I = \int_0^\infty x^2 + ru^2 dt \; ; \quad r \to 0$$

where $u(t)$ is the force acting on the unit mass because of additional spring and viscous damper (Figure P3.16b). Calculate the optimal values of k_1 and c_1.

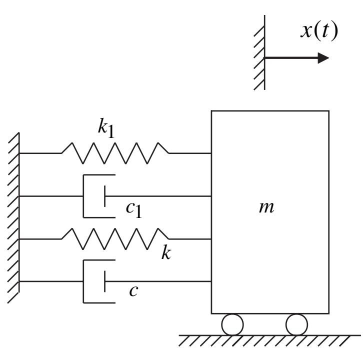

FIGURE P3.16B Additional spring and damper.

P3.17 Consider a plant with the following transfer function:

$$\frac{y(s)}{u(s)} = \frac{(s+1)(s-1)}{(s^2 - s + 1)(s+2)}$$

It is desired to develop a state feedback law such that the following objective function is minimized:

$$I = \frac{1}{2}\int_0^\infty y^2(t) + ru^2(t)dt \quad \text{where } r > 0$$

a. Draw the symmetric root locus plot.
b. Find the optimal closed-loop poles when $r \to 0$.
c. Find the optimal closed-loop poles when $r \to \infty$.

P3.18 Consider the system shown in Figure 5.11.4 in which the torque $T(t)$ is the control input. Design a full state feedback controller such that the following objective function is minimized:

$$I = \frac{1}{2}\int_0^\infty \theta_1^2 + rT^2 dt \quad \text{where } r > 0$$

Construct the symmetric root locus plot to determine optimal closed-loop pole locations. Show the performance of your controller as r changes from 0 to ∞.

Parameters (ECP Manual):

$$J_1 = 0.0024 \text{ kg-m}^2,\ J_2 = 0.0019 \text{ kg-m}^2,\ J_3 = 0.0019 \text{ kg-m}^2$$

$$k_1 = k_2 = 2.8 \text{ N-m/rad}$$

$$c_1 = 0.007 \text{ N-m/rad/sec},\ c_2 = c_3 = 0.001 \text{ N-m/rad/sec}$$

$$k_{hw} = \frac{100}{6} \text{ units}$$

P3.19 Refer to Example 3.11. Select $q_1 = 10$ and $q_2 = 1$.

a. Find optimal state feedback gain for three different values of ρ: 10^{-7}, 10^{-8}, and 10^{-9}. In each case, assume that the initial condition on state vector is $-[1 \quad 1 \quad 0 \quad 0]^T$ and plots $x_1(t)$, $x_2(t)$, and $u(t)$ vs. time. Discuss your results.
b. Refer to Equation 3.5.88 and Figure 3.10. For all the three aforementioned values of ρ, find λ_2 and β_2. Also, plot $f(t)$ vs. time for initial state vector chosen in part a. Discuss your results.

P3.20 The following model is developed for longitudinal pressure oscillation in a uniform chamber (Yang et al., 1992):

$$\ddot{\eta}_i + \omega_i^2\eta_i + \sum_{\ell=1}^{n}(D_{i\ell}\dot{\eta}_\ell + E_{i\ell}\eta_\ell) = v_i(t)\ ;\ i = 1, 2, \ldots, n$$

$$v_i(t) = \frac{2\bar{a}^2}{\alpha} \sum_{\ell=1}^{m} \psi_i(z_{a\ell}) \frac{\bar{u}_\ell(t)}{\bar{p}}$$

where $\bar{p}$ and $\bar{u}_\ell(t)$ are the mean pressure and the pressure excitation supplied by the actuator# ℓ located at $z_{a\ell}$, respectively. The normal mode# i of the chamber is described by

$$\psi_i(z) = \cos(i\pi z/\alpha); \quad 0 \le z \le \alpha$$

where α is the length of the chamber.

$D_{i\ell}$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$i = 1$	–0.01	0.007	–0.001	0.007
$i = 2$	0.01	0.1	0.007	–0.001
$i = 3$	–0.01	0.01	0.75	0.008
$i = 4$	0.02	–0.005	0.01	1.50

$E_{i\ell}$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$i = 1$	–0.005	–0.005	0.0025	0.0016
$i = 2$	–0.0025	–0.015	0.01	0.01
$i = 3$	–0.005	0.0	–0.02	0.02
$i = 4$	0.01	0.02	0.02	–0.025

a. Develop a state space realization and find the open-loop poles with $n = 4$.
b. Select a suitable location of a single actuator with $n = 4$, and find the state feedback gain vector to locate the poles such that closed-loop frequencies are same as those of the open loop, and there is at least a 5% damping ratio in each mode.
c. With the location of the actuator in part b, draw the symmetric root locus for the following objective function:

$$\int_0^{\infty} \left(\sum_{i=1}^{n} \eta_i \right)^2 + ru_1^2 dt$$

d. Select suitable locations for two actuators, and find a state feedback gain matrix to locate the poles, such that closed-loop frequencies are same as those of open loop, and there is at least a 5% damping ratio in each mode.
e. With the location of the actuator in part b, draw the symmetric root locus for the following objective function:

$$\int_0^\infty \left[\left(\sum_{i=1}^{n} \eta_i \right)^2 + u_1^2 + u_2^2 \right] dt$$

P3.21 Consider the flexible tetrahedral truss structure (Appendix F). The controller is to be designed on the basis of first four modes of vibration. Design a full state feedback LQR controller such that

$$\frac{1}{2} \int_0^\infty \mathbf{y}^T(t)\mathbf{y}(t) + \rho \mathbf{u}^T(t)\mathbf{u}(t) dt$$

is minimized. Demonstrate your controller performance for two values of $\rho =$ 0.1 and 1.

P3.22 Consider the single-link flexible manipulator described in Problem P3.4. Consider two vibratory modes for the link. It is given that $u < 1$ N-m. If the link is to be rotated by 30°, find the switching instants and move time for the bang-bang (minimum-time) control. The link starts from the rest and there should not be any vibration when the arm reaches its final position.