Al-Mustansiriya University
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M.SC. Lecture No. 2

Numerical Analysis

Eigen Value Problems.

Iteration method (Jacobi method).

Householder (tridiagonal) Technique.

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2. Eigen Value Problems:

2.1 Iteration method (Jacobi method):

This method permits the transformation of [A] into a diagonal one having the same e-values as the original [A] matrix. This is accomplished by eliminating each off-diagonal element.

Consider $[A][X]=\lambda[X]$ Then [A] will have e-values $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ and the corresponding e-vectors are $[v_1],[v_2],\ldots\ldots,[v_n]$

$$[A][v_1]=\lambda_1\ [v_1]$$
$$[A][v_2]=\lambda_2\ [v_2]$$
$$\vdots$$
$$\vdots$$
$$[A][v_n]=\lambda_n\ [v_n]$$

Or in compact form:

Where $[A][V]=[V][\lambda]\ [V]=[v_1][v_2]\ldots\ldots[v_n]$

And
$$[\lambda]=\begin{bmatrix}\lambda_1 & 0 & \ldots\ldots & 0\\ 0 & \lambda_2 & \ldots\ldots & 0\\ & & & \\ 0 & 0 & \ldots\ldots & \lambda_n\end{bmatrix}$$

Now since [A] is symmetric then their e-vectors are

Hence: $[V^T]=[V^{-1}]$ orthogonal

$$[V^T][A][V]=[V^T][V][\lambda]=[\lambda]$$

(e-vector is normalized).the basic computational scheme is to reduce the matrix [A] into diagonal using real plane rotation matrices as follows:

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$$[A_1]=[A]$$
$$[A_2]=[p_1]^T[A_1][p_1]$$
$$[A_3]=[p_2]^T[A_2][p_2]=[p_2]^T[p_1]^T[A][p_1][p_2]$$
$$\vdots$$
$$[A_k]=[p_{k-1}]^T\ldots\ldots[p_1]^T[A][p_1]\ldots\ldots\ldots[p_{k-1}]$$

$as \quad k\to\infty$ then $[A_k]\to[\lambda]$ and $[p_1]\ldots[p_{k-1}]\to[V]$

Is a plane rotation matrix constructed such $[p_k]$ the matrix that The **off diagonal** in $[A_k]$ Is **reduced** to **zero**.

Coefficient is to be reduced to zero then: a_{rs} hence if

$$[p_k]=\begin{bmatrix} 1 & & & & 0 \\ 0 & \ldots. & & \ldots & 0 \\ & & \cos\vartheta & -\sin\vartheta & \\ & & \sin\vartheta & \cos\vartheta & \\ 0 & \ldots. & & & 1 \end{bmatrix}\begin{matrix} \\ \\ r \\ s \\ \\ \end{matrix}$$

(r, s)

is an orthogonal having $[p_k]$ hence

$$p_{rr}=\cos\vartheta=p_{ss} \quad \text{and} \quad p_{sr}=-p_{sr}=\sin\vartheta$$
$$p_{ii}=1 \quad \text{for} \quad i\neq r,s \quad \text{and} \quad p_{ij}=0 \quad \text{else where}$$

The value of ϑ is chosen such that a_{rs} in $[A_k]$ is zero, hence:

$$\tan 2\vartheta=\frac{2a_{rs}}{a_{rr}-a_{ss}}$$

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Ex: show that above eq. of ϑ is true for 2x2 symmetric matrix

$$[A]=\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}=[A_1]$$

The **off diagonal** element is a_{12} **hence** r =1 , s =2 and:

$$[p_1]=\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$ And :

$$[A_2]=[p_1]^T[A_1][p_1]$$

$$\begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix}=\begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

Our task is to set $a'_{12}=0$,hence carry out multiplication ,simplify you get $\tan 2\vartheta=\dfrac{2a_{12}}{a_{11}-a_{22}}$

Ex: Use iteration method to approximate the e-values for:

$$[A]=\begin{bmatrix} 8 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

Solution: $[A_1]=[A]$ we begin **reducing** $a_{12}=1$ to zero **.thus** $\tan 2\vartheta=\dfrac{2(1)}{8-5}=\dfrac{2}{3}$, $\vartheta=16.845^\circ$

$$[p_1]=\begin{bmatrix} 0.957092 & -0.289784 & 0 \\ 0.289784 & 0.957092 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$[A_2]=[p_1]^T[A_1][p_1]$$ **Gives**

$$=\begin{bmatrix} 8.3027 & 0 & 1.53666 \\ 0 & 4.69722 & 1.6244 \\ 1.53666 & 1.6244 & 5 \end{bmatrix}$$

Next reduce $a_{13}=1.53666$ to **zero** by

$$\tan 2\vartheta=\frac{2(1.53666)}{8.3027-5}\ ,\ \vartheta=21.469^{\circ} \qquad r=1,s=3$$

$$[A_3]=[p_2]^T[A_2][p_2]$$ Gives

$$=\begin{bmatrix} 8.89 & 0.594 & 0 \\ 0.594 & 4.69 & 1.511 \\ 0 & 1.511 & 4.39 \end{bmatrix}$$

Note that as a result of reducing a_{13} to **zero** a_{12} becomes 0.594 in stead of zero. This is the weak point of **Jacobi method**. Complete this sweep by **reducing** $a_{23}=1.511$ to **zero** to get

$$[A_4]=\begin{bmatrix} 8.89=\lambda_1 & 0.44 & -0.39 \\ 0.44 & 6.05=\lambda_2 & 0 \\ -0.39 & 0 & 3.001=\lambda_3 \end{bmatrix}$$

And so on to reduce the off-diagonal to zero in next step.

2.2 Householder (tridiagonal) Technique :

This permits the transformation into a **tridiagonal form** having the same **e-values** .this is more efficient than previous **Jacobi method**, since it required a finite no. of iteration. The tridiagonal form can be used to construct a sequence of polynomials that can be solved for e-values. The

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Householder method requires the construction of reflection matrices and **(n-2)** transformations of the form:

$$[A_1]=[\mathrm{A}]$$
$$[\mathrm{A}_2]=[p_1]^T[A_1][p_1]$$
.
.
$$[\mathrm{A}_{n-2}]=[p_{n-3}]^T[A_{n-3}][p_{n-3}]$$

Where $[p_k]$ matrices are the **Householder** transformation matrices and are construct as:

$$[p_k]=[I]-s_k[w_k][w_k]^T$$

$$\mathrm{s_k}=\frac{2}{[w_k]^T[w_k]} \qquad \text{and the}$$

Coefficients of the vector $[w_k]$

$$w_{ik}=\begin{cases}0 & \text{for } \mathrm{i}=1,2,\ldots,\mathrm{k}\\ \mathrm{a_{ik}} & \text{for } \mathrm{i}=\mathrm{k}+2,\ldots,\mathrm{n}\end{cases}$$

$$w_{k+1,k}=a_{k+1,k}\pm\sqrt{\sum_{i=k+1}^{n}a_{ik}^2}$$

Where the **sign** preceding the square root is taken to be the **same** $a_{k+1,k}$ as that for the coefficient

Once the **tridiagonal** $[T]$ of the form:

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$$[T]=\begin{bmatrix} A_1 & B_2 & 0 & \cdots & 0 \\ B_2 & A_2 & B_3 & \cdots & 0 \\ 0 & B_3 & A_3 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & \vdots & & B_n \\ 0 & 0 & 0 & \cdots & B_n \quad A_n \end{bmatrix}$$

Then

$$\det[T-\lambda I]=\det\begin{bmatrix} A_1-\lambda & B_2 & 0 & \ldots & 0 \\ B_2 & A_2-\lambda & B_3 & & 0 \\ 0 & B_3 & A_3-\lambda & & 0 \\ 0 & 0 & & & \\ & & & & B_n \\ 0 & 0 & & B_n & A_n-\lambda \end{bmatrix}=0$$

This determinant is given by the following polynomial:

$f_0(\lambda)=1$

$f_1(\lambda)=A_1-\lambda$

.

.

$f_m(\lambda)=(A_m-\lambda)f_{m-1}(\lambda)-B_m^2 f_{m-2}(\lambda)$

Ex: Reduce the matrix:

$$[A]=\begin{bmatrix} 30 & 6 & 5 \\ 6 & 30 & 9 \\ 5 & 9 & 30 \end{bmatrix}$$

Into **tridiagonal form** using **Householder technique**, then approximate **e-values**.

Solution:

$$[A_1]=[A]=\begin{bmatrix} 30 & 6 & 5 \\ 6 & 30 & 9 \\ 5 & 9 & 30 \end{bmatrix}$$

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Consequently, for n =3, k =1, we construct $[W_1]$ vector as follows:

$$W_{11} = 0$$

$$W_{31} = a_{31} = 5$$

$$W_{21} = a_{21} \pm \sqrt{a_{21}^2 + a_{31}^2} = 6 + \sqrt{6^2 + 5^2} = 13.81$$

$$[W_1] = \begin{bmatrix} w_{11} \\ w_{21} \\ w_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 13.81 \\ 5 \end{bmatrix}, \; s_1 = \frac{2}{0 + 13.81^2 + 5^2} = 0.00927$$

$$[p_1] = [I] - s_1 [W_1][W_1]^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 0.00927 \begin{bmatrix} 0 \\ 13.81 \\ 5 \end{bmatrix} \begin{bmatrix} 0 & 13.81 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.7680 & -0.6402 \\ 0 & -0.6402 & -0.7680 \end{bmatrix}$$

$$[A_2] = [p_1]^T [A_1][p_1]$$

$$= \begin{bmatrix} 30 & -7.81 & 0 \\ -7.81 & 38.85 & -1.622 \\ 0 & -1.622 & 21.15 \end{bmatrix} = [T]$$

This is the **tridiagonal** form, next

$$|T - \lambda I| = \begin{vmatrix} 30 - \lambda & -7.81 & 0 \\ -7.81 & 38.85 - \lambda & -1.622 \\ 0 & -1.622 & 21.15 - \lambda \end{vmatrix} = 0$$

$$f_0(\lambda) = 1$$

$$f_1(\lambda) = 30 - \lambda$$

$$f_2(\lambda) = (38.85 - \lambda) f_1(\lambda) - (-7.81)^2 f_0(\lambda)$$

$$or \quad = \lambda^2 - 68.85\lambda + 1104.5 = 0$$

$$\lambda_1 = 43.4 \quad , \quad \lambda_2 = 25.45$$

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For m = 3 then:

$$f_3(\lambda) = (21.1 - \lambda) f_2(\lambda) - (-1.622)^2 f_1(\lambda)$$

$$= (21.15 - \lambda)[\lambda^2 - 68.85\lambda + 1104.5] - 2.63(30 - \lambda)$$

or $\lambda_3 = 21.15$

Ex: Tridiagonalize the real **symmetric matrix**:

$$[A] = \begin{bmatrix} 6 & 4 & 1 & 1 \\ 4 & 6 & 1 & 1 \\ 1 & 1 & 5 & 2 \\ 1 & 1 & 2 & 5 \end{bmatrix}$$

.

Solution:

For n =4, k =1

$$W_{11} = 0$$

$$W_{31} = a_{31} = 1$$

$$W_{41} = a_{41} = 1$$

$$W_{21} = a_{21} \pm \sqrt{a_{21}^2 + a_{31}^2 + a_{41}^2} = 8.24264$$

.

$$[W_1] = \begin{bmatrix} w_{11} \\ w_{21} \\ w_{31} \\ w_{41} \end{bmatrix} = \begin{bmatrix} 0 \\ 8.24264 \\ 1 \\ 1 \end{bmatrix}, \quad s_1 = \frac{2}{0 + 8.24^2 + 1^2 + 1^2} = 0.028595$$

$$[p_1] = [I] - s_1 [W_1][W_1]^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 0.028595 \begin{bmatrix} 0 \\ 8.242 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 8.242 & 1 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.9428 & -0.2357 & -0.0285 \\ 0 & -0.2357 & -0.09428 & -0.0285 \\ 0 & -0.02357 & -0.0285 & 0.971 \end{bmatrix}$$

And $[A_2] = [p_1]^T [A_1][p_1]$

$$[A_2] = \begin{bmatrix} 6 & -\sqrt{18} & 0 & 0 \\ -\sqrt{18} & 7 & -1 & -1 \\ 0 & -1 & 4.5 & 1.5 \\ 0 & -1 & 1.5 & 4.5 \end{bmatrix} \text{(Check)}$$

Next for n = 4, k = 2

$W_{12} = 0$

$W_{22} = 0$

$W_{42} = a_{42} = -1$

$W_{32} = a_{32} \pm \sqrt{a_{32}^2 + a_{42}^2} = -1 - \sqrt{(-1)^2 + (-1)^2} = -2.414$

$$[W_2] = \begin{bmatrix} w_{12} \\ w_{22} \\ w_{32} \\ w_{42} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2.414 \\ -1 \end{bmatrix}, \quad s_2 = \frac{2}{1^2 + (-2.414)^2} = 0.29293$$

$$[p_2] = [I] - s_2 [W_2][W_2]^T$$

$$[p_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

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And $[A_3]=[p_2]^T[A_2][p_2]$

$$[A_3]=\begin{bmatrix} 6 & -\sqrt{18} & 0 & 0 \\ -\sqrt{18} & 7 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 6 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}=[T]$$

Which is tridiagonal matrix, we need **(n-2) = 4-2 = 2** steps to accomplish this reduction.

Note: it is possible to use the **Jacobi iteration** on the reduced **tridiagonal** matrix and the process is stopped up by noticing that once the coefficient is reduced to zero, it remains so in subsequent iterations.

Ex: find **e-values** of the previous example?

Let [B] = tridiagonal = $[B_1]$, start with reducing

$a_{12}=-\sqrt{18}$ to zero.

$$\tan 2\vartheta=\frac{2(-\sqrt{18})}{6-7}\Rightarrow \vartheta=41.639^{o}$$

$$[p_1]=\begin{bmatrix} 0.7473 & -0.6644 & 0 & 0 \\ 0.6644 & 0.7473 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B_2]=[p_1]^T[B_1][p_1]$$

$$=\begin{bmatrix} 2.196 & 0 & 0.94 & 0 \\ 0 & 10.77 & 1.056 & 0 \\ 0 & 1.056 & 6 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Continue the reduction. It is clear that the **e-values** are **2, 11, 6&3** respectively

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