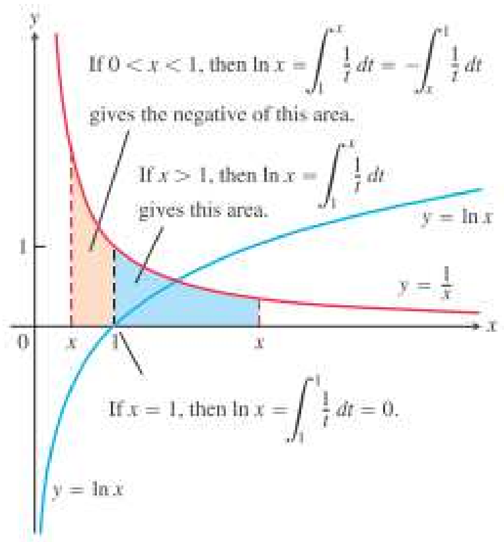
Definition of the Natural Logarithm Function

The natural logarithm of a positive number x, written as ln x, is defined as an integral.

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

From the Zero Width Interval Rule for definite integrals, we also have:

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$$



x	ln x
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

The Derivative of $y = \ln x$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

For every positive value of x, we have:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

If u is a differentiable function of x whose values are positive, so that ln u is defined,then applying the Chain Rule we obtain:

$$\frac{d}{dx}\ln u = \frac{1}{u}.\frac{du}{dx}$$

Example 1: Find derivatives of the following functions:

a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}(2) = \frac{1}{x}$

Notice from Example 2a that the function $y = \ln 2x$ has the same derivative as the function $y = \ln x$.

b) $\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}(2x) = \frac{2x}{x^2+3}$

c) $\frac{d}{dx}\ln|x|,\ x \neq 0$

$$\frac{1}{|x|}.\frac{x}{|x|} = \frac{x}{x^2} = \frac{1}{x}$$

Notice that the derivative of $|x| = 1 = \frac{x}{|x|}$

Algebraic Properties of the Natural Logarithm

For any numbers b > 0 and x > 0, the natural logarithm satisfies the following rules:

1. Product Rule: $\ln bx = \ln b + \ln x$
2. Quotient Rule: $\ln\frac{b}{x} = \ln b - \ln x$
3. Reciprocal Rule: $\ln\frac{1}{x} = -\ln x$
4. Power Rule: $\ln x^r = r\ln x$ For r rational

Example 2: Interpreting the properties of Logarithms

a) $\ln 6 = \ln(2 * 3) = \ln 2 + \ln 3$

b) $\ln 4 - \ln 5 = \ln\frac{4}{5} = \ln 0.8$

c) $\ln\frac{1}{8} = -\ln 8 = -\ln 2^3 = -3\ln 2$

Example 3: Applying the properties of Logarithms to function formulas

a) $\ln 4 + \ln(\sin x) = \ln(4\sin x)$

b) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$

c) $\ln(\sec x) = \ln\frac{1}{\cos x} = -\ln\cos x$

d) $\ln\sqrt[3]{x+1} = \ln(x+1)^{1/3} = \frac{1}{3}\ln(x+1)$

The Integral $\int(1/u)du$

If u is a differentiable function that is never zero, then:

$$\int \frac{1}{u}du = \ln|u| + C$$

Whenever $u = f(x)$ is a differentiable function that is never zero, we have that $du = f'(x)\,dx$ and:

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Example4: Calculate the following integral:

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}d\theta$$

Let: $u = 3 + 2\sin\theta\,, du = 2\cos\theta$

Therefore,

$$\int_1^5 \frac{2}{u}du = 2\ln|u|]_1^5 = 2\ln|5| - 2\ln|1| = 2\ln 5$$

The Integrals of $\tan x, \cot x, \sec x,$ and $\csc x$

To integrate these trigonometric functions,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C = -\ln|\csc x| + C$$

To integrate sec x, we multiply and divide by (sec x + tan x) as an algebraic form of 1.

$$\int \sec x \; dx = \int \sec x . \frac{\sec x + \tan x}{\sec x + \tan x} \; dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} = \ln|\sec x + \tan x| + C$$

For csc x, we multiply and divide by (csc x + cot x).

$$\int \csc x \; dx = \int \csc x . \frac{\csc x + \cot x}{\csc x + \cot x} \; dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} = -\ln|\csc x + \cot x| + C$$

Example 5: Calculate the following integral:

$$\int_0^{\pi/6} \tan 2x \, dx = \int_0^{\pi/3} \tan u \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u \, du = \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Example 6: Using the Logarithmic Differentiation, find *dy/dx* if:

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1$$

Solution: the Natural logarithm of both sides is taken:

$$\ln y = \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$\ln y = \ln(x^2+1)(x+3)^{1/2} - \ln(x-1)$$

$$\ln y = \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1)$$

$$\ln y = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$$

The derivatives of both sides are taken with respect to *x*:

$$\frac{1}{y} . \frac{dy}{dx} = \frac{1}{x^2+1} . 2x + \frac{1}{2} . \frac{1}{x+3} - \frac{1}{x-1}$$

Then, solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

Finally, substitute the function of *y*:

$$\frac{dy}{dx}=\frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1}+\frac{1}{2x+6}-\frac{1}{x-1}\right)$$

Example 7: Using the Logarithmic Differentiation, find *dy/dx* if:

$$y=\ln x.\frac{1}{x}.\sqrt{x}$$

Solution: the Natural logarithm of both sides is taken:

$$\ln y=\ln\left(\ln x.\frac{1}{x}.\sqrt{x}\right)$$

$$\ln y=\ln(\ln x)+\ln\frac{1}{x}+\ln\sqrt{x}$$

$$\ln y=\ln(\ln x)+\ln\frac{1}{x}+\frac{1}{2}\ln x$$

$$\frac{1}{y}.\frac{dy}{dx}=\frac{\frac{1}{x}}{\ln x}+\frac{-\frac{1}{x^2}}{\frac{1}{x}}+\frac{1}{2}.\frac{1}{x}$$

$$\frac{1}{y}.\frac{dy}{dx}=\frac{1}{x.\ln x}-\frac{1}{x}+\frac{1}{2x}$$

$$\frac{dy}{dx}=y.\left(\frac{1}{x.\ln x}-\frac{1}{2x}\right)$$

Finally, substitute the function of *y*:

$$\frac{dy}{dx}=\left(\ln x.\frac{1}{x}.\sqrt{x}\right).\left(\frac{1}{x.\ln x}-\frac{1}{2x}\right)$$

Homework

1. Use the properties of the natural logarithms to simplify the following expressions:
   a) $\ln\sin\theta-\ln\left(\frac{\sin\theta}{5}\right)$
   b) $\ln(3x^2-9x)+\ln\left(\frac{1}{3x}\right)$
   c) $\frac{1}{2}\ln(4t^4)-\ln 2$
2. Find the first derivative of the following functions with respect to *x*:

a) $y = \ln x^3$

b) $y = \ln\left(\frac{10}{x}\right)$

c) $y = \ln(\ln(\ln x))$

3. Evaluate the following integrals:

a) $\int_{-1}^{0} \frac{3\,dx}{3x-2}$

b) $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x+\tan x)}}$