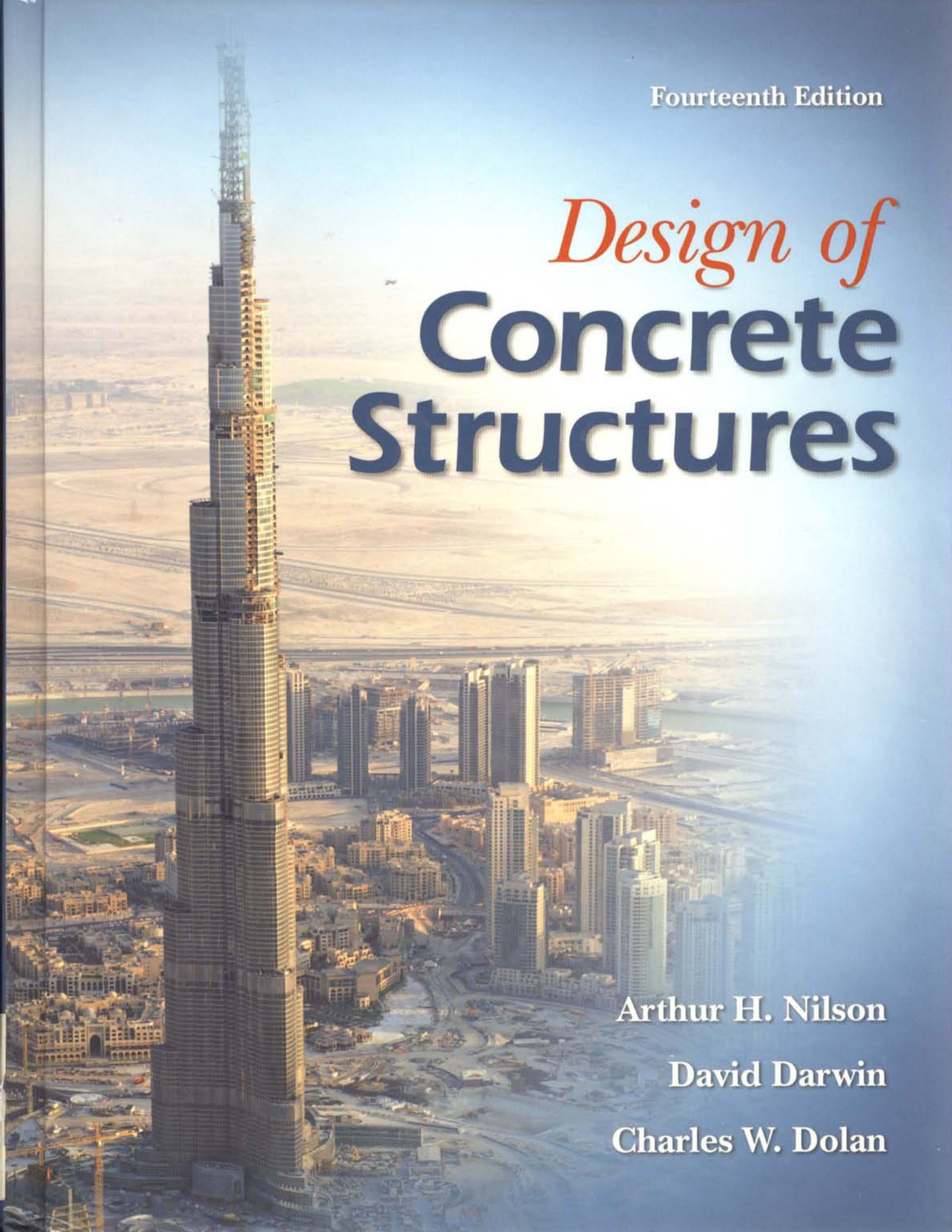

Fourteenth Edition

Design of Concrete Structures

Arthur H. Nilson

David Darwin

Charles W. Dolan

DESIGN of CONCRETE STRUCTURES

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DESIGN OF CONCRETE STRUCTURES, FOURTEENTH EDITION

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About the Authors

Arthur H. Nilson was engaged in research, teaching, and consulting relating to structural concrete for over 40 years. He has been a member of the faculty of the College of Engineering at Cornell University since 1956, in charge of undergraduate and graduate courses in the design of reinforced concrete and prestressed concrete structures until his retirement in 1991. He served as Chairman of the Department of Structural Engineering from 1978 to 1985. Dr. Nilson has served on many professional committees, including Building Code Subcommittee 318-D of the American Concrete Institute (ACI). His pioneering work on high-strength concrete has been widely recognized. He was awarded the ACI Wason Medal for materials research in 1974, the ACI Wason Medal for best technical paper in 1986 and 1987, and the ACI Structural Research Award in 1993. Professor Nilson is an Honorary Member of ACI and a Fellow in the American Society of Civil Engineers (ASCE). He has been honored by the civil engineering student body at Cornell for outstanding teaching. He was elected Professor Emeritus in 1991. He has held research appointments or lectureships at the University of Manchester, Salford University, and the Technical University of Milan. He is a registered professional engineer in several states and, prior to entering teaching, was engaged in full-time professional practice. He received the B.S. degree from Stanford University in 1948, the M.S. from Cornell in 1956, and the Ph.D. from the University of California at Berkeley in 1967.

David Darwin has been a member of the faculty at the University of Kansas since 1974 and has been director of the Structural Engineering and Materials Laboratory since 1982. He was appointed the Deane E. Ackers Distinguished Professor of Civil Engineering in 1990. Dr. Darwin served as President of the American Concrete Institute in 2007–2008 and is a member and past chair of ACI Committees 224 on Cracking and 408 on Bond and Development of Reinforcement. He is also a member of ACI Building Code Subcommittee 318-B on Reinforcement and Development and of ACI-ASCE Committee 445 on Shear and Torsion. Dr. Darwin is an acknowledged expert on concrete crack control and bond between steel reinforcement and concrete. He received the ACI Arthur R. Anderson Award in 1992 for his research efforts in plain and reinforced concrete, the ACI Structural Research Award in 1996, and the ACI Joe W. Kelly Award in 2005 for his contributions to teaching and design. He has also received a number of awards from the American Society of Civil Engineers, including the Walter L. Huber Civil Engineering Research Prize in 1985; the Moisseiff Award in 1991; the State-of-the-Art of Civil Engineering Award in 1996 and 2000; the Richard R. Torrens Award in 1997; and the Dennis L. Tewksbury Award in 2008. He has been honored for his teaching by the civil engineering students at the University of Kansas. He is past editor of the ASCE *Journal of Structural Engineering*. Professor Darwin is a Fellow of ACI and ASCE. He is a licensed professional engineer and serves as a consultant in the fields of concrete materials and structures. He was honored with the Distinguished Alumnus Award from the University of Illinois Civil and Environmental Engineering Alumni Association in 2003. Between his M.S. and Ph.D. degrees, he served four years with the U.S. Army Corps of Engineers. He received the B.S. and M.S. degrees from Cornell University in 1967 and 1968 and the Ph.D. from the University of Illinois at Urbana-Champaign in 1974.

Charles W. Dolan has been on the faculty at the University of Wyoming since 1991, serving as Department Head from 1998 to 2001. He was appointed the H. T. Person Professor of Engineering in 2002. He is currently chair of Building Code Subcommittee 318-R of the American Concrete Institute. He has served as chair of the Technical Activities Committee, of ACI Committee 358 on Transit Guideways, and of ACI-ASCE Committee 423 on Prestressed Concrete. In private design practice for nearly 20 years, he was the project engineer on the Walt Disney World Monorail, the Detroit Downtown Peoplemover guideway, and the Dallas–Fort Worth Airport transit system guideway and is responsible for the conceptual design of the Dubai Palm Island monorail. He received the T. Y. Lin Award from ASCE in 1973 for outstanding contributions to the field of prestressed concrete and the Arthur R. Anderson award from ACI in 2005 for advancements in the design of reinforced and prestressed concrete structures. A Fellow in ACI and the Prestressed Concrete Institute (PCI), he is an internationally recognized leader in the development of fiber reinforced polymers for concrete reinforcement. He is a registered professional engineer and a consultant in the design of structural concrete. He received the B.S. from the University of Massachusetts in 1965 and the M.S. and Ph.D. from Cornell University in 1967 and 1989.

Contents

Preface

The fourteenth edition of *Design of Concrete Structures* has the same dual objectives as the previous work: first to establish a firm understanding of the behavior of structural concrete, then to develop proficiency in the methods used in current design practice. It has been updated in accordance with the provisions of the 2008 American Concrete Institute (ACI) Building Code.

It is generally recognized that mere training in special design skills and codified procedures is inadequate for successful professional practice. As new research becomes available and new design methods are continually introduced, these procedures are subject to frequent changes. To understand and keep abreast of these rapid developments and to engage safely in innovative design, the engineer needs a thorough grounding in the basic performance of concrete and steel as structural materials, and in the behavior of reinforced concrete members and structures. On the other hand, the main business of the structural engineer is to design structures safely, economically, and efficiently. Consequently, with this basic understanding as a firm foundation, familiarity with current design procedures is essential. This edition, like the preceding ones, addresses both needs.

The text not only presents the basic mechanics of structural concrete and methods for the design of individual members for bending, shear, torsion, and axial forces, but also provides much detail pertaining to applications in the various types of structural systems, including an extensive presentation of slabs, footings, foundations, and retaining walls. The important topic of joint design is included. The chapter on flexural design has been expanded to improve the presentation of both the basic material and the example problems, coverage of seismic design is updated, and an introduction to prestressed concrete is included, as in previous editions.

There have been a number of significant changes in the 2008 ACI Building Code, which governs design practice in most of the United States and serves as a model code in many other countries as well. Among these are a reorganization of the provisions for both slender column and earthquake design, the former with some simplification compared to earlier Codes and the latter with some important additions; and the addition of headed studs for use as shear reinforcement in two-way slabs and headed deformed bars as another option for use in anchoring reinforcement.

In addition to changes in the ACI Code, the text includes the modified compression field theory method of shear design as updated in the 2008 Interim Revisions to the American Association of State Highway and Transportation Officials (AASHTO) *LRFD Bridge Design Specifications*.

A feature of the text is the comprehensive presentation of all aspects of slab design. A chapter covering one-way and two-way edge-supported and column-supported slabs, including the new Code material on headed studs, is followed by chapters on slab analysis and design based on the theory of plasticity covering, respectively, the yield line method for analysis and the strip method for design of slabs, both particularly useful for innovative structures.

A special strength of the text is the analysis chapter, which includes load combinations for use in design, a description of envelope curves for moment and shear, guidelines for proportioning members under both gravity and lateral loads, and procedures for developing preliminary designs of reinforced concrete structures.

Most present-day design is carried out using computer programs, either general-purpose, commercially available software or individual programs written for special needs. Step-by-step procedures are given throughout the book to guide the student and engineer through the increasingly complex methodology of current design, with the emphasis on understanding the design process. Once mastered, these procedures are easily converted into flowcharts to aid in programming. References are given, where appropriate, to the more widely used commercial programs.

The text will be found suitable for either a one or two-semester course in the design of concrete structures. If the curriculum permits only a single course (probably taught in the fourth undergraduate year), the following will provide a good basis: the introduction and treatment of materials found in Chapters 1 and 2, respectively; the material on flexure, shear, and anchorage in Chapters 3, 4, and 5; Chapter 6 on serviceability; Chapter 8 on short columns; and the introduction to one and two-way slabs found in the first four sections of Chapter 13. Time may or may not permit classroom coverage of frame analysis or building systems, Chapters 12 and 18, but these could well be assigned as independent reading, concurrent with the earlier work of the course. In the authors' experience, such complementary outside reading tends to enhance student motivation.

The text is more than adequate for a second course, most likely taught in the first year of graduate study. The authors have found that this is an excellent opportunity to provide students with a more general understanding of reinforced concrete structural design, often beginning with Chapters 12 and 18 and followed by the increasingly important topics of torsion, Chapter 7; slender columns, Chapter 9; the strut-and-tie method, Chapter 10; and the design and detailing of joints, Chapter 11. It should also offer an opportunity for a much expanded study of slabs, including the remaining sections of Chapter 13, plus the methods for slab analysis and design based on plasticity theory found in Chapters 14 and 15, yield line analysis and the strip method of design. Other topics appropriate to a second course include foundations and retaining walls, Chapters 16 and 17, and the introduction to seismic design in Chapter 20. Prestressed concrete is sufficiently important to justify a separate course. If time constraints do not permit this, Chapter 19 provides an introduction and can be used as the text for a one-credit-hour course.

At the end of each chapter, the user will find extensive reference lists, which provide an entry into the literature for those wishing to increase their knowledge through independent study. For professors, the instructor's solution manual is available online at www.mhhe.com/concrete.

A word must be said about units. In the United States, regrettably, the transition from U.S. Customary System units to the metric system has proceeded very slowly, and in many quarters not at all. This is in part because of the expense to the construction industry of the conversion, but perhaps also because of perceived shortcomings in the SI metric system (use of derived units such as the pascal, elimination of the convenient centimeter, etc.) compared with the traditional European metric system. Although most basic science courses are taught using SI units, in most upper-class and graduate design courses, inch-pound units are customarily used, reflecting conditions of practice here. Accordingly, inch-pound units are used throughout the text, although graphs and basic data in Chapter 2 are given in dual units. Appendix B gives the SI equivalents of inch-pound units. An SI version of the ACI Building Code is available.

A brief historical note may be of interest. This book is the fourteenth edition of a textbook originated in 1923 by Leonard C. Urquhart and Charles E. O'Rourke, both professors of structural engineering at Cornell University at that time. Over its remarkable 86-year history, new editions have kept pace with research, improved materials, and new methods of analysis and design. The second, third, and fourth editions firmly established the work as a leading text for elementary courses in the subject area. Professor George Winter, also of Cornell, collaborated with Urquhart in preparing the fifth and sixth editions. Winter and the present senior author were responsible for the seventh, eighth, and ninth editions, which substantially expanded both the scope and the depth of the presentation. The tenth, eleventh, and twelfth editions were prepared by Professor Nilson subsequent to Professor Winter's passing in 1982, the latter with Professor David Darwin of the University of Kansas serving as a contributor.

Professors Nilson and Darwin were joined by Professor Charles Dolan of the University of Wyoming beginning with the thirteenth edition. All three have been deeply involved in research and teaching in the fields of reinforced and prestressed concrete, as well as professional Code-writing committees, and have spent significant time in professional practice, invaluable in developing the perspective and structural judgement that sets this book apart.

Special thanks are due to reviewers and former students for their many helpful comments and suggestions for this and previous editions. In particular, the authors would like to thank the following reviewers: Paul Barr, Utah State University; Robert N. Emerson, Oklahoma State University; A. Fafitis, Arizona State University; R. Craig Henderson, Tennessee Technological University; Max Porter, Iowa State University; Pizhong Qiao, The University of Akron; Aziz Saber, Louisiana Tech University; and Eric Steinberg, Ohio University. Thanks are also due to the McGraw-Hill project team, notably Debra Hash, Sponsoring Editor; Lorraine Buczek, Developmental Editor; and Melissa Leick, Project Manager.

We gladly acknowledge our indebtedness to the original authors. Although it is safe to say that neither Urquhart nor O'Rourke would recognize very much of the detail, the approach to the subject and the educational philosophy that did so much to account for the success of the early editions would be familiar. We acknowledge with particular gratitude the influence of Professor George Winter in developing a point of view that has shaped the work in the chapters that follow.

Arthur H. Nilson
David Darwin
Charles W. Dolan

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1 Introduction

1.1 CONCRETE, REINFORCED CONCRETE, AND PRESTRESSED CONCRETE

Concrete is a stonelike material obtained by permitting a carefully proportioned mixture of cement, sand and gravel or other aggregate, and water to harden in forms of the shape and dimensions of the desired structure. The bulk of the material consists of fine and coarse aggregate. Cement and water interact chemically to bind the aggregate particles into a solid mass. Additional water, over and above that needed for this chemical reaction, is necessary to give the mixture the workability that enables it to fill the forms and surround the embedded reinforcing steel prior to hardening. Concretes with a wide range of properties can be obtained by appropriate adjustment of the proportions of the constituent materials. Special cements (such as high early strength cements), special aggregates (such as various lightweight or heavyweight aggregates), admixtures (such as plasticizers, air-entraining agents, silica fume, and fly ash), and special curing methods (such as steam-curing) permit an even wider variety of properties to be obtained.

These properties depend to a very substantial degree on the proportions of the mix, on the thoroughness with which the various constituents are intermixed, and on the conditions of humidity and temperature in which the mix is maintained from the moment it is placed in the forms until it is fully hardened. The process of controlling conditions after placement is known as *curing*. To protect against the unintentional production of substandard concrete, a high degree of skillful control and supervision is necessary throughout the process, from the proportioning by weight of the individual components, through mixing and placing, until the completion of curing.

The factors that make concrete a universal building material are so pronounced that it has been used, in more primitive kinds and ways than at present, for thousands of years, starting with lime mortars from 12,000 to 6000 BCE in Crete, Cyprus, Greece, and the Middle East. The facility with which, while plastic, it can be deposited and made to fill forms or molds of almost any practical shape is one of these factors. Its high fire and weather resistance is an evident advantage. Most of the constituent materials, with the exception of cement and additives, are usually available at low cost locally or at small distances from the construction site. Its compressive strength, like that of natural stones, is high, which makes it suitable for members primarily subject to compression, such as columns and arches. On the other hand, again as in natural stones, it is a relatively brittle material whose tensile strength is small compared with its compressive strength. This prevents its economical use in structural members that are subject to tension either entirely (such as in tie-rods) or over part of their cross sections (such as in beams or other flexural members).

To offset this limitation, it was found possible, in the second half of the nineteenth century, to use steel with its high tensile strength to reinforce concrete, chiefly in those places where its low tensile strength would limit the carrying capacity of the member. The reinforcement, usually round steel rods with appropriate surface deformations to provide interlocking, is placed in the forms in advance of the concrete. When completely surrounded by the hardened concrete mass, it forms an integral part of the member. The resulting combination of two materials, known as *reinforced concrete,* combines many of the advantages of each: the relatively low cost, good weather and fire resistance, good compressive strength, and excellent formability of concrete and the high tensile strength and much greater ductility and toughness of steel. It is this combination that allows the almost unlimited range of uses and possibilities of reinforced concrete in the construction of buildings, bridges, dams, tanks, reservoirs, and a host of other structures.

In more recent times, it has been found possible to produce steels, at relatively low cost, whose yield strength is 3 to 4 times and more that of ordinary reinforcing steels. Likewise, it is possible to produce concrete 4 to 5 times as strong in compression as the more ordinary concretes. These high-strength materials offer many advantages, including smaller member cross sections, reduced dead load, and longer spans. However, there are limits to the strengths of the constituent materials beyond which certain problems arise. To be sure, the strength of such a member would increase roughly in proportion to those of the materials. However, the high strains that result from the high stresses that would otherwise be permissible would lead to large deformations and consequently large deflections of such members under ordinary loading conditions. Equally important, the large strains in such high-strength reinforcing steel would induce large cracks in the surrounding low tensile strength concrete, cracks that not only would be unsightly but also could significantly reduce the durability of the structure. This limits the useful yield strength of high-strength reinforcing steel to 80 ksi† according to many codes and specifications; 60 ksi steel is most commonly used.

A special way has been found, however, to use steels and concretes of very high strength in combination. This type of construction is known as *prestressed concrete.* The steel, in the form of wires, strands, or bars, is embedded in the concrete under high tension that is held in equilibrium by compressive stresses in the concrete after hardening. Because of this precompression, the concrete in a flexural member will crack on the tension side at a much larger load than when not so precompressed. Prestressing greatly reduces both the deflections and the tensile cracks at ordinary loads in such structures, and thereby enables these high-strength materials to be used effectively. Prestressed concrete has extended, to a very significant extent, the range of spans of structural concrete and the types of structures for which it is suited.

1.2 STRUCTURAL FORMS

The figures that follow show some of the principal structural forms of reinforced concrete. Pertinent design methods for many of them are discussed later in this volume.

Floor support systems for buildings include the monolithic slab-and-beam floor shown in Fig. 1.1, the one-way joist system of Fig. 1.2, and the flat plate floor, without beams or girders, shown in Fig. 1.3. The flat slab floor of Fig. 1.4, frequently used for more heavily loaded buildings such as warehouses, is similar to the flat plate floor, but makes use of increased slab thickness in the vicinity of the columns, as well

† Abbreviation for kips per square inch, or thousands of pounds per square inch.

FIGURE 1.1
One-way reinforced concrete floor slab with monolithic supporting beams. (*Portland Cement Association.*)

FIGURE 1.2
One-way joist floor system, with closely spaced ribs supported by monolithic concrete beams; transverse ribs provide for lateral distribution of localized loads. (*Portland Cement Association.*)

as flared column tops, to reduce stresses and increase strength in the support region. The choice among these and other systems for floors and roofs depends upon functional requirements, loads, spans, and permissible member depths, as well as on cost and esthetic factors.

Where long clear spans are required for roofs, concrete shells permit use of extremely thin surfaces, often thinner, relatively, than an eggshell. The folded plate roof of Fig. 1.5 is simple to form because it is composed of flat surfaces; such roofs have been employed for spans of 200 ft and more. The cylindrical shell of Fig. 1.6 is also relatively easy to form because it has only a single curvature; it is similar to the folded plate in its structural behavior and range of spans and loads. Shells of this type were once quite popular in the United States and remain popular in other parts of the world.

Doubly curved shell surfaces may be generated by simple mathematical curves such as circular arcs, parabolas, and hyperbolas, or they may be composed of complex combinations of shapes. The hyperbolic paraboloid shape, defined by a concave downward parabola moving along a concave upward parabolic path, has been widely

FIGURE 1.3
Flat plate floor slab, carried directly by columns without beams or girders. (*Portland Cement Association.*)

FIGURE 1.4
Flat slab floor, without beams but with slab thickness increased at the columns and with flared column tops to provide for local concentration of forces. (*University of Southern Maine.*)

used. It has the interesting property that the doubly curved surface contains two systems of straight-line generators, permitting straight-form lumber to be used. The complex dome of Fig. 1.7, which provides shelter for performing arts events, consists essentially of a circular dome but includes monolithic, upwardly curved edge surfaces to provide stiffening and strengthening in that critical region.

FIGURE 1.5
Folded plate roof of 125 ft span that, in addition to carrying ordinary roof loads, carries the second floor as well from a system of cable hangers; the ground floor is kept free of columns.

FIGURE 1.6
Cylindrical shell roof providing column-free interior space.

Bridge design has provided the opportunity for some of the most challenging and creative applications of structural engineering. The award-winning Napoleon Bonaparte Broward Bridge, shown in Fig. 1.8, is a six-lane, cable-stayed structure that spans St. John's River at Dame Point, Jacksonville, Florida. Its 1300 ft center span is the

FIGURE 1.7
Spherical shell in Lausanne, Switzerland. Upwardly curved edges provide stiffening for the central dome.

FIGURE 1.8
Napoleon Bonaparte Broward Bridge, with a 1300 ft center span at Dame Point, Jacksonville, Florida. *(HNTB Corporation, Kansas City, Missouri.)*

second longest of its type in the western hemisphere. Figure 1.9 shows the Bennett Bay Centennial Bridge, a four-span continuous, segmentally cast-in-place box girder structure. Special attention was given to esthetics in this award-winning design. The spectacular Natchez Trace Parkway Bridge in Fig. 1.10, a two-span arch structure using hollow precast concrete elements, carries a two-lane highway 155 ft above the valley

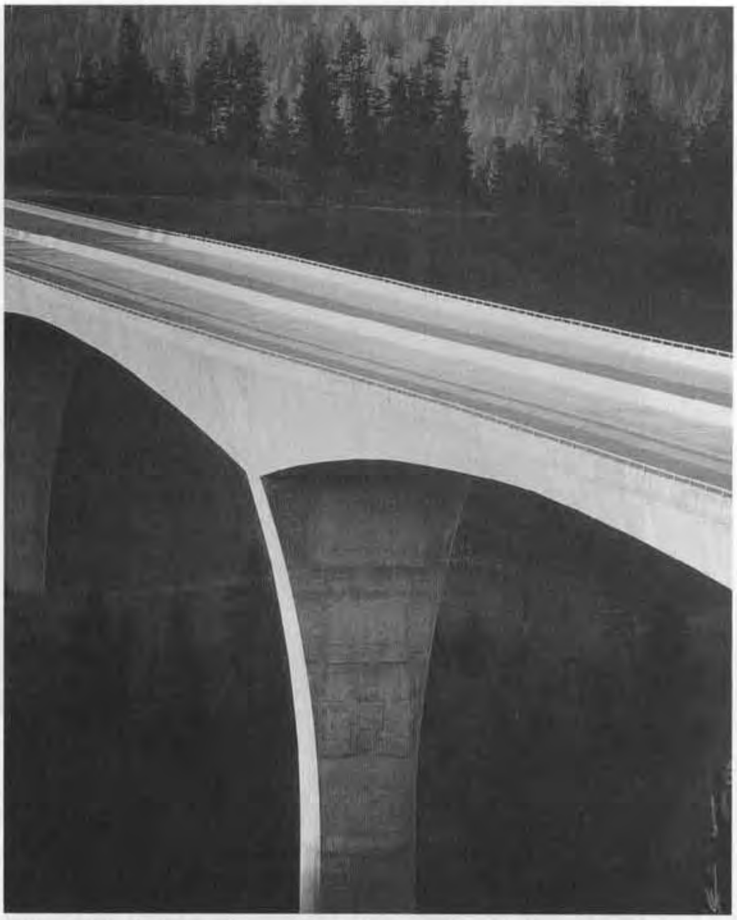

FIGURE 1.9
Bennett Bay Centennial Bridge, Coeur d'Alene, Idaho, a four-span continuous concrete box girder structure of length 1730 ft. (*HNTB Corporation, Kansas City, Missouri.*)

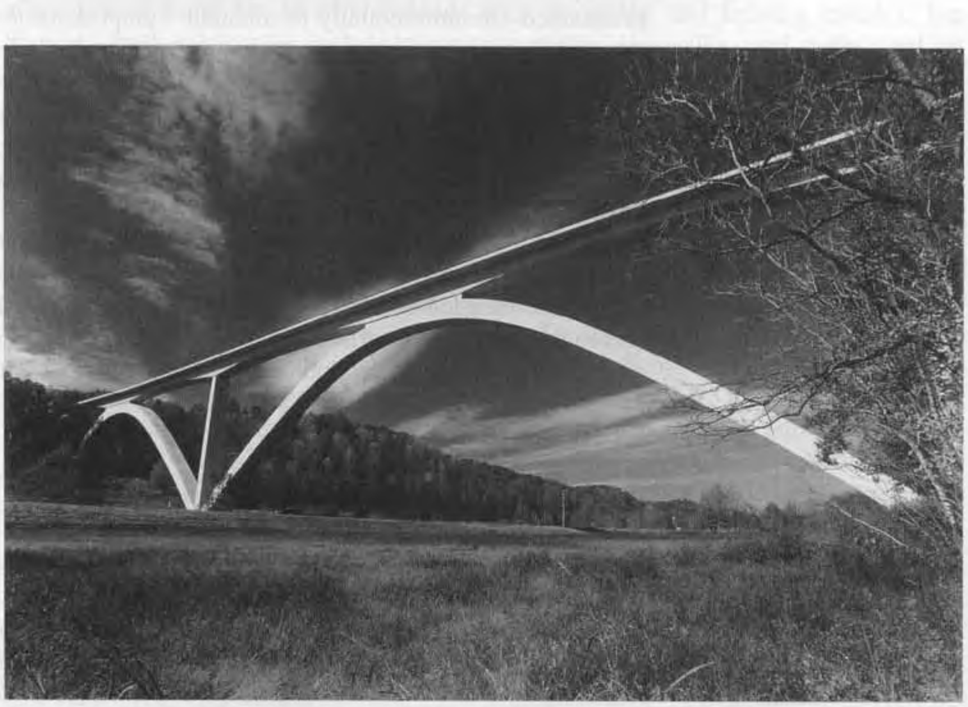

FIGURE 1.10
Natchez Trace Parkway Bridge near Franklin, Tennessee, an award-winning two-span concrete arch structure rising 155 ft above the valley floor. (*Figg Engineering Group, Tallahassee, Florida.*)

FIGURE 1.11
Circular concrete tanks used as a part of the wastewater purification facility at Howden, England. (*Northumbrian Water Authority with Luder and Jones, Architects.*)

floor. This structure has won many honors, including awards from the American Society of Civil Engineers and the National Endowment for the Arts.

Cylindrical concrete tanks are widely used for storage of water or in waste purification plants. The design shown in Fig. 1.11 is proof that a sanitary engineering facility can be esthetically pleasing as well as functional. Cylindrical tanks are often prestressed circumferentially to maintain compression in the concrete and eliminate the cracking that would otherwise result from internal pressure.

Concrete structures may be designed to provide a wide array of surface textures, colors, and structural forms. Figure 1.12 shows a precast concrete building containing both color changes and architectural finishes.

The forms shown in Figs. 1.1 to 1.12 hardly constitute a complete inventory but are illustrative of the shapes appropriate to the properties of reinforced or prestressed concrete. They illustrate the adaptability of the material to a great variety of one-dimensional (beams, girders, columns), two-dimensional (slabs, arches, rigid frames), and three-dimensional (shells, tanks) structures and structural components. This variability allows the shape of the structure to be adapted to its function in an economical manner, and furnishes the architect and design engineer with a wide variety of possibilities for esthetically satisfying structural solutions.

1.3 LOADS

Loads that act on structures can be divided into three broad categories: dead loads, live loads, and environmental loads.

Dead loads are those that are constant in magnitude and fixed in location throughout the lifetime of the structure. Usually the major part of the dead load is the weight of the structure itself. This can be calculated with good accuracy from the design configuration, dimensions of the structure, and density of the material. For buildings, floor

FIGURE 1.12
Concrete structures can be produced in a wide range of colors, finishes, and architectural detailing. (*Courtesy of Rocky Mountain Prestress Corp.*)

fill, finish floors, and plastered ceilings are usually included as dead loads, and an allowance is made for suspended loads such as piping and lighting fixtures. For bridges, dead loads may include wearing surfaces, sidewalks, and curbs, and an allowance is made for piping and other suspended loads.

Live loads consist chiefly of occupancy loads in buildings and traffic loads on bridges. They may be either fully or partially in place or not present at all, and may also change in location. Their magnitude and distribution at any given time are uncertain, and even their maximum intensities throughout the lifetime of the structure are not known with precision. The minimum live loads for which the floors and roof of a building should be designed are usually specified in the building code that governs at the site of construction. Representative values of minimum live loads to be used in a wide variety of buildings are found in *Minimum Design Loads for Buildings and Other Structures* (Ref. 1.1), a portion of which is reprinted in Table 1.1. The table gives uniformly distributed live loads for various types of occupancies; these include impact provisions where necessary. These loads are expected maxima and considerably exceed average values.

In addition to these uniformly distributed loads, it is recommended that, as an alternative to the uniform load, floors be designed to support safely certain concentrated loads if these produce a greater stress. For example, according to Ref. 1.1, office floors are to be designed to carry a load of 2000 lb distributed over an area 2.5 ft square (6.25 ft^2), to allow for the weight of a safe or other heavy equipment, and stair treads must safely support a 300 lb load applied on the center of the tread. Certain reductions are often permitted in live loads for members supporting large areas, on the premise that it is not likely that the entire area would be fully loaded at one time (Refs. 1.1 and 1.2).

TABLE 1.1
Minimum uniformly distributed live loads

Occupancy or Use	Live Load, psf[a]	Occupancy or Use	Live Load, psf[a]
Apartments (see residential)		Dining rooms and restaurants	100
Access floor systems		Dwellings (see residential)	
Office use	50	Fire escapes	100
Computer use	100	On single-family dwellings only	40
Armories and drill rooms	150	Garages (passenger cars only)	40
Assembly areas and theaters		Trucks and buses[b]	
Fixed seats (fastened to floor)	60	Grandstands (see stadium and arena bleachers)	
Lobbies	100	Gymnasiums, main floors and balconies[c]	100
Movable seats	100	Hospitals	
Platforms (assembly)	100	Operating rooms, laboratories	60
Stage floors	150	Patient rooms	40
Balconies (exterior)	100	Corridors above first floor	80
On one and two-family residences only, and not exceeding 100 ft^2	60	Hotels (see residential)	
		Libraries	
Bowling alleys, poolrooms, and similar recreational areas	75	Reading rooms	60
		Stack rooms[d]	150
Catwalks for maintenance access	40	Corridors above first floor	80
Corridors		Manufacturing	
First floor	100	Light	125
Other floors, same as occupancy served except as indicated		Heavy	250
		Marquees and canopies	75
Dance halls and ballrooms	100	Office buildings	
Decks (patio and roof)		File and computer rooms shall be designed for heavier loads based on anticipated occupancy	
Same as area served, or for the type of occupancy accommodated		Lobbies and first-floor corridors	100

(continued)

Tabulated live loads cannot always be used. The type of occupancy should be considered and the probable loads computed as accurately as possible. Warehouses for heavy storage may be designed for loads as high as 500 psf or more; unusually heavy operations in manufacturing buildings may require an increase in the 250 psf value specified in Table 1.1; special provisions must be made for all definitely located heavy concentrated loads.

Live loads for highway bridges are specified by the American Association of State Highway and Transportation Officials (AASHTO) in its *LRFD Bridge Design Specifications* (Ref. 1.3). For railway bridges, the American Railway Engineering and Maintenance-of-Way Association (AREMA) has published the *Manual of Railway Engineering* (Ref. 1.4), which specifies traffic loads.

Environmental loads consist mainly of snow loads, wind pressure and suction, earthquake loads (i.e., inertia forces caused by earthquake motions), soil pressures on subsurface portions of structures, loads from possible ponding of rainwater on flat surfaces, and forces caused by temperature differentials. Like live loads, environmental loads at any given time are uncertain in both magnitude and distribution. Reference 1.1 contains much information on environmental loads, which is often modified locally depending, for instance, on local climatic or seismic conditions.

Figure 1.13, from the 1972 edition of Ref. 1.1, gives snow loads for the continental United States and is included here for illustration only. The 2005 edition

TABLE 1.1 (Continued)

Occupancy or Use	Live Load, psf[a]	Occupancy or Use	Live Load, psf[a]
Offices	50	Schools	
Corridors above first floor	80	Classrooms	40
Penal institutions		Corridors above first floor	80
Cell blocks	40	First-floor corridors	100
Corridors	100	Sidewalks, vehicular driveways, and yards subject to trucking[e]	250
Residential		Stadiums and arenas	
Dwellings (one and two-family)		Bleachers[c]	100
Uninhabitable attics without storage	10	Fixed seats (fastened to floor)[c]	60
Uninhabitable attics with storage	20	Stairs and exit ways	100
Habitable attics and sleeping areas	30	One and two-family residences only	40
All other areas except stairs and balconies	40	Storage areas above ceilings	20
Hotels and multifamily houses		Storage warehouses (shall be designed for heavier loads if required for anticipated storage)	
Private rooms and corridors serving them	40	Light	125
Public rooms and corridors serving them	100	Heavy	250
Reviewing stands, grandstands, and bleachers[c]		Stores	
Roofs		Retail	
Ordinary flat, pitched, and curved roofs	20	First floor	100
Roofs used for promenade purposes	60	Upper floors	73
Roofs used for roof gardens or assembly purpose	100	Wholesale, all floors	125
Roofs used for other special purposes[f]		Walkways and elevated platforms (other than exitways)	60
Awnings and canopies		Yards and terraces, pedestrians	100
Fabric construction supported by a lightweight rigid skeleton structure[g]	5		
All other construction	20		

[a] Pounds per square foot.
[b] Garages accommodating trucks and buses shall be designed in accordance with an approved method that contains provisions for truck and bus loadings.
[c] In addition to the vertical live loads, the design shall include horizontal swaying forces applied to each row of seats as follows: 24 lb per linear seat applied in the direction parallel to each row of seats and 10 lb per linear ft of seat applied in the direction perpendicular to each row of seats. The parallel and perpendicular horizontal swaying forces need not be applied simultaneously.
[d] The loading applies to stack room floors that support nonmobile, double-faced library bookstacks subject to the following limitations: (a) The nominal bookstack unit height shall not exceed 90 in.; (b) the nominal shelf depth shall not exceed 12 in. for each face; and (c) parallel rows of double-faced bookstacks shall be separated by aisles not less than 36 in. wide.
[e] Other uniform loads in accordance with an approved method that contains provisions for truck loadings shall also be considered where appropriate.
[f] Roofs used for other special purposes shall be designed for appropriate loads as approved by the authority having jurisdiction.
[g] Nonreducible.

Source: From Ref. 1.1. Used by permission of the American Society of Civil Engineers.

of Ref. 1.1 gives much more detailed information. In either case, specified values represent not average values, but expected upper limits. A minimum roof load of 20 psf is often specified to provide for construction and repair loads and to ensure reasonable stiffness.

Much progress has been made in developing rational methods for predicting horizontal forces on structures due to wind and seismic action. Reference 1.1 summarizes current thinking regarding wind forces and has much information pertaining to earthquake loads as well. Reference 1.5 presents detailed recommendations for lateral forces from earthquakes.

Reference 1.1 specifies design wind pressures per square foot of vertical wall surface. Depending upon locality, these equivalent static forces vary from about 10 to 50 psf.

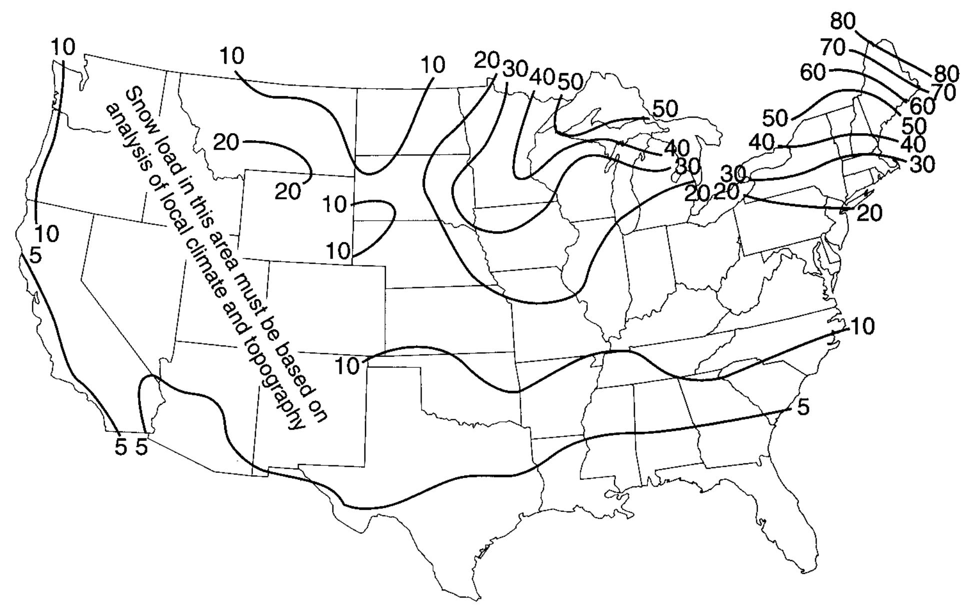

FIGURE 1.13
Snow load in pounds per square foot (psf) on the ground, 50-year mean recurrence interval. (*From* Minimum Design Loads for Buildings and Other Structures, *ANSI A58.1-1972, American National Standards Institute, New York, NY, 1972.*)

Factors include basic wind speed, exposure (urban vs. open terrain, for example), height of the structure, the importance of the structure (i.e., consequences of failure), and gust effect factors to account for the fluctuating nature of the wind and its interaction with the structure.

Seismic forces may be found for a particular structure by elastic or inelastic dynamic analysis, considering expected ground accelerations and the mass, stiffness, and damping characteristics of the construction. However, often the design is based on equivalent static forces calculated from provisions such as those of Refs. 1.1 and 1.5. The base shear is found by considering such factors as location, type of structure and its occupancy, total dead load, and the particular soil condition. The total lateral force is distributed to floors over the entire height of the structure in such a way as to approximate the distribution of forces obtained from a dynamic analysis.

1.4 SERVICEABILITY, STRENGTH, AND STRUCTURAL SAFETY

To serve its purpose, a structure must be safe against collapse and serviceable in use. Serviceability requires that deflections be adequately small; that cracks, if any, be kept to tolerable limits; that vibrations be minimized; etc. Safety requires that the strength of the structure be adequate for all loads that may foreseeably act on it. If the strength of a structure, built as designed, could be predicted accurately, and if the loads and their internal effects (moments, shears, axial forces) were known accurately, safety could be ensured by providing a carrying capacity just barely in excess of the known loads. However, there are a number of sources of uncertainty in the analysis, design, and construction of reinforced concrete structures. These sources of uncertainty, which require a definite margin of safety, may be listed as follows:

1. Actual loads may differ from those assumed.
2. Actual loads may be distributed in a manner different from that assumed.

3. The assumptions and simplifications inherent in any analysis may result in calculated load effects—moments, shears, etc.—different from those that, in fact, act in the structure.
4. The actual structural behavior may differ from that assumed, owing to imperfect knowledge.
5. Actual member dimensions may differ from those specified.
6. Reinforcement may not be in its proper position.
7. Actual material strength may be different from that specified.

In addition, in the establishment of a safety specification, consideration must be given to the consequences of failure. In some cases, a failure would be merely an inconvenience. In other cases, loss of life and significant loss of property may be involved. A further consideration should be the nature of the failure, should it occur. A gradual failure with ample warning permitting remedial measures is preferable to a sudden, unexpected collapse.

It is evident that the selection of an appropriate margin of safety is not a simple matter. However, progress has been made toward rational safety provisions in design codes (Refs. 1.6 to 1.11).

a. Variability of Loads

Since the maximum load that will occur during the life of a structure is uncertain, it can be considered a random variable. In spite of this uncertainty, the engineer must provide an adequate structure. A probability model for the maximum load can be devised by means of a probability density function for loads, as represented by the frequency curve of Fig. 1.14*a*. The exact form of this distribution curve, for any particular type of loading such as office loads, can be determined only on the basis of statistical data obtained from large-scale load surveys. A number of such surveys have been completed. For types of loads for which such data are scarce, fairly reliable information can be obtained from experience, observation, and judgment.

In such a frequency curve (Fig. 1.14*a*), the area under the curve between two abscissas, such as loads Q_1 and Q_2, represents the probability of occurrence of loads Q of magnitude $Q_1 < Q < Q_2$. A specified service load Q_d for design is selected conservatively in the upper region of Q in the distribution curve, as shown. The probability of occurrence of loads larger than Q_d is then given by the shaded area to the right of Q_d. It is seen that this specified service load is considerably larger than the mean load $\bar{Q}$ acting on the structure. This mean load is much more typical of average load conditions than the design load Q_d.

b. Strength

The strength of a structure depends on the strength of the materials from which it is made. For this purpose, minimum material strengths are specified in standardized ways. Actual material strengths cannot be known precisely and therefore also constitute random variables (see Section 2.6). Structural strength depends, furthermore, on the care with which a structure is built, which in turn reflects the quality of supervision and inspection. Member sizes may differ from specified dimensions, reinforcement may be out of position, poorly placed concrete may show voids, etc.

Strength of the entire structure or of a population of repetitive structures, e.g., highway overpasses, can also be considered a random variable with a probability density function of the type shown in Fig. 1.14*b*. As in the case of loads, the exact form

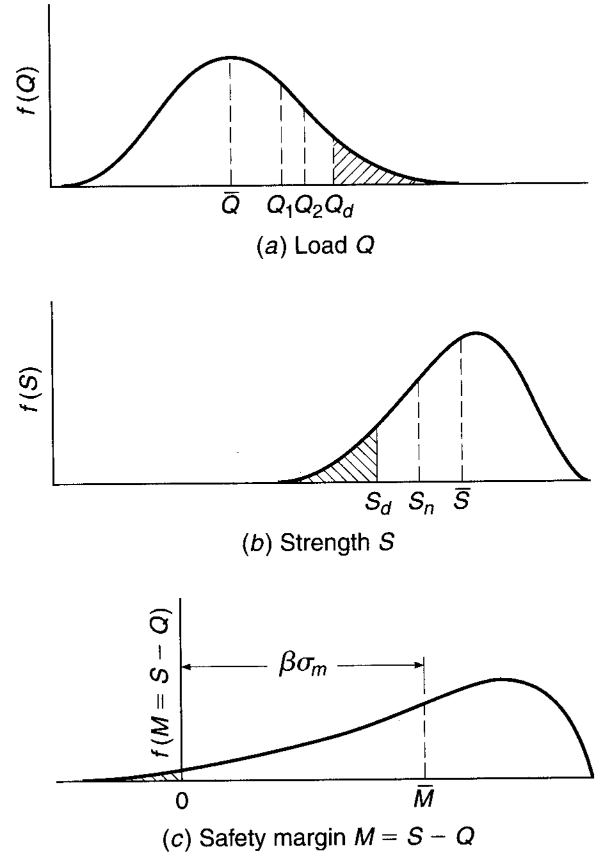

FIGURE 1.14
Frequency curves for (*a*) loads *Q*, (*b*) strengths *S*, and (*c*) safety margin *M*.

of this function cannot be known but can be approximated from known data, such as statistics of actual, measured materials and member strengths and similar information. Considerable information of this type has been, or is being, developed and used.

c. Structural Safety

A given structure has a *safety margin M* if

$$M = S - Q > 0 \qquad (1.1)$$

i.e., if the strength of the structure is larger than the load acting on it. Since S and Q are random variables, the safety margin $M = S - Q$ is also a random variable. A plot of the probability function of M may appear as in Fig. 1.14*c*. Failure occurs when M is less than zero. Thus, the probability of failure is represented by the shaded area in the figure.

Even though the precise form of the probability density functions for S and Q, and therefore for M, is not known, much can be achieved in the way of a rational approach to structural safety. One such approach is to require that the mean safety margin M be a specified number β of standard deviations σ_m above zero. It can be demonstrated that this results in the requirement that

$$\psi_s \bar{S} \geq \psi_L \bar{Q} \qquad (1.2)$$

where ψ_s is a partial safety coefficient smaller than one applied to the mean strength $\bar{S}$ and ψ_L is a partial safety coefficient larger than one applied to the mean load $\bar{Q}$.

The magnitude of each partial safety coefficient depends on the variance of the quantity to which it applies, S or Q, and on the chosen value of β, the reliability index of the structure. As a general guide, a value of the safety index β between 3 and 4 corresponds to a probability of failure of the order of 1:100,000 (Ref. 1.8). The value of β is often established by calibration against well-proved and established designs.

In practice, it is more convenient to introduce partial safety coefficients with respect to code-specified loads which, as already noted, considerably exceed average values, rather than with respect to mean loads as in Eq. (1.2); similarly, the partial safety coefficient for strength is applied to nominal strength generally computed somewhat conservatively, rather than to mean strengths as in Eq. (1.2). A restatement of the safety requirement in these terms is

$$\phi S_n \geq \gamma Q_d \tag{1.3a}$$

in which ϕ is a strength reduction factor applied to nominal strength S_n and γ is a load factor applied to calculated or code-specified design loads Q_d. Furthermore, recognizing the differences in variability between, say, dead loads D and live loads L, it is both reasonable and easy to introduce different load factors for different types of loads. The preceding equation can thus be written

$$\phi S_n \geq \gamma_d D + \gamma_l L \tag{1.3b}$$

in which γ_d is a load factor somewhat greater than 1.0 applied to the calculated dead load D and γ_l is a larger load factor applied to the code-specified live load L. When additional loads, such as the wind load W, are to be considered, the reduced probability that maximum dead, live, and wind or other loads will act simultaneously can be incorporated by using modified load factors such that

$$\phi S_n \geq \gamma_{d_i} D + \gamma_{l_i} L + \gamma_{w_i} W + \cdots \tag{1.3c}$$

Present U.S. design specifications follow the format of Eqs. (1.3*b*) and (1.3*c*).

1.5 DESIGN BASIS

The single most important characteristic of any structural member is its actual strength, which must be large enough to resist, with some margin to spare, all foreseeable loads that may act on it during the life of the structure, without failure or other distress. It is logical, therefore, to proportion members, i.e., to select concrete dimensions and reinforcement, so that member strengths are adequate to resist forces resulting from certain hypothetical overload stages, significantly above loads expected actually to occur in service. This design concept is known as *strength design.*

For reinforced concrete structures at loads close to and at failure, one or both of the materials, concrete and steel, are invariably in their nonlinear inelastic range. That is, concrete in a structural member reaches its maximum strength and subsequent fracture at stresses and strains far beyond the initial elastic range in which stresses and strains are fairly proportional. Similarly, steel close to and at failure of the member is usually stressed beyond its elastic domain into and even beyond the yield region. Consequently, the nominal strength of a member must be calculated on the basis of this inelastic behavior of the materials.

A member designed by the strength method must also perform in a satisfactory way under normal service loading. For example, beam deflections must be limited to acceptable values, and the number and width of flexural cracks at service loads must be controlled. Serviceability limit conditions are an important part of the total design, although attention is focused initially on strength.

Historically, members were proportioned so that stresses in the steel and concrete resulting from normal service loads were within specified limits. These limits, known as *allowable stresses*, were only fractions of the failure stresses of the materials. For members proportioned on such a service load basis, the margin of safety was provided by stipulating allowable stresses under service loads that were appropriately small fractions of the compressive concrete strength and the steel yield stress. We now refer to this basis for design as *service load design*. Allowable stresses, in practice, were set at about one-half the concrete compressive strength and one-half the yield stress of the steel.

Because of the difference in realism and reliability, the strength design method has displaced the older service load design method. However, the older method provides the basis for some serviceability checks and is the design basis for many older structures. Throughout this text, strength design is presented almost exclusively.

1.6 DESIGN CODES AND SPECIFICATIONS

The design of concrete structures such as those of Figs. 1.1 to 1.12 is generally done within the framework of codes giving specific requirements for materials, structural analysis, member proportioning, etc. The International Building Code (Ref. 1.2) is an example of a consensus code governing structural design and is often adopted by local municipalities. The responsibility of preparing material-specific portions of the codes rests with various professional groups, trade associations, and technical institutes. In contrast with many other industrialized nations, the United States does not have an official, government-sanctioned, national code.

The American Concrete Institute (ACI) has long been a leader in such efforts. As one part of its activity, the American Concrete Institute has published the widely recognized *Building Code Requirements for Structural Concrete and Commentary* (Ref. 1.12), which serves as a guide in the design and construction of reinforced concrete buildings. The ACI Code has no official status in itself. However, it is generally regarded as an authoritative statement of current good practice in the field of reinforced concrete. As a result, it has been incorporated into the International Building Code and similar codes, which in turn are adopted by law into municipal and regional building codes that do have legal status. Its provisions thereby attain, in effect, legal standing. Most reinforced concrete buildings and related construction in the United States are designed in accordance with the current ACI Code. It has also served as a model document for many other countries. The commentary incorporated in Ref. 1.12 provides background material and rationale for the Code provisions. The American Concrete Institute also publishes important journals and standards, as well as recommendations for the analysis and design of special types of concrete structures such as the tanks shown in Fig. 1.11.

Most highway bridges in the United States are designed according to the requirements of the AASHTO bridge specifications (Ref. 1.3) which not only contain the provisions relating to loads and load distributions mentioned earlier, but also include detailed provisions for the design and construction of concrete bridges. Many of the provisions follow ACI Code provisions closely, although a number of significant differences will be found.

The design of railway bridges is done according to the specifications of the AREMA *Manual of Railway Engineering* (Ref. 1.4). It, too, is patterned after the ACI

Code in most respects, but it contains much additional material pertaining to railway structures of all types.

No code or design specification can be construed as a substitute for sound engineering judgment in the design of concrete structures. In structural practice, special circumstances are frequently encountered where code provisions can serve only as a guide, and the engineer must rely upon a firm understanding of the basic principles of structural mechanics applied to reinforced or prestressed concrete, and an intimate knowledge of the nature of the materials.

1.7 SAFETY PROVISIONS OF THE ACI CODE

The safety provisions of the ACI Code are given in the form of Eqs. (1.3*b*) and (1.3*c*) using strength reduction factors and load factors. These factors are based to some extent on statistical information but to a larger degree on experience, engineering judgment, and compromise. In words, the design strength ϕS_n of a structure or member must be at least equal to the required strength U calculated from the factored loads, i.e.,

$$\text{Design strength} \geq \text{required strength}$$

or

$$\phi S_n \geq U \tag{1.4}$$

The nominal strength S_n is computed (usually somewhat conservatively) by accepted methods. The required strength U is calculated by applying appropriate load factors to the respective service loads: dead load D, live load L, wind load W, earthquake load E, earth pressure H, fluid pressure F, snow load S, rain load R, and environmental effects T that may include settlement, creep, shrinkage, and temperature change. Loads are defined in a general sense, to include either loads or the related internal effects such as moments, shears, and thrusts. Thus, in specific terms for a member subjected, say, to moment, shear, and axial load

$$\phi M_n \geq M_u \tag{1.5a}$$

$$\phi V_n \geq V_u \tag{1.5b}$$

$$\phi P_n \geq P_u \tag{1.5c}$$

where the subscripts n denote the nominal strengths in flexure, shear, and axial load, respectively, and the subscripts u denote the factored load moment, shear, and axial load. In computing the factored load effects on the right, load factors may be applied either to the service loads themselves or to the internal load effects calculated from the service loads.

The load factors specified in the ACI Code, to be applied to calculated dead loads and those live and environmental loads specified in the appropriate codes or standards, are summarized in Table 1.2. These are consistent with the concepts introduced in Section 1.4 and with SEI/ASCE 7, *Minimum Design Loads for Buildings and Other Structures* (Ref. 1.1), and allow design of composite structures using combinations of structural steel and reinforced concrete. For individual loads, lower factors are used for loads known with greater certainty, e.g., dead load, compared with loads of greater variability, e.g., live loads. Further, for load combinations such as dead plus live loads plus wind forces, reductions are applied to one load or the other that reflect the improbability that an excessively large live load coincides with an unusually high windstorm. The

TABLE 1.2
Factored load combinations for determining required strength *U* in the ACI Code

Condition[a]	Factored Load or Load Effect *U*
Basic[b]	$U = 1.2D + 1.6L$
Dead plus fluid[b]	$U = 1.4(D + F)$
Snow, rain, temperature, and wind	$U = 1.2(D + F + T) + 1.6(L + H) + 0.5(L_r \text{ or } S \text{ or } R)$
	$U = 1.2D + 1.6(L_r \text{ or } S \text{ or } R) + (1.0L \text{ or } 0.8W)$
	$U = 1.2D + 1.6W + 1.0L + 0.5(L_r \text{ or } S \text{ or } R)$
	$U = 0.9D + 1.6W + 1.6H$
Earthquake	$U = 1.2D + 1.0E + 1.0L + 0.2S$
	$U = 0.9D + 1.0E + 1.6H$

[a] Where the following represent the loads or related internal moments or forces resulting from the listed factors: D = dead load; E = earthquake; F = fluids; H = weight or pressure from soil; L = live load; L_r = roof live load; R = rain; S = snow; T = cumulative effects of temperature, creep, shrinkage, and differential settlement; W = wind.

[b] The ACI Code includes F or H loads in the load combinations. The "basic" load condition of $1.2D + 1.6L$ reflects the fact that most buildings have neither F nor H loads present and that $1.4D$ rarely governs design.

factors also reflect, in a general way, uncertainties with which internal load effects are calculated from external loads in systems as complex as highly indeterminate, inelastic reinforced concrete structures which, in addition, consist of variable-section members (because of tension cracking, discontinuous reinforcement, etc.). Finally, the load factors also distinguish between two situations, particularly when horizontal forces are present in addition to gravity, i.e., the situation where the effects of all simultaneous loads are additive, as distinct from that in which various load effects counteract one another. For example, in a retaining wall the soil pressure produces an overturning moment, and the gravity forces produce a counteracting stabilizing moment.

In all cases in Table 1.2, the controlling equation is the one that gives the largest factored load effect U.

The strength reduction factors ϕ in the ACI Code are given different values depending on the state of knowledge, i.e., the accuracy with which various strengths can be calculated. Thus, the value for bending is higher than that for shear or bearing. Also, ϕ values reflect the probable importance, for the survival of the structure, of the particular member and of the probable quality control achievable. For both these reasons, a lower value is used for columns than for beams. Table 1.3 gives the ϕ values specified in the ACI Code.

The joint application of strength reduction factors (Table 1.3) and load factors (Table 1.2) is aimed at producing approximate probabilities of understrength of the order of 1/100 and of overloads of 1/1000. This results in a probability of structural failure of the order of 1/100,000.

In addition to the values given in Table 1.3, ACI Code Appendix B, "Alternative Provisions for Reinforced and Prestressed Concrete Flexural and Compression Members," allows the use of load factors and strength reduction factors from previous editions of the ACI Code. The load factors and strength reduction factors of ACI Code Appendix B are calibrated in conjunction with the detailed requirements of that appendix. Consequently, they may not be interchanged with the provisions of the main body of the Code.

TABLE 1.3
Strength reduction factors in the ACI Code

Strength Condition	Strength Reduction Factor ϕ
Tension-controlled sections[a]	0.90
Compression-controlled sections[b]	
Members with spiral reinforcement	0.75
Other reinforced members	0.65
Shear and torsion	0.75
Bearing on concrete	0.65
Post-tensioned anchorage zones	0.85
Strut-and-tie models[c]	0.75

[a] Chapter 19 discusses reductions in ϕ for pretensioned members where strand embedment is less than the development length.
[b] Chapter 3 contains a discussion of the linear variation of ϕ between tension and compression-controlled sections. Chapter 8 discusses the conditions that allow an increase in ϕ for spirally reinforced columns.
[c] Strut-and-tie models are described in Chapter 10.

1.8 FUNDAMENTAL ASSUMPTIONS FOR REINFORCED CONCRETE BEHAVIOR

The chief task of the structural engineer is the design of structures. *Design* is the determination of the general shape and all specific dimensions of a particular structure so that it will perform the function for which it is created and will safely withstand the influences that will act on it throughout its useful life. These influences are primarily the loads and other forces to which it will be subjected, as well as other detrimental agents, such as temperature fluctuations, foundation settlements, and corrosive influences. *Structural mechanics* is one of the main tools in this process of design. As here understood, it is the body of knowledge that permits one to predict with a good degree of certainty how a structure of given shape and dimensions will behave when acted upon by known forces or other mechanical influences. The chief items of behavior that are of practical interest are (1) the strength of the structure, i.e., that magnitude of loads of a given distribution which will cause the structure to fail, and (2) the deformations, such as deflections and extent of cracking, that the structure will undergo when loaded under service conditions.

The fundamental propositions on which the mechanics of reinforced concrete is based are as follows:

1. The internal forces, such as bending moments, shear forces, and normal and shear stresses, at any section of a member are in equilibrium with the effects of the external loads at that section. This proposition is not an assumption but a fact, because any body or any portion thereof can be at rest only if all forces acting on it are in equilibrium.
2. The strain in an embedded reinforcing bar (unit extension or compression) is the same as that of the surrounding concrete. Expressed differently, it is assumed that perfect bonding exists between concrete and steel at the interface, so that no slip can occur between the two materials. Hence, as the one deforms, so must the other. With modern deformed bars (see Section 2.14), a high degree of mechanical

interlocking is provided in addition to the natural surface adhesion, so this assumption is very close to correct.

3. Cross sections that were plane prior to loading continue to be plane in the member under load. Accurate measurements have shown that when a reinforced concrete member is loaded close to failure, this assumption is not absolutely accurate. However, the deviations are usually minor, and the results of theory based on this assumption check well with extensive test information.
4. In view of the fact that the tensile strength of concrete is only a small fraction of its compressive strength (see Section 2.9), the concrete in that part of a member which is in tension is usually cracked. While these cracks, in well-designed members, are generally so narrow as to be hardly visible (they are known as *hairline* cracks), they evidently render the cracked concrete incapable of resisting tension stress. Correspondingly, it is assumed that concrete is not capable of resisting any tension stress whatever. This assumption is evidently a simplification of the actual situation because, in fact, concrete prior to cracking, as well as the concrete located between cracks, does resist tension stresses of small magnitude. Later in discussions of the resistance of reinforced concrete beams to shear, it will become apparent that under certain conditions this particular assumption is dispensed with and advantage is taken of the modest tensile strength that concrete can develop.
5. The theory is based on the actual stress-strain relationships and strength properties of the two constituent materials (see Sections 2.8 and 2.14) or some reasonable equivalent simplifications thereof. The fact that nonelastic behavior is reflected in modern theory, that concrete is assumed to be ineffective in tension, and that the joint action of the two materials is taken into consideration results in analytical methods which are considerably more complex, and also more challenging, than those that are adequate for members made of a single, substantially elastic material.

These five assumptions permit one to predict by calculation the performance of reinforced concrete members only for some simple situations. Actually, the joint action of two materials as dissimilar and complicated as concrete and steel is so complex that it has not yet lent itself to purely analytical treatment. For this reason, methods of design and analysis, while using these assumptions, are very largely based on the results of extensive and continuing experimental research. They are modified and improved as additional test evidence becomes available.

1.9 BEHAVIOR OF MEMBERS SUBJECT TO AXIAL LOADS

Many of the fundamentals of the behavior of reinforced concrete, through the full range of loading from zero to ultimate, can be illustrated clearly in the context of members subject to simple axial compression or tension. The basic concepts illustrated here will be recognized in later chapters in the analysis and design of beams, slabs, eccentrically loaded columns, and other members subject to more complex loadings.

a. Axial Compression

In members that sustain chiefly or exclusively axial compression loads, such as building columns, it is economical to make the concrete carry most of the load. Still, some steel reinforcement is always provided for various reasons. For one, very few members are truly axially loaded; steel is essential for resisting any bending that may exist. For another, if part of the total load is carried by steel with its much greater strength, the cross-sectional dimensions of the member can be reduced—the more so, the larger the amount of reinforcement.

The two chief forms of reinforced concrete columns are shown in Fig. 1.15. In the square column, the four longitudinal bars serve as main reinforcement. They are held in place by transverse small-diameter steel ties that prevent displacement of the main bars during construction operations and counteract any tendency of the compression-loaded bars to buckle out of the concrete by bursting the thin outer cover. On the left is shown a round column with eight main reinforcing bars. These are surrounded by a closely spaced spiral that serves the same purpose as the more widely spaced ties but also acts to confine the concrete within it, thereby increasing its resistance to axial compression. The discussion that follows applies to tied columns.

When axial load is applied, the compression strain is the same over the entire cross section and, in view of the bonding between concrete and steel, is the same in the two materials (see propositions 2 and 3 in Section 1.8). To illustrate the action of such a member as load is applied, Fig. 1.16 shows two typical stress-strain curves, one for a concrete with compressive strength $f_c' = 4000$ psi and the other for a steel with yield stress $f_y = 60{,}000$ psi. The curves for the two materials are drawn on the same graph using different vertical stress scales. Curve *b* has the shape that would be obtained in a concrete cylinder test. The rate of loading in most structures is considerably slower than that in a cylinder test, and this affects the shape of the curve. Curve *c*, therefore, is drawn as being characteristic of the performance of concrete under slow loading. Under these conditions, tests have shown that the maximum reliable compressive strength of reinforced concrete is about $0.85f_c'$, as shown.

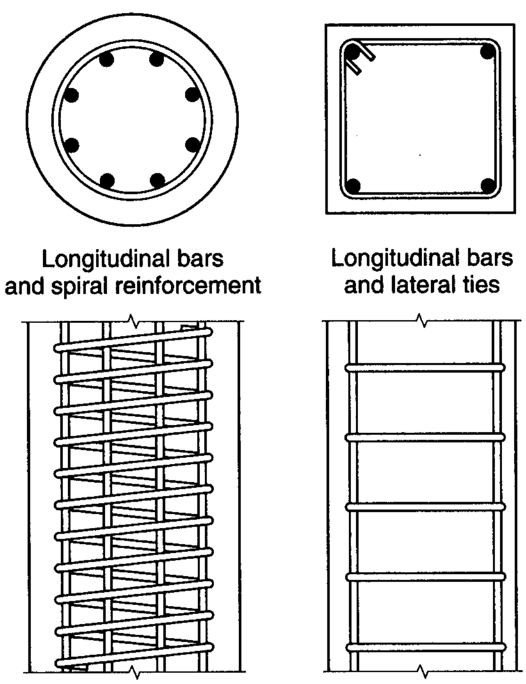

FIGURE 1.15
Reinforced concrete columns.

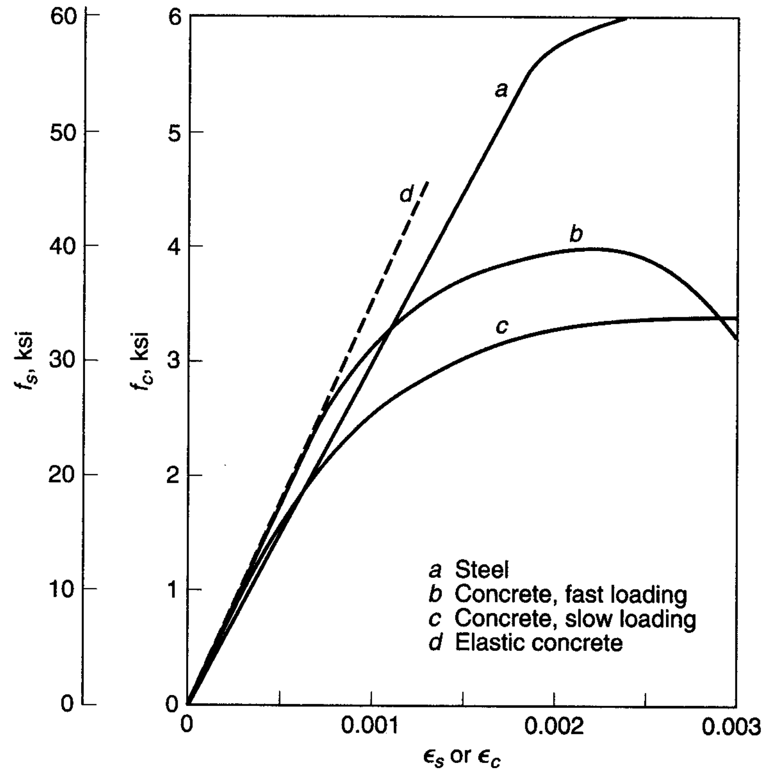

FIGURE 1.16
Concrete and steel stress-strain curves.

Elastic Behavior At low stresses, up to about $f_c'/2$, the concrete is seen to behave nearly elastically, i.e., stresses and strains are quite closely proportional; the straight line d represents this range of behavior with little error for both rates of loading. For the given concrete, the range extends to a strain of about 0.0005. The steel, on the other hand, is seen to be elastic nearly to its yield point of 60 ksi, or to the much greater strain of about 0.002.

Because the compression strain in the concrete, at any given load, is equal to the compression strain in the steel,

$$\epsilon_c = \frac{f_c}{E_c} = \epsilon_s = \frac{f_s}{E_s}$$

from which the relation between the steel stress f_s and the concrete stress f_c is obtained as

$$f_s = \frac{E_s}{E_c} f_c = nf_c \tag{1.6}$$

where $n = E_s/E_c$ is known as the *modular ratio.*

Let

A_c = net area of concrete, i.e., gross area minus area occupied by reinforcing bars
A_g = gross area
A_{st} = total area of reinforcing bars
P = axial load

Then

$$P = f_c A_c + f_s A_{st} = f_c A_c + nf_c A_{st}$$

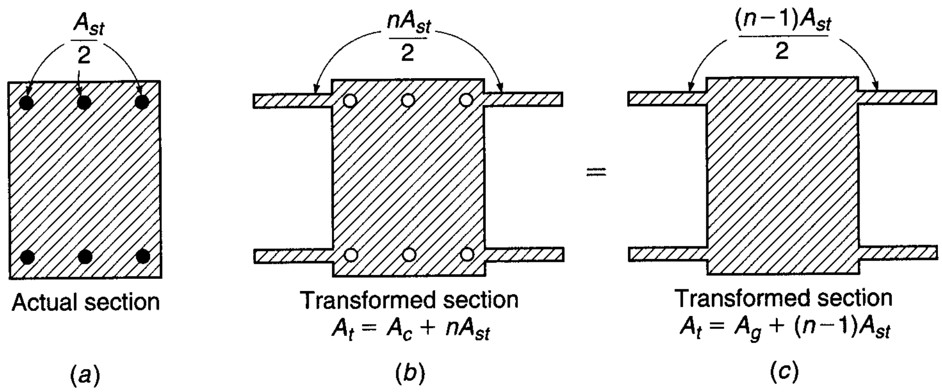

FIGURE 1.17
Transformed section in axial compression.

or

$$P = f_c(A_c + nA_{st}) \tag{1.7}$$

The term $A_c + nA_{st}$ can be interpreted as the area of a fictitious concrete cross section, the *transformed area*, which when subjected to the particular concrete stress f_c results in the same axial load P as the actual section composed of both steel and concrete. This transformed concrete area is seen to consist of the actual concrete area plus n times the area of the reinforcement. It can be visualized as shown in Fig. 1.17. That is, in Fig. 1.17*b* the three bars along each of the two faces are thought of as being removed and replaced, at the same distance from the axis of the section, with added areas of fictitious concrete of total amount nA_{st}. Alternatively, as shown in Fig. 1.17*c*, one can think of the area of the steel bars as replaced with concrete, in which case one has to add to the gross concrete area A_g so obtained only $(n - 1)A_{st}$ to obtain the same total transformed area. Therefore, alternatively,

$$P = f_c[A_g + (n - 1)A_{st}] \tag{1.8}$$

If load and cross-sectional dimensions are known, the concrete stress can be found by solving Eq. (1.7) or (1.8) for f_c, and the steel stress can be calculated from Eq. (1.6). These relations hold in the range in which the concrete behaves nearly elastically, i.e., up to about 50 to 60 percent of f'_c. For reasons of safety and serviceability, concrete stresses in structures under normal conditions are kept within this range. Therefore, these relations permit one to calculate *service load stresses.*

EXAMPLE 1.1 A column made of the materials defined in Fig. 1.16 has a cross section of 16 × 20 in. and is reinforced by six No. 9 (No. 29) bars, disposed as shown in Fig. 1.17. (See Tables A.1 and A.2 of Appendix A for bar diameters and areas and Section 2.14 for a description of bar size designations.) Determine the axial load that will stress the concrete to 1200 psi. The modular ratio n may be assumed equal to 8. (In view of the scatter inherent in E_c, it is customary and satisfactory to round off the value of n to the nearest integer.)

Solution. One finds $A_g = 16 \times 20 = 320 \text{ in}^2$, and from Appendix A, Table A.2, two No. 9 (No. 29) bars provide steel area $A_{st} = 6.00 \text{ in}^2$ or 1.88 percent of the gross area. The load on the column, from Eq. (1.8), is $P = 1200[320 + (8 - 1)6.00] = 434{,}000$ lb. Of this total load, the concrete is seen to carry $P_c = f_cA_c = f_c(A_g - A_{st}) = 1200(320 - 6) = 377{,}000$ lb, and the steel $P_s = f_sA_{st} = (nf_c)A_{st} = 9600 \times 6 = 57{,}600$ lb, which is 13.3 percent of the total axial load.

Inelastic Range Inspection of Fig. 1.16 shows that the elastic relationships that have been used so far cannot be applied beyond a strain of about 0.0005 for the given concrete. To obtain information on the behavior of the member at larger strains and, correspondingly, at larger loads, it is therefore necessary to make direct use of the information in Fig. 1.16.

EXAMPLE 1.2 One may want to calculate the magnitude of the axial load that will produce a strain or unit shortening $\epsilon_c = \epsilon_s = 0.0010$ in the column of Example 1.1. At this strain the steel is seen to be still elastic, so that the steel stress $f_s = \epsilon_s E_s = 0.001 \times 29{,}000{,}000 = 29{,}000$ psi. The concrete is in the inelastic range, so that its stress cannot be directly calculated, but it can be read from the stress-strain curve for the given value of strain.

1. If the member has been loaded at a fast rate, curve *b* holds at the instant when the entire load is applied. The stress for $\epsilon = 0.001$ can be read as $f_c = 3200$ psi. Consequently, the total load can be obtained from

$$P = f_c A_c + f_s A_{st} \tag{1.9}$$

which applies in the inelastic as well as in the elastic range. Hence, $P = 3200(320 - 6) + 29{,}000 \times 6 = 1{,}005{,}000 + 174{,}000 = 1{,}179{,}000$ lb. Of this total load, the steel is seen to carry 174,000 lb, or 14.7 percent.

2. For slowly applied or sustained loading, curve *c* represents the behavior of the concrete. Its stress at a strain of 0.001 can be read as $f_c = 2400$ psi. Then $P = 2400 \times 314 + 29{,}000 \times 6 = 754{,}000 + 174{,}000 = 928{,}000$ lb. Of this total load, the steel is seen to carry 18.8 percent.

Comparison of the results for fast and slow loading shows the following. Owing to creep of concrete, a given shortening of the column is produced by a smaller load when slowly applied or sustained over some length of time than when quickly applied. More important, the farther the stress is beyond the proportional limit of the concrete, and the more slowly the load is applied or the longer it is sustained, the smaller the share of the total load carried by the concrete and the larger the share carried by the steel. In the sample column, the steel was seen to carry 13.3 percent of the load in the elastic range, 14.7 percent for a strain of 0.001 under fast loading, and 18.8 percent at the same strain under slow or sustained loading.

Strength The one quantity of chief interest to the structural designer is *strength*, i.e., the maximum load that the structure or member will carry. Information on stresses, strains, and similar quantities serves chiefly as a tool for determining carrying capacity. The performance of the column discussed so far indicates two things: (1) in the range of large stresses and strains that precede attainment of the maximum load and subsequent failure, elastic relationships cannot be used; (2) the member behaves differently under fast and under slow or sustained loading and shows less resistance to the latter than to the former. In usual construction, many types of loads, such as the weight of the structure and any permanent equipment housed therein, are sustained, and others are applied at slow rates. For this reason, to calculate a reliable magnitude of compressive strength, curve *c* of Fig. 1.16 must be used as far as the concrete is concerned.

The steel reaches its tensile strength (peak of the curve) at strains on the order of 0.08 (see Fig. 2.15). Concrete, on the other hand, fails by crushing at the much smaller strain of about 0.003 and, as seen from Fig. 1.16 (curve *c*), reaches its maximum stress in the strain range of 0.002 to 0.003. Because the strains in steel and concrete are equal in axial compression, the load at which the steel begins to yield can be calculated from the information in Fig. 1.16.

If the small knee prior to yielding of the steel is disregarded, i.e., if the steel is assumed to be sharp-yielding, the strain at which it yields is

$$\epsilon_y = \frac{f_y}{E_s} \tag{1.10}$$

or

$$\epsilon_y = \frac{60{,}000}{29{,}000{,}000} = 0.00207$$

At this strain, curve c of Fig. 1.16 indicates a stress of 3200 psi in the concrete; therefore, by Eq. (1.9), the load in the member when the steel starts yielding is $P_y = 3200 \times 314 + 60{,}000 \times 6 = 1{,}365{,}000$ lb. At this load the concrete has not yet reached its full strength, which, as mentioned before, can be assumed as $0.85f_c' = 3400$ psi for slow or sustained loading, and therefore the load on the member can be further increased. During this stage of loading, the steel keeps yielding at constant stress. Finally, the nominal capacity† of the member is reached when the concrete crushes while the steel yields, i.e.,

$$P_n = 0.85f'_c A_c + f_y A_{st} \tag{1.11}$$

Numerous careful tests have shown the reliability of Eq. (1.11) in predicting the ultimate strength of a concentrically loaded reinforced concrete column, provided its slenderness ratio is small so that buckling will not reduce its strength.

For the particular numerical example, $P_n = 3400 \times 314 + 60{,}000 \times 6 = 1{,}068{,}000 + 360{,}000 = 1{,}428{,}000$ lb. At this stage the steel carries 25.2 percent of the load.

Summary In the elastic range, the steel carries a relatively small portion of the total load of an axially compressed member. As member strength is approached, there occurs a redistribution of the relative shares of the load resisted by concrete and steel, the latter taking an increasing amount. The nominal capacity, at which the member is on the point of failure, consists of the contribution of the steel when it is stressed to the yield point plus that of the concrete when its stress has attained a value of $0.85f_c'$, as reflected in Eq. (1.11).

b. Axial Tension

The tension strength of concrete is only a small fraction of its compressive strength. It follows that reinforced concrete is not well suited for use in tension members because the concrete will contribute little, if anything, to their strength. Still, there are situations in which reinforced concrete is stressed in tension, chiefly in tie-rods in structures such as arches. Such members consist of one or more bars embedded in concrete in a symmetric arrangement similar to compression members (see Figs. 1.15 and 1.17).

When the tension force in the member is small enough for the stress in the concrete to be considerably below its tensile strength, both steel and concrete behave elastically. In this situation, all the expressions derived for elastic behavior in compression in Section 1.9a are identically valid for tension. In particular, Eq. (1.7) becomes

$$P = f_{ct}(A_c + nA_{st}) \tag{1.12}$$

where f_{ct} is the tensile stress in the concrete.

† Throughout this book quantities that refer to the strength of members, calculated by accepted analysis methods, are furnished with the subscript n, which stands for "nominal." This notation is in agreement with the ACI Code. It is intended to convey that the actual strength of any member is bound to deviate to some extent from its calculated, nominal value because of inevitable variations of dimensions, materials properties, and other parameters. Design in all cases is based on this nominal strength, which represents the best available estimate of the actual member strength.

However, when the load is further increased, the concrete reaches its tensile strength at a stress and strain on the order of one-tenth of what it could sustain in compression. At this stage, the concrete cracks across the entire cross section. When this happens, it ceases to resist any part of the applied tension force, since, evidently, no force can be transmitted across the air gap in the crack. At any load larger than that which caused the concrete to crack, the steel is called upon to resist the entire tension force. Correspondingly, at this stage,

$$P = f_s A_{st} \tag{1.13}$$

With further increased load, the tensile stress f_s in the steel reaches the yield point f_y. When this occurs, the tension members cease to exhibit small, elastic deformations but instead stretch a sizable and permanent amount at substantially constant load. This does not impair the strength of the member. Its elongation, however, becomes so large (on the order of 1 percent or more of its length) as to render it useless. Therefore, the maximum useful strength P_{nt} of a tension member is the force that will just cause the steel stress to reach the yield point. That is,

$$P_{nt} = f_y A_{st} \tag{1.14}$$

To provide adequate safety, the force permitted in a tension member under normal service loads should be limited to about $\frac{1}{2}P_{nt}$. Because the concrete has cracked at loads considerably smaller than this, concrete does not contribute to the carrying capacity of the member in service. It does serve, however, as fire and corrosion protection and often improves the appearance of the structure.

There are situations, though, in which reinforced concrete is used in axial tension under conditions in which the occurrence of tension cracks must be prevented. A case in point is a circular tank (see Fig. 1.11). To provide watertightness, the hoop tension caused by the fluid pressure must be prevented from causing the concrete to crack. In this case, Eq. (1.12) can be used to determine a safe value for the axial tension force P by using, for the concrete tension stress f_{ct}, an appropriate fraction of the tensile strength of the concrete, i.e., of the stress that would cause the concrete to crack.

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PROBLEMS

1.1. A 16 × 20 in. column is made of the same concrete and reinforced with the same six No. 9 (No. 29) bars as the column in Examples 1.1 and 1.2, except that a steel with yield strength f_y = 40 ksi is used. The stress-strain diagram of this reinforcing steel is shown in Fig. 2.15 for f_y = 40 ksi. For this column determine (*a*) the axial load that will stress the concrete to 1200 psi; (*b*) the load at which the steel starts yielding; (*c*) the maximum load; and (*d*) the share of the total load carried by the reinforcement at these three stages of loading. Compare results with those calculated in the examples for f_y = 60 ksi, keeping in mind, in regard to relative economy, that the price per pound for reinforcing steels with 40 and 60 ksi yield points is about the same.

1.2. The area of steel, expressed as a percentage of gross concrete area, for the column of Problem 1.1 is lower than would often be used in practice. Recalculate the comparisons of Problem 1.1, using f_y of 40 ksi and 60 ksi as before, but for a 16 × 20 in. column reinforced with eight No. 11 (No. 36) bars. Compare your results with those of Problem 1.1.

1.3. A square concrete column with dimensions 22 × 22 in. is reinforced with a total of eight No. 10 (No. 32) bars arranged uniformly around the column perimeter. Material strengths are f_y = 60 ksi and f'_c = 4000 psi, with stress-strain curves as given by curves *a* and *c* of Fig. 1.16. Calculate the percentages of total load carried by the concrete and by the steel as load is gradually increased from 0 to failure, which is assumed to occur when the concrete strain reaches a limit value of 0.0030. Determine the loads at strain increments of 0.0005 up to the failure strain, and graph your results, plotting load percentages vs. strain. The modular ratio may be assumed at n = 8 for these materials.

1.4. A 20 × 24 in. column is made of the same concrete as used in Examples 1.1 and 1.2. It is reinforced with six No. 11 (No. 36) bars with f_y = 60 ksi. For this column section, determine (*a*) the axial load that the section will carry at a concrete stress of 1400 psi; (*b*) the load on the section when the steel begins to yield; (*c*) the maximum load if the section is loaded slowly; and (*d*) the maximum load if the section is loaded rapidly. The area of one No. 11 (No. 36) bar is 1.56 in^2. Determine the percent of the load carried by the steel and the concrete for each combination.

1.5. A 24 in. diameter column is made of the same concrete as used in Examples 1.1 and 1.2. The area of reinforcement equals 2.1 percent of the gross cross section (that is, $A_s = 0.021A_g$) and f_y = 60 ksi. For this column section, determine (*a*) the axial load the section will carry at a concrete stress of 1200 psi; (*b*) the load on the section when the steel begins to yield; (*c*) the maximum load if the section is loaded slowly; (*d*) the maximum load if the section is loaded rapidly; and (*e*) the maximum load if the reinforcement in the column is raised to 6.5 percent of the gross cross section and the column is loaded slowly. Comment on your answer, especially the percent of the load carried by the steel and the concrete for each combination.

2 Materials

2.1 INTRODUCTION

The structures and component members treated in this text are composed of concrete reinforced with steel bars, and in some cases prestressed with steel wire, strand, or alloy bars. An understanding of the materials characteristics and behavior under load is fundamental to understanding the performance of structural concrete, and to safe, economical, and serviceable design of concrete structures. Although prior exposure to the fundamentals of material behavior is assumed, a brief review is presented in this chapter, as well as a description of the types of bar reinforcement and prestressing steels in common use. Numerous references are given as a guide for those seeking more information on any of the topics discussed.

2.2 CEMENT

A cementitious material is one that has the adhesive and cohesive properties necessary to bond inert aggregates into a solid mass of adequate strength and durability. This technologically important category of materials includes not only cements proper but also limes, asphalts, and tars as they are used in road building, and others. For making structural concrete, *hydraulic cements* are used exclusively. Water is needed for the chemical process (hydration) in which the cement powder sets and hardens into one solid mass. Of the various hydraulic cements that have been developed, *portland cement*, which was first patented in England in 1824, is by far the most common.

Portland cement is a finely powdered, grayish material that consists chiefly of calcium and aluminum silicates.[†] The common raw materials from which it is made are limestones, which provide CaO, and clays or shales, which furnish SiO_2 and Al_2O_3. These are ground, blended, fused to clinkers in a kiln, and cooled. Gypsum is added and the mixture is ground to the required fineness. The material is shipped in bulk or in bags containing 94 lb of cement.

Over the years, five standard types of portland cement have been developed. Type I, *normal* portland cement, is used for over 90 percent of construction in the United States. Concretes made with Type I portland cement generally need one to two weeks to reach sufficient strength so that forms of beams and slabs can be removed and

[†] See ASTM C150, "Standard Specification for Portland Cement." This and other ASTM references are published and periodically updated by ASTM International (formerly the American Society for Testing and Materials), West Conshohoken, PA.

reasonable loads applied; they reach their design strength after 28 days and continue to gain strength thereafter at a decreasing rate. To speed construction when needed, *high early strength cements* such as Type III have been developed. They are costlier than ordinary portland cement, but within 7 to 14 days they reach the strength achieved using Type I at 28 days. Type III portland cement contains the same basic compounds as Type I, but the relative proportions differ and it is ground more finely.

When cement is mixed with water to form a soft paste, it gradually stiffens until it becomes a solid. This process is known as *setting* and *hardening*. The cement is said to have set when it has gained sufficient rigidity to support an arbitrarily defined pressure, after which it continues for a long time to harden, i.e., to gain further strength. The water in the paste dissolves material at the surfaces of the cement grains and forms a gel that gradually increases in volume and stiffness. This leads to a rapid stiffening of the paste 2 to 4 hours after water has been added to the cement. *Hydration* continues to proceed deeper into the cement grains, at decreasing speed, with continued stiffening and hardening of the mass. The principal products of hydration are calcium silicate hydrate, which is insoluble, and calcium hydroxide, which is soluble.

In ordinary concrete, the cement is probably never completely hydrated. The gel structure of the hardened paste seems to be the chief reason for the volume changes that are caused in concrete by variations in moisture, such as the shrinkage of concrete as it dries.

For complete hydration of a given amount of cement, an amount of water equal to about 25 percent of that of cement, by weight—i.e., a *water-cement ratio* of 0.25—is needed chemically. An additional amount must be present, however, to provide mobility for the water in the cement paste during the hydration process so that it can reach the cement particles and to provide the necessary workability of the concrete mix. For normal concretes, the water-cement ratio is generally in the range of about 0.40 to 0.60, although for high-strength concretes, ratios as low as 0.21 have been used. In this case, the needed workability is obtained through the use of admixtures.

Any amount of water above that consumed in the chemical reaction produces pores in the cement paste. The strength of the hardened paste decreases in inverse proportion to the fraction of the total volume occupied by pores. Put differently, since only the solids, and not the voids, resist stress, strength increases directly as the fraction of the total volume occupied by the solids. That is why the strength of the cement paste depends primarily on, and decreases directly with, an increasing water-cement ratio.

The chemical process involved in the setting and hardening liberates heat, known as *heat of hydration*. In large concrete masses, such as dams, this heat is dissipated very slowly and results in a temperature rise and volume expansion of the concrete during hydration, with subsequent cooling and contraction. To avoid the serious cracking and weakening that may result from this process, special measures must be taken for its control.

2.3 AGGREGATES

In ordinary structural concretes the aggregates occupy 65 to 75 percent of the volume of the hardened mass. The remainder consists of hardened cement paste, uncombined water (i.e., water not involved in the hydration of the cement), and air voids. The latter two do not contribute to the strength of the concrete. In general, the more densely the aggregate can be packed, the better the durability and economy of the concrete. For this reason the gradation of the particle sizes in the aggregate, to produce close packing, is of considerable importance. It is also important that the aggregate have good strength, durability, and weather resistance; that its surface be free from impurities such as loam, clay, silt, and organic matter that may weaken the bond with cement paste; and that no unfavorable chemical reaction take place between it and the cement.

Natural aggregates are generally classified as fine and coarse. *Fine aggregate* (typically natural sand) is any material that will pass a No. 4 sieve, i.e., a sieve with four openings per linear inch. Material coarser than this is classified as *coarse aggregate*. When favorable gradation is desired, aggregates are separated by sieving into two or three size groups of sand and several size groups of coarse aggregate. These can then be combined according to grading charts to result in a densely packed aggregate. The *maximum size of coarse aggregate* in reinforced concrete is governed by the requirement that it shall easily fit into the forms and between the reinforcing bars. For this purpose it should not be larger than one-fifth of the narrowest dimension of the forms or one-third of the depth of slabs, nor three-quarters of the minimum distance between reinforcing bars. Requirements for satisfactory aggregates are found in ASTM C33, "Standard Specification for Concrete Aggregates," and authoritative information on aggregate properties and their influence on concrete properties, as well as guidance in selection, preparation, and handling of aggregate, is found in Ref. 2.1.

The unit weight of *stone concrete*, i.e., concrete with natural stone aggregate, varies from about 140 to 152 pounds per cubic foot (pcf) and can generally be assumed to be 145 pcf. For special purposes, lightweight concretes, on one hand, and heavy concretes, on the other, are used.

A variety of *lightweight* aggregates are available. Some unprocessed aggregates, such as pumice or cinders, are suitable for insulating concretes, but for structural lightweight concrete, *processed aggregates* are used because of better control. These consist of expanded shales, clays, slates, slags, or pelletized fly ash. They are light in weight because of the porous, cellular structure of the individual aggregate particle, which is achieved by gas or steam formation in processing the aggregates in rotary kilns at high temperatures (generally in excess of 2000°F). Requirements for satisfactory lightweight aggregates are found in ASTM C330, "Standard Specification for Lightweight Aggregates for Structural Concrete."

Three classes of lightweight concrete are distinguished in Ref. 2.2: low-density concretes, which are chiefly employed for insulation and whose unit weight rarely exceeds 50 pcf; moderate strength concretes, with unit weights from about 60 to 85 pcf and compressive strengths of 1000 to 2500 psi, which are chiefly used as fill, e.g., over light-gage steel floor panels; and structural concretes, with unit weights from 90 to 120 pcf and compressive strengths comparable to those of stone concretes. Similarities and differences in structural characteristics of lightweight and stone concretes are discussed in Sections 2.8 and 2.9.

Heavyweight concrete is sometimes required for shielding against gamma and X-radiation in nuclear reactors and similar installations, for protective structures, and for special purposes, such as counterweights of lift bridges. Heavy aggregates are used for such concretes. These consist of heavy iron ores or barite (barium sulfate) rock crushed to suitable sizes. Steel in the form of scrap, punchings, or shot (as fines) is also used. Unit weights of heavyweight concretes with natural heavy rock aggregates range from about 200 to 230 pcf; if iron punchings are added to high-density ores, weights as high as 270 pcf are achieved. The weight may be as high as 330 pcf if ores are used for the fines only and steel for the coarse aggregate.

2.4 PROPORTIONING AND MIXING CONCRETE

The various components of a mix are proportioned so that the resulting concrete has adequate strength, proper workability for placing, and low cost. The third calls for use of the minimum amount of cement (the most costly of the components) that will achieve adequate properties. The better the gradation of aggregates, i.e., the smaller the volume of voids, the less cement paste is needed to fill these voids. In addition to the water required for hydration, water is needed for wetting the surface of the aggregate. As water is added, the plasticity and fluidity of the mix increase (i.e., its workability improves), but the strength decreases because of the larger volume of voids created by the free water. To reduce the free water while retaining the workability, cement must be added. Therefore, as for the cement paste, the *water-cement ratio* is the chief factor that controls the strength of the concrete. For a given water-cement ratio, one selects the minimum amount of cement that will secure the desired workability.

Figure 2.1 shows the decisive influence of the water-cement ratio on the compressive strength of concrete. Its influence on the tensile strength, as measured by the nominal flexural strength or modulus of rupture, is seen to be pronounced but much

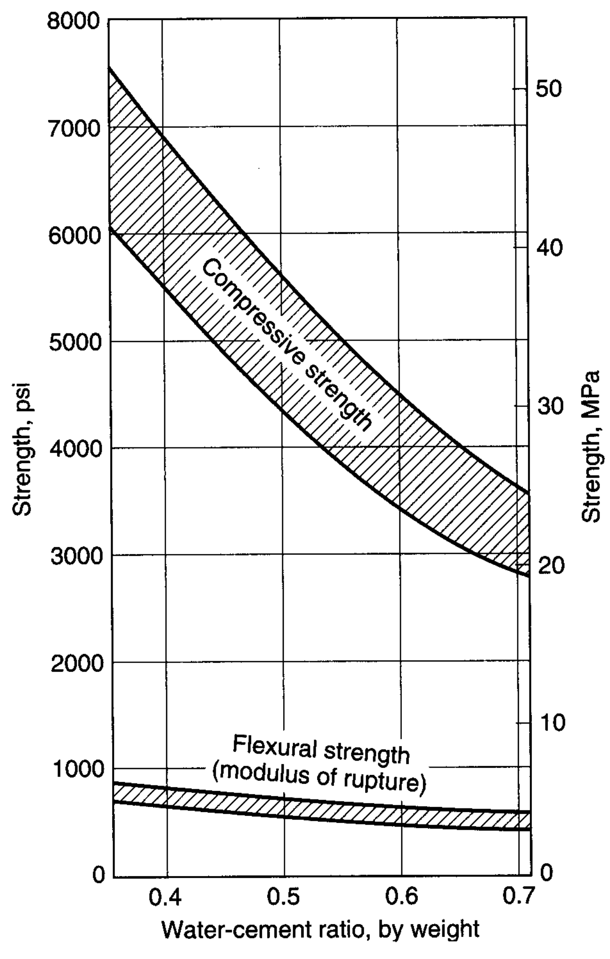


FIGURE 2.1
Effect of water-cement ratio on 28-day compressive and flexural tensile strength.
(*Adapted from Ref. 2.3.*)

smaller than its effect on the compressive strength. This seems to be so because, in addition to the void ratio, the tensile strength depends strongly on the strength of bond between coarse aggregate and cement mortar (i.e., cement paste plus fine aggregate). According to tests at Cornell University, this bond strength is only slightly affected by the water-cement ratio (Ref. 2.4).

It is customary to define the *proportions* of a concrete mix in terms of the total weight of each component needed to make up 1 yd^3 of wet concrete, such as 517 lb of cement, 300 lb of water, 1270 lb of sand, and 1940 lb of coarse aggregate, plus the total volume of air, in percent, when air is deliberately *entrained* in the mix (typically 4 to 7 percent). The weights of the fine and coarse aggregates are based on material in the *saturated surface dry condition*, in which, as the description implies, the aggregates are fully saturated but have no water on the exterior of the particles.

Various methods of proportioning are used to obtain mixes of the desired properties from the cements and aggregates at hand. One is the *trial-batch method*. Selecting a water-cement ratio from information such as that in Fig. 2.1, one produces several small trial batches with varying amounts of aggregate to obtain the required strength, consistency, and other properties with a minimum amount of paste. Concrete *consistency* is most frequently measured by the *slump test*. A metal mold in the shape of a truncated cone 12 in. high is filled with fresh concrete in a carefully specified manner. Immediately upon being filled, the mold is lifted off, and the slump of the concrete is measured as the difference in height between the mold and the pile of concrete. The slump is a good measure of the total water content in the mix and should be kept as low as is compatible with workability. Slumps for concretes in building construction generally range from 2 to 5 in., although higher slumps are used with the aid of chemical admixtures.

The so-called ACI method of proportioning makes use of the slump test in connection with a set of tables that, for a variety of conditions (types of structures, dimensions of members, degree of exposure to weathering, etc.), permit one to estimate proportions that will result in the desired properties (Ref. 2.5). These preliminary selected proportions are checked and adjusted by means of trial batches to result in concrete of the desired quality. Inevitably, strength properties of a concrete of given proportions scatter from batch to batch. It is therefore necessary to select proportions that will furnish an average strength sufficiently greater than the specified design strength for even the accidentally weaker batches to be of adequate quality (for details, see Section 2.6). Discussion in detail of practices for proportioning concrete is beyond the scope of this volume; this topic is treated fully in Refs. 2.5 and 2.6, both for stone concrete and for lightweight aggregate concrete.

If the results of trial batches or field experience are not available, the ACI Code allows concrete to be proportioned based on other experience or information, if approved by the registered design professional overseeing the project. This alternative may not be applied for specified compressive strengths greater than 5000 psi.

On all but the smallest jobs, *batching* is carried out in special batching plants. Separate hoppers contain cement and the various fractions of aggregate. Proportions are controlled, by weight, by means of manually operated or automatic scales connected to the hoppers. The mixing water is batched either by measuring tanks or by water meters.

The principal purpose of *mixing* is to produce an intimate mixture of cement, water, fine and coarse aggregate, and possible admixtures of uniform consistency throughout each batch. This is achieved in machine mixers of the revolving-drum type. Minimum mixing time is 1 min for mixers of not more than 1 yd^3 capacity, with an additional 15 sec for each additional 1 yd^3. Mixing can be continued for a considerable time without adverse effect. This fact is particularly important in connection with ready mixed concrete.

On large projects, particularly in the open country where ample space is available, movable mixing plants are installed and operated at the site. On the other hand, in construction under congested city conditions, on smaller jobs, and frequently in highway construction, *ready mixed concrete* is used. Such concrete is batched in a stationary plant and then hauled to the site in trucks in one of three ways: (1) mixed completely at the stationary plant and hauled in a truck agitator, (2) transit-mixed, i.e., batched at the plant but mixed in a truck mixer, or (3) partially mixed at the plant with mixing completed in a truck mixer. Concrete should be discharged from the mixer or agitator within a limited time after the water is added to the batch. Although specifications often provide a single value for all conditions, the maximum mixing time should be based on the concrete temperature because higher temperatures lead to increased rates of *slump loss* and rapid setting. Conversely, lower temperatures increase the period during which the concrete remains workable. A good guide for maximum mixing time is to allow 1 hour at a temperature of 70°F, plus (or minus) 15 min for each 5°F drop (or rise) in concrete temperature for concrete temperatures between 40 and 90°F. Ten minutes may be used at 95°F, the practical upper limit for normal mixing and placing.

Much information on proportioning and other aspects of design and control of concrete mixtures will be found in Refs. 2.7 and 2.8.

2.5 CONVEYING, PLACING, COMPACTING, AND CURING

Conveying of most building concrete from the mixer or truck to the form is done in bottom-dump buckets or by pumping through steel pipelines. The chief danger during conveying is that of *segregation*. The individual components of concrete tend to segregate because of their dissimilarity. In overly wet concrete standing in containers or forms, the heavier coarse aggregate particles tend to settle, and the lighter materials, particularly water, tend to rise. Lateral movement, such as flow within the forms, tends to separate the coarse gravel from the finer components of the mix.

Placing is the process of transferring the fresh concrete from the conveying device to its final place in the forms. Prior to placing, loose rust must be removed from reinforcement, forms must be cleaned, and hardened surfaces of previous concrete lifts must be cleaned and treated appropriately. Placing and consolidating are critical in their effect on the final quality of the concrete. Proper placement must avoid segregation, displacement of forms or of reinforcement in the forms, and poor bond between successive layers of concrete. Immediately upon placing, the concrete should be *consolidated*, usually by means of vibrators. Consolidation prevents honeycombing, ensures close contact with forms and reinforcement, and serves as a partial remedy to possible prior segregation. Consolidation is achieved by high-frequency, power-driven *vibrators*. These are of the *internal* type, immersed in the concrete, or of the *external* type, attached to the forms. The former are preferable but must be supplemented by the latter where narrow forms or other obstacles make immersion impossible (Ref. 2.9).

Fresh concrete gains strength most rapidly during the first few days and weeks. Structural design is generally based on the *28-day strength*, about 70 percent of which is reached at the end of the first week after placing. The final concrete strength depends greatly on the conditions of moisture and temperature during this initial period. The maintenance of proper conditions during this time is known as *curing*. Thirty percent of the strength or more can be lost by premature drying out of the concrete; similar amounts may be lost by permitting the concrete temperature to drop to 40°F or lower during the first few days unless the concrete is kept continuously moist for a long time thereafter. Freezing of fresh concrete may reduce its strength by 50 percent or more.

To prevent such damage, concrete should be protected from loss of moisture for at least 7 days and, in more sensitive work, up to 14 days. When high early strength cements are used, curing periods can be cut in half. Curing can be achieved by keeping exposed surfaces continually wet through sprinkling, ponding, or covering with plastic film or by the use of sealing compounds, which, when properly used, form evaporation-retarding membranes. In addition to improving strength, proper moist-curing provides better shrinkage control. To protect the concrete against low temperatures during cold weather, the mixing water, and occasionally the aggregates, is heated; temperature insulation is used where possible; and special admixtures are employed. When air temperatures are very low, external heat may have to be supplied in addition to insulation (Refs. 2.7, 2.8, 2.10, and 2.11).

2.6 QUALITY CONTROL

The quality of mill-produced materials, such as structural or reinforcing steel, is ensured by the producer, who must exercise systematic quality controls, usually specified by pertinent ASTM standards. Concrete, in contrast, is produced at or close to the site, and its final qualities are affected by a number of factors, which have been discussed briefly. Thus, systematic quality control must be instituted at the construction site.

The main measure of the structural quality of concrete is its *compressive strength*. Tests for this property are made on cylindrical specimens of height equal to twice the diameter, usually 6 × 12 in. or 4 × 8 in. Impervious molds of this shape are filled with concrete during the operation of placement as specified by ASTM C172, "Standard Method of Sampling Freshly Mixed Concrete," and ASTM C31, "Standard Practice for Making and Curing Concrete Test Specimens in the Field." The cylinders are moist-cured at about 70°F, generally for 28 days, and then tested in the laboratory at a specified rate of loading. The compressive strength obtained from such tests is known as the *cylinder strength* f'_c and is the main property specified for design purposes.

To provide structural safety, continuous control is necessary to ensure that the strength of the concrete as furnished is in satisfactory agreement with the value called for by the designer. The ACI Code specifies that two 6 × 12 in. or three 4 × 8 in. cylinders must be tested for each 150 yd^3 of concrete or for each 5000 ft^2 of surface area actually placed, but not less than once a day. As mentioned in Section 2.4, the results of strength tests of different batches mixed to identical proportions show inevitable scatter. The scatter can be reduced by closer control, but occasional tests below the cylinder strength specified in the design cannot be avoided.

To ensure adequate concrete strength in spite of such scatter, the ACI Code stipulates that concrete quality is satisfactory if

1. No individual strength test result (the average of two or three cylinder tests depending on cylinder size) falls below the required f'_c by more than 500 psi when f'_c is 5000 psi or less or by more than $0.10f'_c$ when f'_c is more than 5000 psi, and
2. Every arithmetic average of any three consecutive strength tests equals or exceeds f'_c.

It is evident that if concrete were proportioned so that its mean strength were just equal to the required strength f'_c, it would not pass these quality requirements, because about one-half of its strength test results would fall below the required f'_c. It is

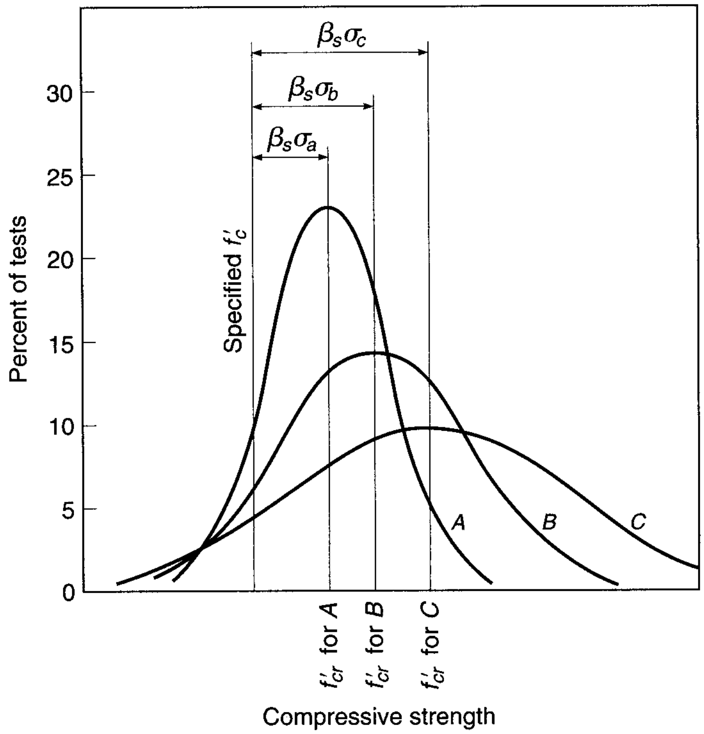


FIGURE 2.2
Frequency curves and average strengths for various degrees of control of concretes with specified design strength f_c'. (*Adapted from Ref. 2.12.*)

therefore necessary to proportion the concrete so that its mean strength f_{cr}', used as the basis for selection of suitable proportions, exceeds the required design strength f_c' by an amount sufficient to ensure that the two quoted requirements are met. The minimum amount by which the required mean strength must exceed f_c' can be determined only by statistical methods because of the random nature of test scatter. Requirements have been derived, based on statistical analysis, to be used as a guide to proper proportioning of the concrete at the plant so that the probability of strength deficiency at the construction site is acceptably low.

The basis for these requirements is illustrated in Fig. 2.2, which shows three normal frequency curves giving the distribution of strength test results. The specified design strength is f_c'. The curves correspond to three different degrees of quality control, curve A representing the best control, i.e., the least scatter, and curve C the worst control, with the most scatter. The degree of control is measured statistically by the standard deviation σ (σ_a for curve A, σ_b for curve B, and σ_c for curve C), which is relatively small for producer A and relatively large for producer C. All three distributions have the same probability of strength less than the specified value f_c'; i.e., each has the same fractional part of the total area under the curve to the left of f_c'. For any normal distribution curve, that fractional part is defined by the index β_s, a multiplier applied to the standard deviation σ; β_s is the same for all three distributions of Fig. 2.2. It is seen that, to satisfy the requirement that, say, 1 test in 100 will fall below f_c' (with the value of β_s thus determined), for producer A with the best quality control the mean strength f_{cr}' can be much closer to the specified f_c' than for producer C with the most poorly controlled operation.

On the basis of such studies, the ACI Code requires that concrete production facilities maintain records from which the standard deviation achieved in the particular facility can be determined. It then stipulates the minimum amount by which the required average compressive strength f'_{cr}, aimed at when selecting concrete proportions, must exceed the specified compressive strength f'_c. In accordance with ACI Code 5.3.1, the value of f'_{cr} is equal to the larger of the values in Eqs. (2.1) and (2.2).

$$f'_{cr} = f'_c + 1.34s_s \tag{2.1}$$

or

$$f'_{cr} = \begin{cases} f'_c + 2.33s_s - 500 & \text{for } f'_c \leq 5000 \text{ psi} \quad (2.2a) \\ 0.9f'_c + 2.33s_s & \text{for } f'_c > 5000 \text{ psi} \quad (2.2b) \end{cases}$$

where s_s is the standard deviation of the test sample.

Equation (2.1) provides a probability of 1 in 100 that averages of three consecutive tests will be below the specified strength f'_c. Equations (2.2*a*) and (2.2*b*) provide a probability of 1 in 100 that an individual strength test will be more than 500 psi below the specified f'_c for f'_c up to 5000 psi or below $0.90f'_c$ for f'_c over 5000 psi.

To use Eqs. (2.1) and (2.2), ACI Code 5.3.1 requires that a minimum of 30 consecutive test results be available. The tests must represent concrete with (1) a specified compressive strength within 1000 psi of f'_c for the project and (2) materials, quality control, and conditions similar to those expected for the building in question. If fewer than 30 but at least 15 tests are available, the equations may still be used, but s_s must be multiplied by a factor from Table 2.1. If fewer than 15 tests have been made, the average strength must exceed f'_c by at least 1000 psi for f'_c less than or equal to 3000 psi, by at least 1200 psi for f'_c between 3000 and 5000 psi, and by $0.10f'_c + 700$ psi for f'_c over 5000 psi, according to the ACI Code.

It is seen that this method of control recognizes the fact that occasional deficient batches are inevitable. The requirements for f'_{cr} ensure (1) a small probability that such strength deficiencies as are bound to occur will be large enough to represent a serious danger and (2) an equally small probability that a sizable portion of the structure, as represented by three consecutive strength tests, will be made of below-par concrete.

Both the requirements described earlier in this section for determining if concrete, as produced, is of satisfactory quality and the process just described of selecting f'_{cr} are based on the same basic considerations but are applied independently, as demonstrated in Examples 2.1 and 2.2.

TABLE 2.1
Modification factors for sample standard deviation s_s when less than 30 tests are available

No. of Tests†	Modification Factor for Sample Standard Deviation‡
Less than 15	See paragraph following Eqs. (2.1) and (2.2)
15	1.16
20	1.08
25	1.03
30 or more	1.00

†Interpolate for intermediate values.
‡The sample standard deviation s_s must be multiplied by the modification factor prior to use in Eqs. (2.1) and (2.2).

EXAMPLE 2.1 A building design calls for specified concrete strength f_c' of 4000 psi. Calculate the average required strength f_{cr}' if (*a*) 30 consecutive tests for concrete with similar strength and materials produce a sample standard deviation s_s of 535 psi, (*b*) 15 consecutive tests for concrete with similar strength and materials produce a sample standard deviation s_s of 510 psi, and (*c*) less than 15 tests are available.

SOLUTION. (*a*) 30 tests available. Using s_s = 535 psi, Eq. (2.1) gives

$$f_{cr}' = f_c' + 1.34s_s = 4000 + 1.34 \times 535 = 4720 \text{ psi}^\dagger$$

Because the specified strength f_c' is less than 4000 psi, Eq. (2.2*a*) must be used.

$$f_{cr}' = f_c' + 2.33s_s - 500 = 4000 + 2.33 \times 535 - 500 = 4750 \text{ psi}$$

The required average strength f_{cr}' is equal to the larger value, 4750 psi.

(*b*) 15 tests available. Because only 15 tests are available, s_s, given as 510 psi, must be multiplied by 1.16, the factor from Table 2.1.

$$1.16 \times s_s = 1.16 \times 510 = 590 \text{ psi}$$

Again, using Eqs. (2.1) and (2.2*a*),

$$f_{cr}' = 4000 + 1.34 \times 590 = 4790 \text{ psi}$$

$$f_{cr}' = 4000 + 2.33 \times 590 - 500 = 4870 \text{ psi}$$

The larger value, 4870 psi, is selected as the required average strength f_{cr}'.

(*c*) Less than 15 tests available. Because f_c' is between 3000 and 5000 psi, the required average strength is

$$f_{cr}' = f_c' + 1200 = 4000 + 1200 = 5200 \text{ psi}$$

This example demonstrates that in cases where test data are available, good quality control, represented by a low sample standard deviation s_s, can be used to reduce the required average strength f_{cr}'. The example also demonstrates that a lack of certainty in the value of the standard deviation due to the limited availability of data results in higher values for f_{cr}', as shown in parts (*b*) and (*c*). As additional test results become available, the higher safety margins can be reduced.

EXAMPLE 2.2 The first eight compressive strength test results for the building described in Example 2.1*c* are 4730, 4280, 3940, 4370, 5180, 4870, 4930, and 4850 psi.

(*a*) Are the test results satisfactory, and (*b*) in what fashion, if any, should the mixture proportions of the concrete be altered?

SOLUTION.

(*a*) For concrete to be considered satisfactory, no individual test may fall below $f_c' - 500$ psi and every arithmetic average of any three consecutive tests must equal f_c'. The eight tests meet these criteria. No test is less than $f_c' - 500$ psi = 4000 − 500 = 3500 psi, and the average of all sets of three consecutive tests exceeds f_c' [for example, (4730 + 4280 + 3940)/3 = 4320, (4280 + 3940 + 4370)/3 = 4200, etc.].

(*b*) To determine if the mixture proportions must be altered, we note that the solution to Example 2.1*c* requires that f_{cr}' equal or exceed 5200 psi. The average of the first eight tests is 4640 psi, well below the value of f_{cr}'. Thus, the mixture proportions should be modified by decreasing the water-cement ratio to increase the concrete strength. Once at least 15 tests are available, the value of f_{cr}' can be recalculated using Eqs. (2.1) and (2.2) with the

† ASTM International specifies that concrete cylinder strengths be recorded to the nearest 10 psi. Hence the values used for test results and f_{cr}' are rounded accordingly.

appropriate factor for S_s from Table 2.1. The mixture proportions can then be adjusted based on the new value of f'_{cr}, the strength of the concrete being produced, and the level of quality control, as represented by the sample standard deviation s_s.

In spite of scientific advances, building in general and concrete making in particular retain some elements of an art; they depend on many skills and imponderables. It is the task of systematic *inspection* to ensure close correspondence between plans and specifications and the finished structure. Inspection during construction should be carried out by a competent engineer, preferably the one who produced the design or one who is responsible to the design engineer. The inspector's main functions in regard to materials quality control are sampling, examination, and field testing of materials; control of concrete proportioning; inspection of batching, mixing, conveying, placing, compacting, and curing; and supervision of the preparation of specimens for laboratory tests. In addition, the inspector must inspect foundations, formwork, placing of reinforcing steel, and other pertinent features of the general progress of work; keep records of all the inspected items; and prepare periodic reports. The importance of thorough inspection to the correctness and adequate quality of the finished structure cannot be emphasized too strongly.

This brief account of concrete technology represents the merest outline of an important subject. Anyone in practice who is actually responsible for any of the phases of producing and placing concrete must be familiar with the details in much greater depth.

2.7 ADMIXTURES

In addition to the main components of concretes, *admixtures* are often used to improve concrete performance. There are admixtures to accelerate or retard setting and hardening, to improve workability, to increase strength, to improve durability, to decrease permeability, and to impart other properties (Ref. 2.13). The beneficial effects of particular admixtures are well established. Chemical admixtures should meet the requirements of ASTM C494, "Standard Specification for Chemical Admixtures for Concrete."

Air-entraining agents are probably the most commonly used admixtures. They cause the entrainment of air in the form of small dispersed bubbles in the concrete. These improve workability and durability (chiefly resistance to freezing and thawing) and reduce segregation during placing. They decrease concrete density because of the increased void ratio and thereby decrease strength; however, this decrease can be partially offset by a reduction of mixing water without loss of workability. The chief use of air-entrained concretes is in pavements, but they are also used for structures, particularly for exposed elements (Ref. 2.14).

Accelerating admixtures are used to reduce setting time and accelerate early strength development. Calcium chloride is the most widely used accelerator because of its cost effectiveness, but it should not be used in prestressed concrete and should be used with caution in reinforced concrete in a moist environment, because of its tendency to promote corrosion of steel. Nonchloride, noncorrosive accelerating admixtures are available, the principal one being calcium nitrite (Ref. 2.13).

Set-retarding admixtures are used primarily to offset the accelerating effect of high ambient temperature and to keep the concrete workable during the entire placing period. This helps to eliminate cracking due to form deflection and also keeps concrete workable long enough that succeeding lifts can be placed without the development of "cold" joints.

Certain organic compounds are used to reduce the water requirement of a concrete mix for a given slump. Such compounds are termed *plasticizers*. Reduction in water demand may result in either a reduction in the water-cement ratio for a given slump and cement content or an increase in slump for the same water-cement ratio and cement content. Plasticizers work by reducing the interparticle forces that exist between cement grains in the fresh paste, thereby increasing the paste fluidity. High-range water-reducing admixtures, or *superplasticizers*, are used to produce high-strength concrete (see Section 2.12) with a very low water-cement ratio while maintaining the higher slumps needed for proper placement and compaction of the concrete. They are also used to produce flowable concrete at conventional water-cement ratios. Superplasticizers differ from conventional water-reducing admixtures in that they do not act as retarders at high dosages; therefore, they can be used at higher dosage rates without severely slowing hydration (Refs. 2.13, 2.15, and 2.16). The specific effects of water-reducing admixtures vary with different cements, changes in water-cement ratio, mixing temperature, ambient temperature, and other job conditions, and trial batches are generally required.

When superplasticizers are combined with *viscosity-modifying admixtures*, they can be used to produce *self-consolidating concrete* (SCC). Self-consolidating concrete is highly fluid and does not require vibration to remove entrapped air. The viscosity modifying agents allow the concrete to remain cohesive even with a very high degree of fluidity. As a result, SCC can be used for members with congested reinforcement, such as beam-column joints in earthquake-resistant structures, and is widely used for precast concrete, especially precast prestressed concrete, a manufactured product (prestressed concrete is discussed in Chapter 19). The high fluidity of the mix, however, has been shown to have a negative impact on the bond strength between the concrete and prestressing steel located in the upper portions of a member, a shortcoming that should be considered in design (Ref. 2.17) but is not currently addressed in the ACI Code, and the composition of SCC mixtures may result in moduli of elasticity, creep, and shrinkage properties that differ from those of more traditional mixtures.

Fly ash and *silica fume* are pozzolans, highly active silicas, that combine with calcium hydroxide, the soluble product of cement hydration (Section 2.2), to form more calcium silicate hydrate, the insoluble product of cement hydration (Refs. 2.18 and 2.19). Pozzolans qualify as *supplementary cementitious materials*, also referred to as *mineral admixtures*, which are used to replace a part of the portland cement in concrete mixes. Fly ash, which is specified under ASTM C618, "Standard Specification for Coal Fly Ash and Raw or Calcified Natural Pozzolan for Use in Concrete," is precipitated electrostatically as a by-product of the exhaust fumes of coal-fired power stations. It is very finely divided and reacts with calcium hydroxide in the presence of moisture to form a cementitious material. It tends to increase the strength of concrete at ages over 28 days. Silica fume, which is specified under ASTM C1240, "Standard Specification for Silica Fume Used in Cementitious Mixtures," is a by-product resulting from the manufacture, in electric-arc furnaces, of ferro-silicon alloys and silicon metal. It is extremely finely divided and is highly cementitious when combined with portland cement. In contrast to fly ash, silica fume contributes mainly to strength gain at early ages, from 3 to 28 days. Both fly ash and silica fume, particularly the latter, have been important in the production of high-strength concrete (see Section 2.12).

Ground granulated blast-furnace slag (GGBFS), which is specified under ASTM C989, "Standard Specification for Ground Granulated Blast-Furnace Slag for Use in Concrete and Mortar," is another supplementary cementitious material. It is produced by water quenching and grinding slag from the production of pig iron, the key ingredient used to make steel (Ref. 2.20). GGBFS consists primarily of calcium silicates, making it very similar to portland cement. As a result of the similarity, slag can be used in higher quantities than fly ash or silica fume, and the resulting material generally has similar or improved properties to those exhibited by concrete made with 100 percent portland cement.

When blast furnace slag, silica fume, fly ash, or a combination is used, it is customary to refer to the *water–cementitious material ratio* rather than the water-cement ratio. This typically may be as low as 0.25 for high-strength concrete, and ratios as low as 0.21 have been used (Refs. 2.21 and 2.22).

Historically, the high durability and high thermal mass of concrete structures have played a key role in *sustainable development*, that is, development that minimizes both its impact on the environment and the resources used both during and after construction. In sustainable development, the "cost" of concrete lies primarily in the manufacture of portland cement. The production of a ton of portland cement requires roughly the energy needed to operate a typical U.S. household for two weeks and generates approximately 0.9 ton of CO_2 (a greenhouse gas). The latter translates to about 250 lb of CO_2 for every cubic yard of concrete that is placed. The energy and greenhouse gases involved in the production of concrete, however, can be viewed as investments because properly designed reinforced concrete structures that take advantage of concrete's thermal mass provide significant reductions in the energy and CO_2 needed for heating and cooling, and concrete's inherent durability results in structures with long service lives. Because by-products, such as the mineral admixtures fly ash and blast furnace slag, involve minimal energy usage or greenhouse gas production, they have the potential to further improve the sustainability of concrete construction when used as a partial replacement for portland cement.

2.8 PROPERTIES IN COMPRESSION

a. Short-Term Loading

Performance of a structure under load depends to a large degree on the stress-strain relationship of the material from which it is made, under the type of stress to which the material is subjected in the structure. Since concrete is used mostly in compression, its compressive stress-strain curve is of primary interest. Such a curve is obtained by appropriate strain measurements in cylinder tests (Section 2.6) or on the compression side in beams. Figure 2.3 shows a typical set of such curves for normal-density concrete, obtained from uniaxial compressive tests performed at normal, moderate testing speeds on concretes that are 28 days old. Figure 2.4 shows corresponding curves for lightweight concretes having a density of 100 pcf.

All of the curves have somewhat similar character. They consist of an initial relatively straight elastic portion in which stress and strain are closely proportional, then begin to curve to the horizontal, reaching the maximum stress, i.e., the compressive strength, at a strain that ranges from about 0.002 to 0.003 for normal-density concretes, and from about 0.003 to 0.0035 for lightweight concretes (Refs. 2.23 and 2.24), the larger values in each case corresponding to the higher strengths. All curves show a descending branch after the peak stress is reached; however, the characteristics of the curves after peak stress are highly dependent upon the method of testing. If special procedures are followed in testing to ensure a constant strain rate while cylinder resistance is decreasing, long stable descending branches can be obtained (Ref. 2.25). In the absence of special devices, unloading past the point of peak stress

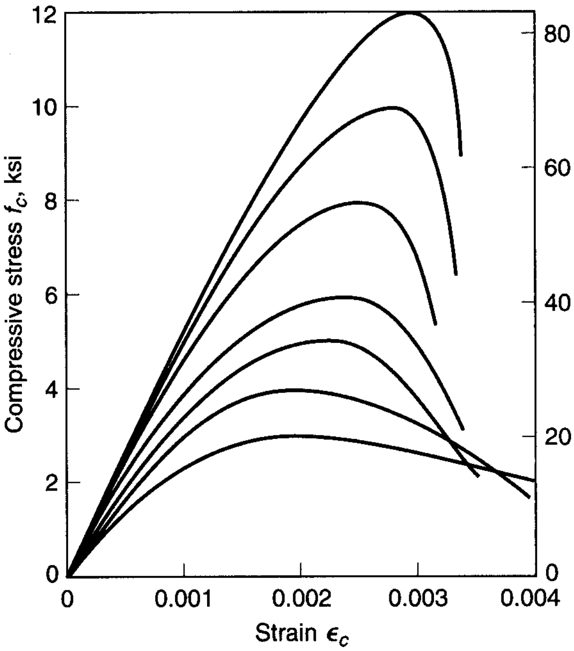


FIGURE 2.3
Typical compressive stress-strain curves for normal-density concrete with w_c = 145 pcf. *(Adapted from Refs. 2.23 and 2.24.)*

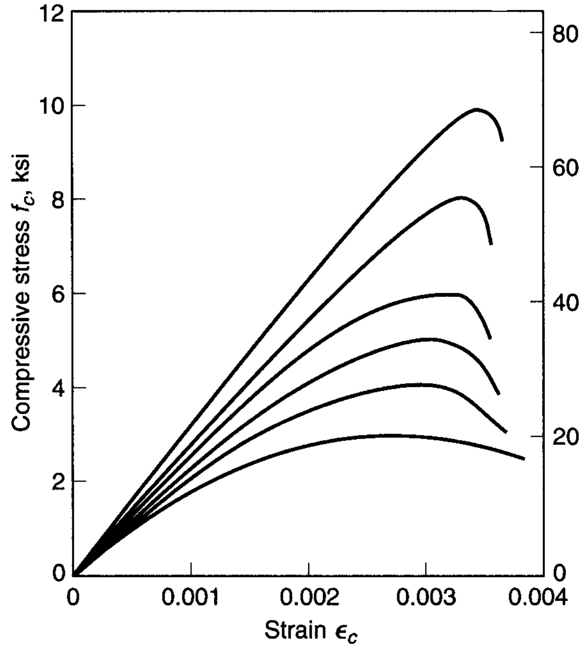


FIGURE 2.4
Typical compressive stress-strain curves for lightweight concrete with w_c = 100 pcf. *(Adapted from Refs. 2.23 and 2.24.)*

may be rapid, particularly for the higher-strength concretes, which are generally more brittle than low-strength concrete.

In present practice, the specified compressive strength f'_c is commonly in the range from 3000 to 5000 psi for normal-density cast-in-place concrete, and up to about 8000 psi for precast prestressed concrete members. Lightweight concrete strengths are somewhat below these values generally. The high-strength concretes, with f'_c to 15,000 psi or more, are used with increasing frequency, particularly for heavily loaded columns in high-rise concrete buildings and for long-span bridges (mostly prestressed) where a significant reduction in dead load may be realized by minimizing member cross section dimensions. (See Section 2.12.)

The *modulus of elasticity* E_c (in psi units), i.e., the slope of the initial straight portion of the stress-strain curve, is seen to be larger as the strength of the concrete increases. For concretes in the strength range to about 6000 psi, it can be computed with reasonable accuracy from the empirical equation found in the ACI Code

$$E_c = 33w_c^{1.5}\sqrt{f'_c} \tag{2.3}$$

where w_c is the unit weight of the hardened concrete in pcf and f'_c is its strength in psi. Equation (2.3) was obtained by testing structural concretes with values of w_c from 90 to 155 pcf. For normal sand-and-stone concretes, with w_c = 145 pcf, E_c may be taken as

$$E_c = 57{,}000\sqrt{f'_c} \tag{2.4}$$

For compressive strengths in the range from 6000 to 12,000 psi, the ACI Code equation may overestimate E_c for both normalweight and lightweight material by as much as 20 percent. Based on research at Cornell University (Refs. 2.23 and 2.24), the following equation is recommended for normal-density concretes with f_c' in the range of 3000 to 12,000 psi, and for lightweight concretes from 3000 to 9000 psi:

$$E_c = (40{,}000\sqrt{f_c'} + 1{,}000{,}000)\left(\frac{w_c}{145}\right)^{1.5} \tag{2.5}$$

where terms and units are as defined above for the ACI Code equations. When coarse aggregates with high moduli of elasticity are used, however, Eq. (2.4) may *underestimate* E_c. Thus, in cases where E_c is a key design criterion, it should be measured, rather than estimated, using Eq. (2.3), (2.4), or (2.5).

Information on concrete strength properties such as those discussed is usually obtained through tests made 28 days after placing. However, cement continues to hydrate, and consequently concrete continues to harden, long after this age, at a decreasing rate. Figure 2.5 shows a typical curve of the gain of concrete strength with age for concrete made using Type I (normal) cement and also Type III (high early strength) cement, each curve normalized with respect to the 28-day compressive strength. High early strength cements produce more rapid strength gain at early ages, although the rate of strength gain at later ages is generally less. Concretes using Type III cement are often used in precasting plants, and often the strength f_c' is specified at 7 days, rather than 28 days.

Note that the shape of the stress-strain curve for various concretes of the same cylinder strength, and even for the same concrete under various conditions of loading, varies considerably. An example of this is shown in Fig. 2.6, where different specimens of the same concrete are loaded at different rates of strain, from one corresponding to a relatively fast loading (0.001 per minute) to one corresponding to an extremely slow application of load (0.001 per 100 days). It is seen that the descending branch of the curve, indicative of internal disintegration of the material, is much more pronounced at fast than at slow rates of loading. It is also seen that the peaks of the curves, i.e., the maximum strengths reached, are somewhat smaller at slower rates of strain.

When compressed in one direction, concrete, like other materials, expands in the direction transverse to that of the applied stress. The ratio of the transverse to the longitudinal strain is known as *Poisson's ratio* and depends somewhat on strength, composition, and other factors. At stresses lower than about $0.7f_c'$, Poisson's ratio for concrete falls within the limits of 0.15 to 0.20.

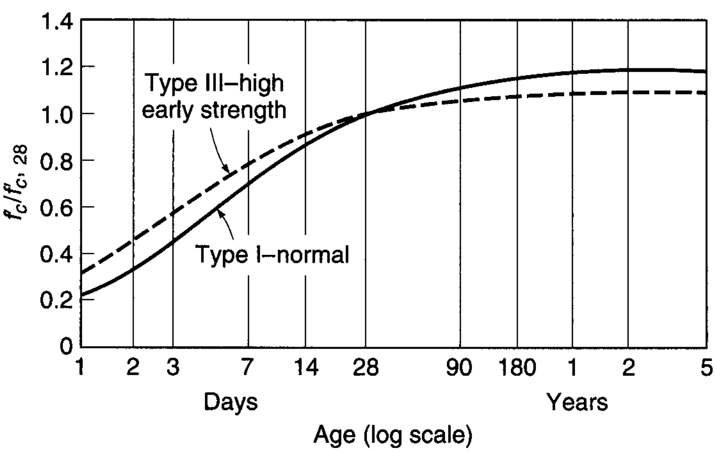


FIGURE 2.5
Effect of age on compressive strength f_c' for moist-cured concrete. (*Adapted from Ref. 2.26.*)

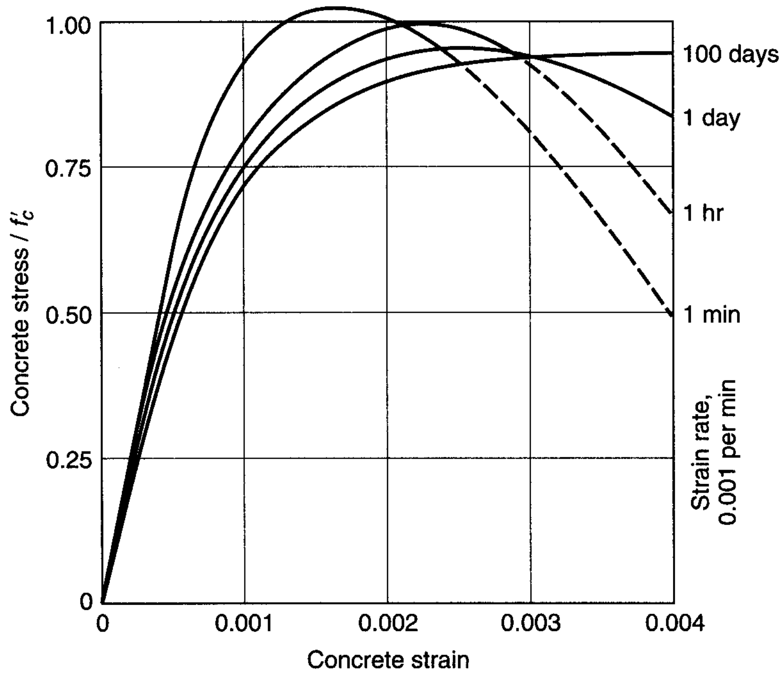


FIGURE 2.6
Stress-strain curves at various strain rates, concentric compression. (*Adapted from Ref. 2.27.*)

b. Long-Term Loading

In some engineering materials, such as steel, strength and the stress-strain relationships are independent of rate and duration of loading, at least within the usual ranges of rate of stress, temperature, and other variables. In contrast, Fig. 2.6 illustrates the fact that the influence of time, in this case of rate of loading, on the behavior of concrete under load is pronounced. The main reason is that concrete creeps under load, while steel does not exhibit creep under conditions prevailing in buildings, bridges, and similar structures.

Creep is the slow deformation of a material over considerable lengths of time at constant stress or load. The nature of the creep process is shown schematically in Fig. 2.7. This particular concrete was loaded after 28 days with resulting instantaneous strain ϵ_{inst}. The load was then maintained for 230 days, during which time creep was seen to have increased the total deformation to almost 3 times its instantaneous value. If the load were maintained, the deformation would follow the solid curve. If the load is removed, as shown by the dashed curve, most of the elastic instantaneous strain ϵ_{inst} is recovered, and some creep recovery is seen to occur. If the concrete is reloaded at some later date, instantaneous and creep deformations develop again, as shown.

Creep deformations for a given concrete are practically proportional to the magnitude of the applied stress; at any given stress, and even at the same ratio of stress to compressive strength, high-strength concretes show less creep than lower-strength concretes (Ref. 2.28). As seen in Fig. 2.7, with elapsing time, creep proceeds at a decreasing rate and ceases after 2 to 5 years at a final value which, depending on concrete strength and other factors, is about 1.2 to 3 times the magnitude of the instantaneous strain. If, instead of being applied quickly and thereafter kept constant, the load is increased slowly and gradually, as is the case in many structures during and after construction, then instantaneous and creep deformations proceed simultaneously.

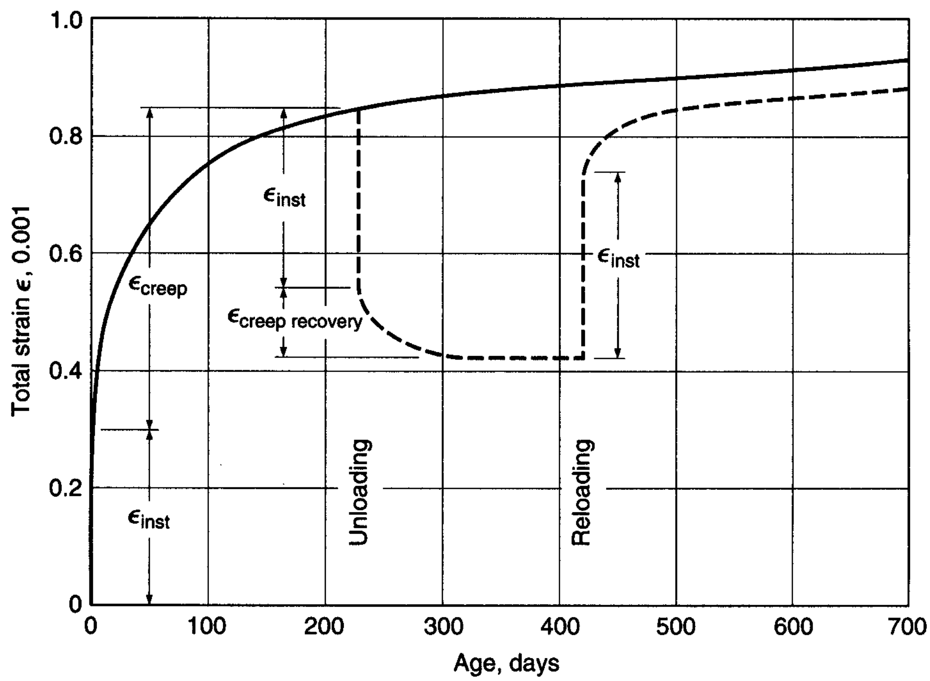


FIGURE 2.7
Typical creep curve (concrete loaded to 600 psi at age 28 days).

The effect is shown in Fig. 2.6; i.e., the previously discussed difference in the shape of the stress-strain curve for various rates of loading is chiefly the result of the creep deformation of concrete.

For stresses not exceeding about one-half the cylinder strength, creep strains are approximately proportional to stress. Because initial elastic strains are also proportional to stress in this range, this permits definition of the *creep coefficient*

$$C_{cu} = \frac{\epsilon_{cu}}{\epsilon_{ci}} \tag{2.6}$$

where ϵ_{cu} is the final asymptotic value of the additional creep strain and ϵ_{ci} is the initial, instantaneous strain when the load is first applied. Creep may also be expressed in terms of the *specific creep* δ_{cu}, defined as the additional time-dependent strain per psi stress. It can easily be shown that

$$C_{cu} = E_c\delta_{cu} \tag{2.7}$$

In addition to the stress level, creep depends on the average ambient relative humidity, being more than twice as large for 50 percent as for 100 percent humidity (Ref. 2.8). This is so because part of the reduction in volume under sustained load is caused by outward migration of free pore water, which evaporates into the surrounding atmosphere. Other factors of importance include the type of cement and aggregate, age of the concrete when first loaded, and concrete strength (Ref. 2.8). The creep coefficient for high-strength concrete is much less than that for low-strength concrete. However, sustained load stresses are apt to be higher so that the creep *deformation* may be as great for high-strength concrete, even though the creep coefficient is less.

The values of Table 2.2, quoted from Ref. 2.29 and extended for high-strength concrete based on research at Cornell University, are typical values for average humidity conditions, for concretes loaded at the age of 7 days.

TABLE 2.2
Typical creep parameters

Compressive Strength		Specific Creep δ_{cu}		
psi	MPa	10^{-6} per psi	10^{-6} per MPa	Creep coefficient C_{cu}
3,000	21	1.00	145	3.1
4,000	28	0.80	116	2.9
6,000	41	0.55	80	2.4
8,000	55	0.40	58	2.0
10,000	69	0.28	41	1.6
12,000	83	0.22	33	1.4

To illustrate, if the concrete in a column with $f_c' = 4000$ psi is subject to a long-time load that causes sustained stress of 1200 psi, then after several years under load the final value of the creep strain will be about $1200 \times 0.80 \times 10^{-6} = 0.00096$. Thus, if the column were 20 ft long, creep would shorten it by about $\frac{1}{4}$ in.

The creep coefficient at any time C_{ct} can be related to the ultimate creep coefficient C_{cu}. In Ref. 2.26, Branson suggests the equation

$$C_{ct} = \frac{t^{0.60}}{10 + t^{0.60}} C_{cu} \tag{2.8}$$

where t = time in days after loading.

In many special situations, e.g., slender members or frames, or in prestressed construction, the designer must take account of the combined effects of creep and shrinkage (Section 2.11). In such cases, rather than rely on the sample values of Table 2.2, more accurate information on creep parameters should be obtained, such as from Ref. 2.26 or 2.29.

Sustained loads affect not only the deformation but also the strength of concrete. The cylinder strength f_c' is determined at normal rates of test loading (about 35 psi/sec). Tests by Rüsch (Ref. 2.27) and at Cornell University (Refs. 2.30 and 2.31) have shown that, for concentrically loaded unreinforced concrete prisms and cylinders, the *strength under sustained load* is significantly smaller than f_c', on the order of 75 percent of f_c' for loads maintained for a year or more. Thus, a member subjected to a sustained overload causing compressive stress of over 75 percent of f_c' may fail after a period of time, even though the load is not increased.

c. Fatigue

When concrete is subject to fluctuating rather than sustained loading, its *fatigue strength*, as for all other materials, is considerably smaller than its static strength. When plain concrete in compression is stressed cyclically from zero to maximum stress, its fatigue limit is from 50 to 60 percent of the static compressive strength, for 2,000,000 cycles. A reasonable estimate can be made for other stress ranges using the modified Goodman diagram (see Ref. 2.29). For other types of applied stress, such as flexural compressive stress in reinforced concrete beams or flexural tension in unreinforced beams or on the tension side of reinforced beams, the fatigue limit likewise appears to be about 55 percent of the corresponding static strength. These figures, however, are for general guidance only. It is known that the fatigue strength of concrete depends not only on its static strength but also on moisture condition, age, and rate of loading (see Ref. 2.32).

2.9 PROPERTIES IN TENSION

While concrete is best employed in a manner that uses its favorable compressive strength, its behavior in tension is also important. The conditions under which cracks form and propagate on the tension side of reinforced concrete flexural members depend strongly on both the tensile strength and the fracture properties of the concrete, the latter dealing with the ease with which a crack progresses once it has formed. Concrete tensile stresses also occur as a result of shear, torsion, and other actions, and in most cases member behavior changes upon cracking. Thus, it is important to be able to predict, with reasonable accuracy, the tensile strength of concrete and to understand the factors that control crack propagation.

a. Tensile Strength

There are considerable experimental difficulties in determining the true tensile strength of concrete. In *direct tension* tests, minor misalignments and stress concentrations in the gripping devices are apt to mar the results. For many years, tensile strength has been measured in terms of the *modulus of rupture* f_r, the computed flexural tensile stress at which a test beam of plain concrete fractures. Because this nominal stress is computed on the assumption that concrete is an elastic material, and because this bending stress is localized at the outermost surface, it is apt to be larger than the strength of concrete in uniform axial tension. It is thus a measure of, but not identical with, the real axial tensile strength.

More recently the result of the *split-cylinder test* has established itself as a measure of the tensile strength of concrete. A concrete cylinder, the same as is used for compressive tests, is inserted in a compression testing machine in the horizontal position, so that compression is applied uniformly along two opposite generators. Pads are inserted between the compression platens of the machine and the cylinder to equalize and distribute the pressure. It can be shown that in an elastic cylinder so loaded, a nearly uniform tensile stress of magnitude $2P/(\pi dL)$ exists at right angles to the plane of load application. Correspondingly, such cylinders, when tested, split into two halves along that plane, at a stress f_{ct} that can be computed from the above expression. P is the applied compressive load at failure, and d and L are the diameter and length of the cylinder, respectively. Because of local stress conditions at the load lines and the presence of stresses at right angles to the aforementioned tension stresses, the results of the split-cylinder tests likewise are not identical with (but are believed to be a good measure of) the true axial tensile strength. The results of all types of tensile tests show considerably more scatter than those of compression tests.

Tensile strength, however determined, does not correlate well with the compressive strength f'_c. It appears that for sand-and-gravel concrete, the tensile strength depends primarily on the strength of bond between hardened cement paste and aggregate, whereas for lightweight concretes it depends largely on the tensile strength of the porous aggregate. The compressive strength, on the other hand, is much less determined by these particular characteristics.

Better correlation is found between the various measures of tensile strength and the square root of the compressive strength. The direct tensile strength, for example, ranges from about 3 to $5\sqrt{f'_c}$ for normal-density concretes, and from about 2 to $3\sqrt{f'_c}$ for all-lightweight concrete. Typical ranges of values for direct tensile strength, split-cylinder strength, and modulus of rupture are summarized in Table 2.3. In these expressions, f'_c is expressed in psi units, and the resulting tensile strengths are obtained in psi.

These approximate expressions show that tensile and compressive strengths are by no means proportional, and that any increase in compressive strength, such as that achieved by lowering the water-cement ratio, is accompanied by a much smaller percentage increase in tensile strength.

The ACI Code recommends that the modulus of rupture f_r be taken to equal $7.5\sqrt{f'_c}$ for normalweight concrete, and that this value be multiplied by 0.85 for "sand-lightweight" and 0.75 for "all-lightweight" concretes, giving values of $6.4\sqrt{f'_c}$ and $5.6\sqrt{f'_c}$, respectively, for those materials.

b. Tensile Fracture

The failure of concrete in tension involves both the formation and the propagation of cracks. The field of fracture mechanics deals with the latter. While reinforced concrete structures have been successfully designed and built for over 150 years without the use of fracture mechanics, the brittle response of high-strength concretes (Section 2.12), in tension as well as compression, increases the importance of the fracture properties of the material as distinct from tensile strength. Research dealing with the shear strength of high-strength concrete beams and the bond between reinforcing steel and high-strength concrete indicates relatively low increases in these structural properties with increases in concrete compressive strength (Refs. 2.33 and 2.34). While shear and bond strength are associated with the $\sqrt{f'_c}$ for normal-strength concrete, tests of high-strength concrete indicate that increases in shear and bond strengths are well below values predicted using $\sqrt{f'_c}$, indicating that concrete tensile strength alone is not the governing factor. An explanation for this behavior is provided by research at the University of Kansas and elsewhere (Refs. 2.35 and 2.36) that demonstrates that the energy required to fully open a crack (i.e., after the crack has started to grow) is largely independent of compressive strength, water-cement ratio, and age. Design expressions reflecting this research are not yet available. The behavior is, however, recognized in the ACI Code by limitations on the maximum value of $\sqrt{f'_c}$ that may be used to calculate shear and bond strength, as will be discussed in Chapters 4 and 5.

TABLE 2.3
Approximate range of tensile strengths of concrete

	Normalweight Concrete, psi	Lightweight Concrete, psi
Direct tensile strength f'_t	3 to $5\sqrt{f'_c}$	2 to $3\sqrt{f'_c}$
Split-cylinder strength f_{ct}	6 to $8\sqrt{f'_c}$	4 to $6\sqrt{f'_c}$
Modulus of rupture f_r	8 to $12\sqrt{f'_c}$	6 to $8\sqrt{f'_c}$

2.10 STRENGTH UNDER COMBINED STRESS

In many structural situations, concrete is subjected simultaneously to various stresses acting in various directions. For instance, in beams much of the concrete is subject simultaneously to compression and shear stresses, and in slabs and footings to compression in two perpendicular directions plus shear. By methods well known from the study of engineering mechanics, any state of combined stress, no matter how complex, can be reduced to three principal stresses acting at right angles to one another on an appropriately oriented elementary cube in the material. Any or all of the principal stresses can be either tension or compression. If any one of them is zero, a state of *biaxial* stress is said to exist; if two of them are zero, the state of stress is *uniaxial*, either simple compression or simple tension. In most cases, only the uniaxial strength properties of a material are known from simple tests, such as the cylinder strength f_c' and the tensile strength f_t'. For predicting the strengths of structures in which concrete is subject to biaxial or triaxial stress, it would be desirable to be able to calculate the strength of concrete in such states of stress, knowing from tests only either f_c' or f_c' and f_t'.

In spite of extensive and continuing research, no general theory of the strength of concrete under combined stress has yet emerged. Modifications of various strength theories, such as maximum stress, maximum strain, the Mohr-Coulomb, and the octahedral shear stress theories, all of which are discussed in structural mechanics texts, have been adapted with varying partial success to concrete. At present, none of these theories has been generally accepted, and many have obvious internal contradictions. The main difficulty in developing an adequate general strength theory lies in the highly nonhomogeneous nature of concrete, and in the degree to which its behavior at high stresses and at fracture is influenced by microcracking and other discontinuity phenomena (Refs. 2.8 and 2.37).

However, the strength of concrete has been well established by tests, at least for the biaxial stress state (Refs. 2.38 and 2.39). Results may be presented in the form of an interaction diagram such as Fig. 2.8, which shows the strength in direction 1 as a function of the stress applied in direction 2. All stresses are normalized in terms of the uniaxial compressive strength f_c'. It is seen that in the quadrant representing biaxial compression a strength increase as great as about 20 percent over the uniaxial compressive strength is attained, the amount of increase depending upon the ratio of f_2 to f_1. In the biaxial tension quadrant, the strength in direction 1 is almost independent of stress in direction 2. When tension in direction 2 is combined with compression in direction 1, the compressive strength is reduced almost linearly, and vice versa. For example, lateral compression of about one-half the uniaxial compressive strength will reduce the tensile strength by almost one-half compared with its uniaxial value. This fact is of great importance in predicting diagonal tension cracking in deep beams or shear walls, for example.

Experimental investigations into the triaxial strength of concrete have been few, due mainly to the practical difficulty of applying load in three directions simultaneously without introducing significant restraint from the loading equipment (Ref. 2.40). From information now available, the following conclusions can be drawn relative to the triaxial strength of concrete: (1) in a state of equal triaxial compression, concrete strength may be an order of magnitude larger than the uniaxial compressive strength; (2) for equal biaxial compression combined with a smaller value of compression in the third direction, a strength increase greater than 20 percent can be expected; and (3) for stress states including compression combined with tension in at least one other direction, the intermediate principal stress is of little consequence, and the compressive strength can be predicted safely based on Fig. 2.8.

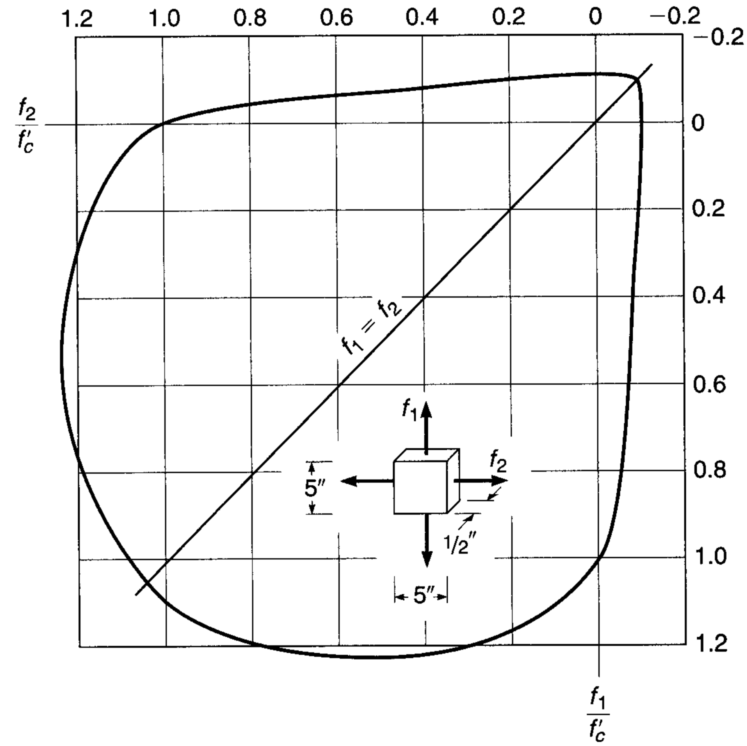


FIGURE 2.8
Strength of concrete in biaxial stress. (*Adapted from Ref. 2.39.*)

In fact, the strength of concrete under combined stress cannot yet be calculated rationally, and, equally important, in many situations in concrete structures it is nearly impossible to calculate all of the acting stresses and their directions; these are two of the main reasons for continued reliance on tests. Because of this, the design of reinforced concrete structures continues to be based more on extensive experimental information than on consistent analytical theory, particularly in the many situations where combined stresses occur.

2.11 SHRINKAGE AND TEMPERATURE EFFECTS

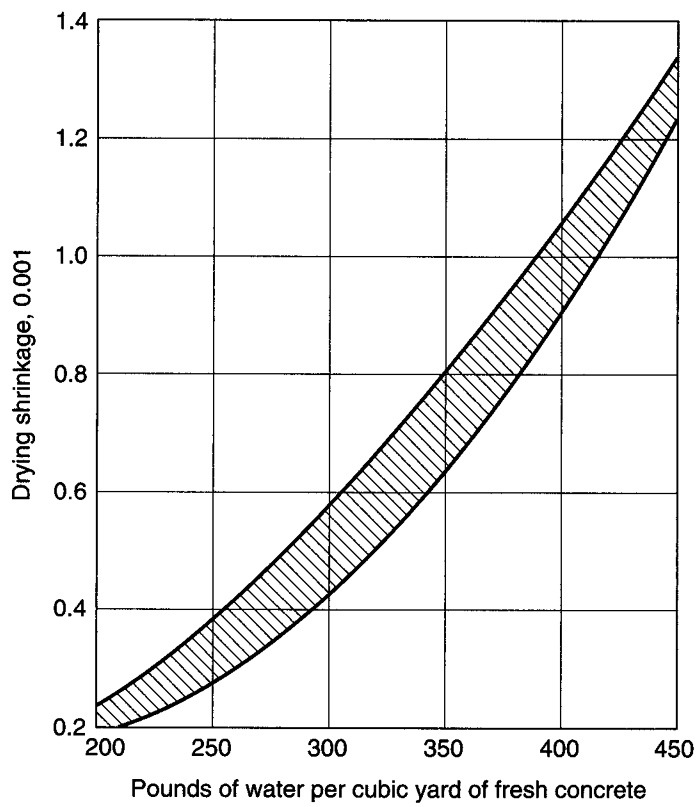
The deformations discussed in Section 2.8 were induced by stresses caused by external loads. Influences of a different nature cause concrete, even when free of any external loading, to undergo deformations and volume changes. The most important of these are shrinkage and the effects of temperature variations.

a. Shrinkage

As discussed in Sections 2.2 and 2.4, any workable concrete mix contains more water than is needed for hydration. If the concrete is exposed to air, the larger part of this free water evaporates in time, the rate and completeness of drying depending on ambient temperature and humidity conditions. As the concrete dries, it shrinks in volume, due initially to the capillary tension that develops in the water remaining in the concrete (Ref. 2.8). Conversely, if dry concrete is immersed in water, it expands, regaining much of the volume loss from prior shrinkage. Shrinkage, which continues at a decreasing rate for several months, depending on the configuration of the member, is a detrimental property of concrete in several respects. When not adequately controlled, it will cause unsightly and often deleterious cracks, as in slabs, walls, etc. In structures that are statically indeterminate (and most concrete structures are), it can cause large and harmful stresses. In prestressed concrete it leads to partial loss of initial prestress. For these reasons it is essential that shrinkage be minimized and controlled.

As is clear from the nature of the process, a key factor in determining the amount of final shrinkage is the unit water content of the fresh concrete. This is illustrated in Fig. 2.9, which shows the amount of shrinkage for varying amounts of mixing water. The same aggregates were used for all tests, but in addition to and independently of the water content, the amount of cement was also varied from 376 to 1034 lb/yd^3 of concrete. This very large variation of cement content causes a 20 to 30 percent variation in shrinkage strain for water contents between 250 to 350 lb/yd^3, the range used for most structural concretes. Increasing the cement content increases the cement paste constituent of the concrete, where the shrinkage actually takes place, while reducing the aggregate content. Since most aggregates do not contribute to shrinkage, an increase in aggregate content can significantly decrease shrinkage. This is shown in Fig. 2.10, which compares the shrinkage of concretes with various aggregate contents with the shrinkage obtained for neat cement paste (cement and water alone). For example, increasing the aggregate content from 71 to 74 percent (at the same water-cement ratio) results in a 20 percent reduction in shrinkage (Ref. 2.29). Increased aggregate content may be obtained through the use of (1) a larger maximum size coarse aggregate (which also reduces the water content required for a given workability), (2) a concrete with lower workability, and (3) chemical admixtures to increase

FIGURE 2.9
Effect of water content on drying shrinkage. *(From Ref. 2.3.)*



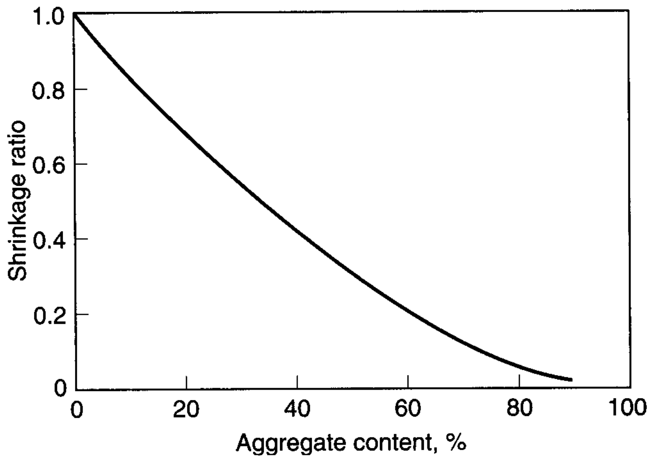


FIGURE 2.10
Influence of aggregate content in concrete (by volume) on the ratio of the shrinkage of concrete to the shrinkage of neat cement paste. (*Adapted from Ref. 2.29, based on data in Ref. 2.41.*)

workability at lower water contents. It is evident that an effective means of reducing shrinkage involves both a reduction in water content and an increase in aggregate content. In addition, prolonged and careful curing is beneficial for shrinkage control.

Values of final shrinkage for ordinary concretes are generally on the order of 400×10^{-6} to 800×10^{-6}, depending on the initial water content, ambient temperature and humidity conditions, and the nature of the aggregate. Highly absorptive aggregates with low moduli of elasticity, such as some sandstones and slates, result in shrinkage values 2 or more times those obtained with less absorptive materials, such as granites and some limestones. Some lightweight aggregates, in view of their great porosity, easily result in much larger shrinkage values than ordinary concretes.

For some purposes, such as predicting the time-dependent loss of force in prestressed concrete beams, it is important to estimate the amount of shrinkage as a function of time. Long-term studies (Ref. 2.26) show that, for moist-cured concrete at any time t after the initial 7 days, shrinkage can be predicted satisfactorily by the equation

$$\epsilon_{sh,t} = \frac{t}{35 + t}\,\epsilon_{sh,u} \tag{2.9}$$

where $\epsilon_{sh,t}$ is the unit shrinkage strain at time t in days and $\epsilon_{sh,u}$ is the ultimate value after a long period of time. Equation (2.9) pertains to "standard" conditions, defined in Ref. 2.26 to exist for humidity not in excess of 40 percent and for an average thickness of member of 6 in., and it applies both for normalweight and lightweight concretes. Modification factors are applied for nonstandard conditions, and separate equations are given for steam-cured members.

For structures in which a reduction in cracking is of particular importance, such as bridge decks, pavement slabs, and liquid storage tanks, the use of *expansive cement concrete* is appropriate. Shrinkage-compensating cement is constituted and proportioned such that the concrete will increase in volume after setting and during hardening. When the concrete is restrained by reinforcement or other means, the tendency to expand will result in compression. With subsequent drying, the shrinkage so produced, instead of causing a tension stress in the concrete that would result in cracking, merely reduces or relieves the expansive strains caused by the initial expansion (Ref. 2.42). Expansive cement is produced by adding a source of reactive aluminate to ordinary portland cement; approximately 90 percent of shrinkage-compensating cement is made up of the constituents of conventional portland cement.

Of the three main types of expansive cements produced, only type K is commercially available in the United States; it is about 20 percent more expensive than ordinary portland cement (Ref. 2.43). Requirements for expansive cement are given in ASTM C845, "Standard Specification for Expansive Hydraulic Cement." The usual admixtures can be used in shrinkage-compensating concrete, but trial mixes are necessary because some admixtures, particularly air-entraining agents, are not compatible with certain expansive cements.

b. Effect of Temperature Change

Like most other materials, concrete expands with increasing temperature and contracts with decreasing temperature. The effects of such volume changes are similar to those caused by shrinkage; i.e., temperature contraction can lead to objectionable cracking, particularly when superimposed on shrinkage. In indeterminate structures, deformations due to temperature changes can cause large and occasionally harmful stresses.

The coefficient of thermal expansion and contraction varies somewhat, depending upon the type of aggregate and richness of the mix. It is generally within the range of 4×10^{-6} to 7×10^{-6} per °F. A value of 5.5×10^{-6} is generally accepted as satisfactory for calculating stresses and deformations caused by temperature changes (Ref. 2.8).

2.12 HIGH-STRENGTH CONCRETE

There are a number of applications in which *high-strength concrete* will provide improved structural performance. Although the exact definition is arbitrary, the term generally refers to concrete having uniaxial compressive strength in the range of about 8000 to 20,000 psi or higher. Such concretes can be made using carefully selected but widely available cements, sands, and stone; certain admixtures including high-range water-reducing superplasticizers, fly ash, and silica fume; plus very careful quality control during production (Refs. 2.44 and 2.45). In addition to higher strength in compression, most other engineering properties are improved, leading to use of the alternative term *high-performance concrete*.

The most common application of high-strength concretes has been in the columns of tall concrete buildings, where normal concrete would result in unacceptably large cross sections, with loss of valuable floor space. It has been shown that the use of the more expensive high-strength concrete mixes in columns not only saves floor area but also is more economical than increasing the amount of steel reinforcement. Concrete of up to 12,000 psi was specified for the lower-story columns of 311 South Wacker Drive in Chicago (see Fig. 2.11), a pioneering structure with a total height of 946 ft. Formerly holding the height record, it has been superseded by taller buildings; the present record is held by the tallest building and the tallest structure of any type in the world, the Burj Dubai in Dubai, United Arab Emirates, shown in Fig. 18.2, which has a total height in excess of 2100 ft.

For bridges, too, smaller cross sections bring significant advantages, and the resulting reduction in dead load permits longer spans. The higher elastic modulus and lower creep coefficient result in reduced initial and long-term deflections, and in the case of prestressed concrete bridges, initial and time-dependent losses of prestress force are less. Other recent applications of high-strength concrete include offshore oil structures, parking garages, bridge deck overlays, dam spillways, warehouses, and heavy industrial slabs (Ref. 2.46).

FIGURE 2.11
311 South Wacker Drive, Chicago, which is among the world's tallest buildings. High-strength concrete with $f_c' =$ 12,000 psi was used in the lower stories. *(Courtesy of Portland Cement Association.)*

FIGURE 2.12
High-strength concrete test cylinder after uniaxial loading to failure; note the typically smooth fracture surface, with little aggregate interlock.

An essential requirement for high-strength concrete is a low water–cementitious material ratio. For normal concretes, this usually falls in the range from about 0.40 to 0.60 by weight, but for high-strength mixes it may be 0.25 or even lower. To permit proper placement of what would otherwise be a zero slump mix, high-range water-reducing admixtures, or superplasticizers, are essential and may increase slumps to as much as 6 or 8 in. Other additives include fly ash and, most notably, silica fume (see Section 2.7).

Much research in recent years has been devoted to establishing the fundamental and engineering properties of high-strength concretes, as well as the engineering characteristics of structural members made with the material (Refs. 2.33, 2.34, and 2.47 to 2.53). A large body of information is now available, permitting the engineer to use high-strength concrete with confidence when its advantages justify the higher cost. The compressive strength curves in Figs. 2.3 and 2.4 illustrate important differences compared with normal concrete, including a higher elastic modulus and an extended range of linear elastic response. Creep coefficients are reduced, as indicated in Table 2.2. Disadvantages include brittle behavior in compression (see Fig. 2.12), somewhat reduced ultimate strain capacity, and an increased tendency to crack when drying shrinkage is restrained (Ref. 2.54), the latter resulting from the lower creep exhibited by the material. Strength under sustained load is a higher fraction of standard cylinder strength (Refs. 2.30 and 2.31), and high-strength concrete exhibits improved durability and abrasion resistance (Refs. 2.51 and 2.55). As broader experience is gained in practical applications, and as design codes are gradually updated to recognize the special properties of higher-strength concretes now available, much wider use can be expected.

2.13 REINFORCING STEELS FOR CONCRETE

The useful strength of ordinary reinforcing steels in tension as well as compression, i.e., the yield strength, is about 15 times the compressive strength of common structural concrete and well over 100 times its tensile strength. On the other hand, steel is a high-cost material compared with concrete. It follows that the two materials are best used in combination if the concrete is made to resist the compressive stresses and the steel the tensile stresses. Thus, in reinforced concrete beams, the concrete resists the compressive force, longitudinal steel reinforcing bars are located close to the tension face to resist the tension force, and usually additional steel bars are so disposed that they resist the inclined tension stresses that are caused by the shear force in the beams. However, reinforcement is also used for resisting compressive forces primarily where it is desired to reduce the cross-sectional dimensions of compression members, as in the lower-floor columns of multistory buildings. Even if no such necessity exists, a minimum amount of reinforcement is placed in all compression members to safeguard them against the effects of small accidental bending moments that might crack and even fail an unreinforced member.

For most effective reinforcing action, it is essential that steel and concrete deform together, i.e., that there be a sufficiently strong *bond* between the two materials to ensure that no relative movements of the steel bars and the surrounding concrete occur. This bond is provided by the relatively large *chemical adhesion* that develops at the steel-concrete interface, by the *natural roughness* of the mill scale of hot-rolled reinforcing bars and by the closely spaced rib-shaped *surface deformations* with which reinforcing bars are furnished to provide a high degree of interlocking of the two materials.

Additional features that make for the satisfactory joint performance of steel and concrete are the following:

1. The *thermal expansion coefficients* of the two materials, about 6.5×10^{-6} for steel vs. an average of 5.5×10^{-6} for concrete, are sufficiently close to forestall cracking and other undesirable effects of differential thermal deformations.
2. While the *corrosion resistance* of bare steel is poor, the concrete that surrounds the steel reinforcement provides excellent corrosion protection, minimizing corrosion problems and corresponding maintenance costs.
3. The *fire resistance* of unprotected steel is impaired by its high thermal conductivity and by the fact that its strength decreases sizably at high temperatures. Conversely, the thermal conductivity of concrete is relatively low. Thus, damage caused by even prolonged fire exposure, if any, is generally limited to the outer layer of concrete, and a moderate amount of concrete cover provides sufficient thermal insulation for the embedded reinforcement.

Steel is used in two different ways in concrete structures: as reinforcing steel and as prestressing steel. Reinforcing steel is placed in the forms prior to casting of the concrete. Stresses in the steel, as in the hardened concrete, are caused only by the loads on the structure, except for possible parasitic stresses from shrinkage or similar causes. In contrast, in prestressed concrete structures, large tension forces are applied to the reinforcement prior to letting it act jointly with the concrete in resisting external loads. The steels for these two uses are very different and will be discussed separately.

2.14 REINFORCING BARS

The most common type of reinforcing steel (as distinct from prestressing steel) is in the form of round bars, often called *rebars*, available in a large range of diameters from about $\frac{3}{8}$ to $1\frac{3}{8}$ in. for ordinary applications and in two heavy bar sizes of about $1\frac{3}{4}$ and $2\frac{1}{4}$ in. These bars are furnished with surface deformations for the purpose of increasing resistance to slip between steel and concrete. Minimum requirements for these deformations (spacing, projection, etc.) have been developed in experimental research. Different bar producers use different patterns, all of which satisfy these requirements. Figure 2.13 shows a variety of current types of deformations.

For many years, bar sizes have been designated by numbers, Nos. 3 to 11 being commonly used and Nos. 14 and 18 representing the two special large-sized bars previously mentioned. Designation by number, instead of by diameter, was introduced because the surface deformations make it impossible to define a single easily measured value of the diameter. The numbers are so arranged that the unit in the number designation corresponds closely to the number of $\frac{1}{8}$ in. of diameter size. A No. 5 bar, for example, has a nominal diameter of $\frac{5}{8}$ in. Bar sizes are rolled into the surface of the bars for easy identification.

For a number of years, ASTM standards have included a second designation for bar size, the International System of Units (SI), with the size being identified using the nominal diameter in millimeters. To limit the number of bar designations, reinforcing bar producers in the United States have converted to SI for marking the bars. Thus, Nos. 3 to 11 bars are marked with Nos. 10 to 36, and Nos. 14 and 18 bars with Nos. 43 and 57. Both systems are still used in the ASTM standards, and the older, customary

FIGURE 2.13
Types of deformed reinforcing bars.

system is used in the 2008 ACI Code. To recognize the dual system of identifying and marking the bars, the customary bar designation system is retained throughout this text, followed by the SI bar designations in parentheses, such as No. 6 (No. 19). Table A.1 of Appendix A gives areas and weights of standard bars. Tables A.2 and A.3 give similar information for groups of bars.

a. Grades and Strengths

In reinforced concrete, a long-term trend is evident toward the use of higher-strength materials, both steel and concrete. Reinforcing bars with 40 ksi yield stress, once standard, have largely been replaced by bars with 60 ksi yield stress, both because they are more economical and because their use tends to reduce steel congestion in the forms. Bars with a yield stress of 75 ksi are often used in columns, and bars with a yield stress of 100 ksi are allowed to be used as confining reinforcement. Table 2.4 lists all presently available reinforcing steels, their grade designations, the ASTM specifications that define their properties (including deformations) in detail, and their two main minimum specified strength values. Grade 40 bars are no longer available in sizes larger than No. 6 (No. 19) and Grade 50 bars are available in sizes up to No. 8 (No. 25).†

The conversion to SI units described above also applies to the strength grades. Thus, Grade 40 is also designated as Grade 280 (for a yield strength of 280 MPa), Grade 60 is designated Grade 420, Grade 75 is designated Grade 520, and Grade 100 is designated Grade 690. The values 280, 420, 520, and 690 result in minimum yield strengths of 40.6, 60.9, 75.4, and 100.1 ksi; i.e., reinforcing steel is slightly stronger than implied by the grade in ksi. Grades based on inch-pound units will be used in this text.

Welding of reinforcing bars in making splices, or for convenience in fabricating reinforcing cages for placement in the forms, may result in metallurgical changes that reduce both strength and ductility, and special restrictions must be placed both on the type of steel used and the welding procedures. The provisions of ASTM A706 relate specifically to welding.

The ACI Code permits reinforcing steels up to f_y = 80 ksi for most applications. Such high-strength steels usually yield gradually but have no yield plateau (see Fig. 2.15). In this situation it is required that at the specified minimum yield strength the total strain not exceed 0.0035. This is necessary to make current design methods, which were developed for sharp-yielding steels with a yield plateau, applicable to such higher-strength steels. Under special circumstances, steel in this higher-strength range has its place, e.g., in lower-story columns of high-rise buildings.

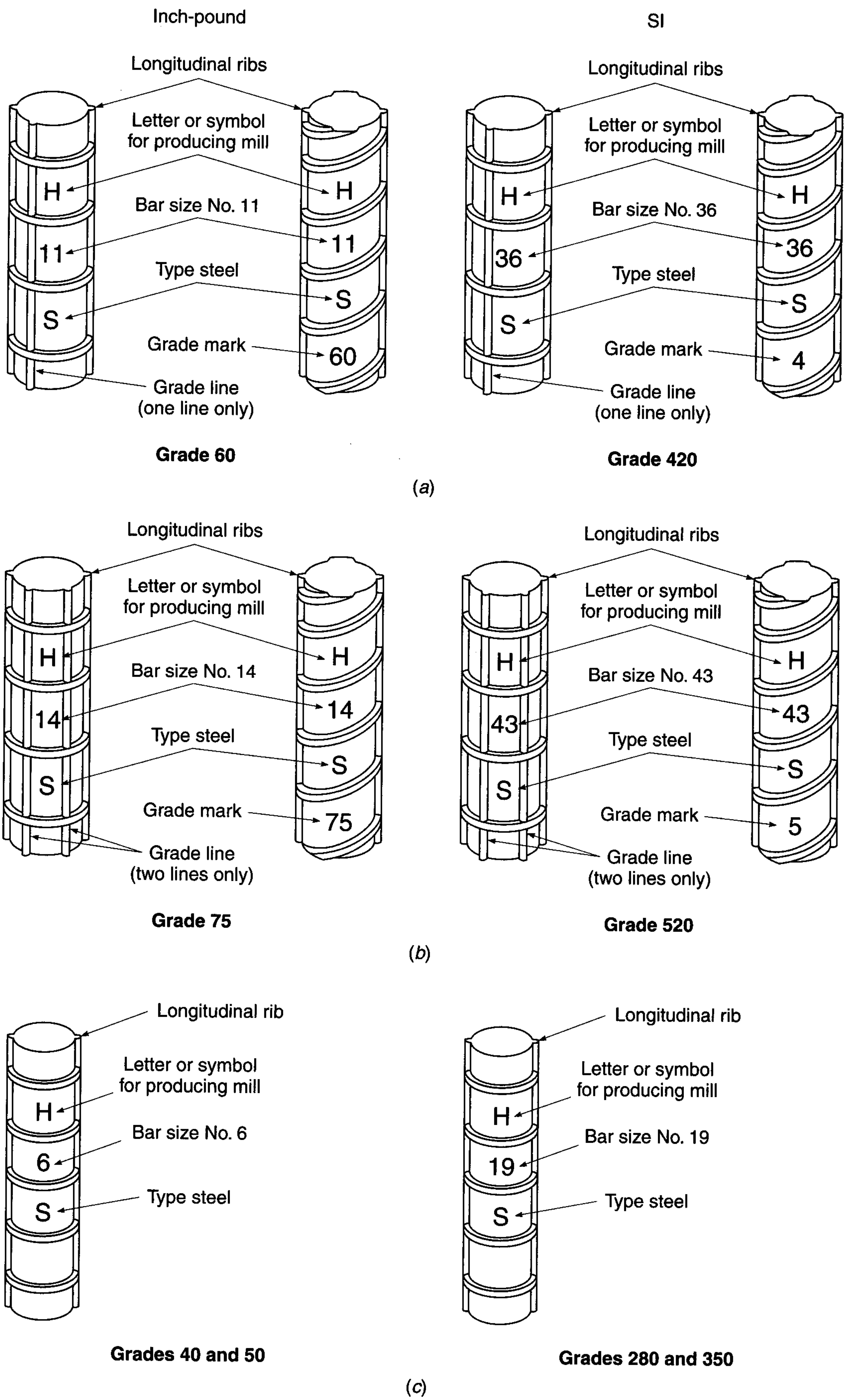
To allow bars of various grades and sizes to be easily distinguished, which is necessary to avoid accidental use of lower-strength or smaller-size bars than called for in the design, all deformed bars are furnished with rolled-in markings. These identify the producing mill (usually with an initial), the bar size (Nos. 3 to 18 under the inch-pound system and Nos. 10 to 57 under the SI), the type of steel (*S* for carbon steel, *W* for low-alloy steel, a rail sign for rail steel, *A* for axle steel, and *CS* for low-carbon chromium steel, corresponding, respectively, to ASTM Specifications A615, A706, A996 for both rail and axle steel, and A1035), and an additional marking to identify higher-strength steels. Grade 60 (420) bars have either one longitudinal line or the number 60 (4); Grade 75 (520) bars have either two longitudinal lines or the number 75 (5); Grade 100 (690) bars have either three longitudinal bars or the number 100 (6). The identification marks are shown in Fig. 2.14. As mentioned earlier, SI markings are used exclusively for bars rolled by mills in the United States.

† In practice, very little Grade 50 reinforcement is produced.

TABLE 2.4
Summary of minimum ASTM strength requirements

Product	ASTM Specification	Designation	Minimum Yield Strength, psi (MPa)	Minimum Tensile Strength, psi (MPa)
Reinforcing bars	A615	Grade 40 Grade 60 Grade 75	40,000 (280) 60,000 (420) 75,000 (520)	60,000 (420) 90,000 (620) 100,000 (690)
	A706	Grade 60	60,000 (420) [78,000 (540) maximum]	80,000 (550)[a]
	A996	Grade 40 Grade 50 Grade 60	40,000 (280) 50,000 (350) 60,000 (420)	60,000 (420) 80,000 (550) 90,000 (620)
	A1035	Grade 100	100,000 (690)	150,000 (1030)
Deformed bar mats	A184	Same as reinforcing bars		
Zinc-coated bars	A767	Same as reinforcing bars		
Epoxy-coated bars	A775, A934	Same as reinforcing bars		
Stainless-steel bars[b]	A955	Same as reinforcing bars		
Wire Plain	A82		70,000 (480)	80,000 (550)
Deformed	A496		75,000 (515)	85,000 (585)
Welded wire reinforcement Plain W1.2 and larger Smaller than W1.2	A185		65,000 (450) 56,000 (385)	75,000 (515) 70,000 (485)
Deformed	A497		70,000 (480)	80,000 (550)
Prestressing tendons Seven-wire strand	A416	Grade 250 (stress-relieved)	212,500 (1465)	250,000 (1725)
		Grade 250 (low-relaxation)	225,000 (1555)	250,000 (1725)
		Grade 270 (stress-relieved)	229,500 (1580)	270,000 (1860)
		Grade 270 (low-relaxation)	243,000 (1675)	270,000 (1860)
Wire	A421	Stress-relieved	199,750 (1375) to 212,500 (1465)[c]	235,000 (1620) to 250,000 (1725)[c]
		Low-relaxation	211,500 (1455) to 225,000 (1550)[c]	235,000 (1620) to 250,000 (1725)[c]
Bars	A722	Type I (plain) Type II (deformed)	127,500 (800) 120,000 (825)	150,000 (1035) 150,000 (1035)
Compacted strand[b]	A779	Type 245 Type 260 Type 270	241,900 (1480) 228,800 (1575) 234,900 (1620)	247,000 (1700) 263,000 (1810) 270,000 (1860)

[a] But not less than 1.25 times the actual yield strength.
[b] Not listed in ACI 318.
[c] Minimum strength depends on wire size.



Inch-pound
SI
Longitudinal ribs
Letter or symbol for producing mill
Bar size No. 11
Type steel
Grade mark
Grade line (one line only)
H
11
S
60
Grade 60
Longitudinal ribs
Letter or symbol for producing mill
Bar size No. 36
Type steel
Grade mark
Grade line (one line only)
H
36
S
4
Grade 420
(a)
Longitudinal ribs
Letter or symbol for producing mill
Bar size No. 14
Type steel
Grade mark
Grade line (two lines only)
H
14
S
75
Grade 75
Longitudinal ribs
Letter or symbol for producing mill
Bar size No. 43
Type steel
Grade mark
Grade line (two lines only)
H
43
S
5
Grade 520
(b)
Longitudinal rib
Letter or symbol for producing mill
Bar size No. 6
Type steel
H
6
S
Grades 40 and 50
Longitudinal rib
Letter or symbol for producing mill
Bar size No. 19
Type steel
H
19
S
Grades 280 and 350
(c)

FIGURE 2.14
Marking system for reinforcing bars meeting ASTM Specifications A615, A706, and A996: (*a*) Grades 60 and 420; (*b*) Grades 75 and 520; (*c*) Grades 40, 50, 280, and 350. (*Adapted from Ref. 2.56.*) (Facing page.)

b. Stress-Strain Curves

The two chief numerical characteristics that determine the character of bar reinforcement are its *yield point* (generally identical in tension and compression) and its *modulus of elasticity* E_s. The latter is practically the same for all reinforcing steels (but not for prestressing steels) and is taken as $E_s = 29{,}000{,}000$ psi.

In addition, however, the shape of the stress-strain curve, and particularly of its initial portion, has significant influence on the performance of reinforced concrete members. Typical stress-strain curves for U.S. reinforcing steels are shown in Fig. 2.15. The complete stress-strain curves are shown in the left part of the figure; the right part gives the initial portions of the curves magnified 10 times.

Low-carbon steels, typified by the Grade 40 curve, show an elastic portion followed by a *yield plateau*, i.e., a horizontal portion of the curve where strain continues to increase at constant stress. For such steels, the yield point is that stress at which the yield plateau establishes itself. With further strains, the stress begins to increase again, though at a slower rate, a process that is known as *strain-hardening*. The curve flattens out when the *tensile strength* is reached; it then turns down until fracture occurs. Higher-strength carbon steels, e.g., those with 60 ksi yield stress or higher, either have a yield plateau of much shorter length or enter strain-hardening immediately without any continued yielding at constant stress. In the latter case, the ACI Code specifies that the yield stress f_y be the stress corresponding to a strain of 0.0035, as shown in Fig. 2.15. Low-alloy, high-strength steels rarely show any yield plateau and usually enter strain-hardening immediately upon beginning to yield.

c. Fatigue Strength

In highway bridges and some other situations, both steel and concrete are subject to large numbers of stress fluctuations. Under such conditions, steel, just like concrete (Section 2.8c), is subject to *fatigue*. In metal fatigue, one or more microscopic cracks form after cyclic stress has been applied a significant number of times. These fatigue

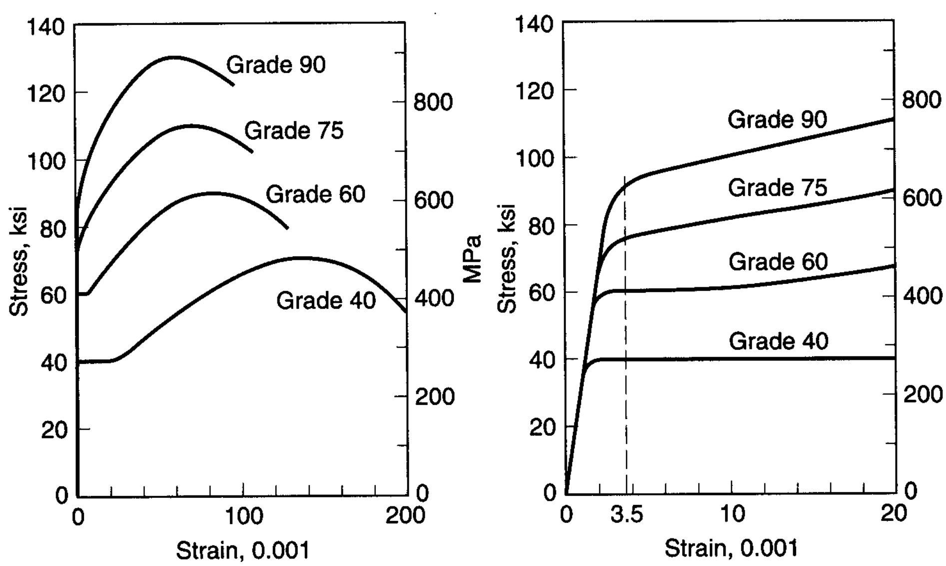


FIGURE 2.15
Typical stress-strain curves for reinforcing bars.

cracks occur at points of stress concentrations or other discontinuities and gradually increase with increasing numbers of stress fluctuations. This reduces the remaining uncracked cross-sectional area of the bar until it becomes too small to resist the applied force. At this point the bar fails in a sudden, brittle manner.

For reinforcing bars it has been found (Refs. 2.32 and 2.57) that the fatigue strength, i.e., the stress at which a given stress fluctuation between f_{max} and f_{min} can be applied 2 million times or more without causing failure, is practically independent of the grade of steel. It has also been found that the stress range, i.e., the algebraic difference between maximum and minimum stress, $f_1 = f_{max} - f_{min}$, that can be sustained without fatigue failure depends on f_{min}. Further, in deformed bars the degree of stress concentration at the location where the deformation joins the main cylindrical body of the bar tends to reduce the safe stress range. This stress concentration depends on the ratio r/h, where r is the base radius of the deformation and h its height. The radius r is the transition radius from the surface of the bar to that of the deformation; it is a fairly uncertain quantity that changes with roll wear as bars are being rolled.

On the basis of extensive tests (Ref. 2.57), the following formula has been developed for design:

$$f_r = 21 - 0.33 f_{min} + 8 \frac{r}{h} \tag{2.10}$$

where f_r = safe stress range, ksi

f_{min} = minimum stress; positive if tension, negative if compression

r/h = ratio of base radius to height of rolled-on deformation (in the common situation where r/h is not known, a value of 0.3 may be used)

Where bars are exposed to fatigue regimes, stress concentrations such as welds or sharp bends should be avoided since they may impair fatigue strength.

d. Coated Reinforcing Bars

Galvanized or epoxy-coated reinforcing bars are often specified to minimize corrosion of reinforcement and consequent spalling of concrete under severe environmental conditions, such as in bridge decks or parking garages subject to deicing chemicals, port and marine structures, and wastewater treatment plants.

ASTM A767, "Standard Specification for Zinc-Coated (Galvanized) Steel Bars for Concrete Reinforcement," includes requirements for the zinc coating material, the galvanizing process, the class or weight of coating, finish and adherence of coating, and the method of fabrication. Bars are usually galvanized after cutting and bending. Supplementary requirements pertain to coating of sheared ends and repair of damaged coating if bars are fabricated after galvanizing.

Epoxy-coated bars, presently more widely used than galvanized bars, are governed by ASTM A775, "Standard Specification for Epoxy-Coated Reinforcing Steel Bars," which includes requirements for the coating material, surface preparation prior to coating, method of application, and limits on coating thickness, and by ASTM A934, "Standard Specification for Epoxy-Coated Prefabricated Steel Reinforcing Bars." Under ASTM A775, the coating is applied to straight bars in a production-line operation, and the bars are cut and bent after coating. Under ASTM A934, bars are bent to final shape prior to coating. Cut ends and small spots of damaged coating are suitably repaired after fabrication. Extra care is required in the field to ensure that the coating is not damaged during shipment and placing and that repairs are made if necessary.

2.15 WELDED WIRE REINFORCEMENT

Apart from single reinforcing bars, *welded wire reinforcement* (also described as *welded wire fabric*) is often used for reinforcing slabs and other surfaces, such as shells, and for shear reinforcement in thin beam webs, particularly in prestressed beams. Welded wire reinforcement consists of sets of longitudinal and transverse cold-drawn steel wires at right angles to each other and welded together at all points of intersection. The size and spacing of wires may be the same in both directions or may be different, depending on the requirements of the design.

The notation used to describe the type and size of welded wire fabric involves a letter-number combination. ASTM uses the letter "W" to designate smooth wire and letter "D" to describe deformed wire. The number following the letter gives the cross-sectional area of the wire in hundredths of a square inch. For example, a W5.0 wire is a smooth wire with a cross-sectional area of 0.05 in^2. A W5.5 wire has a cross-sectional area of 0.055 in^2. D6.0 indicates a deformed wire with a cross-sectional area of 0.06 in^2. Welded wire fabric having a designation $4 \times 4 -$ W5.0 $\times$ W5.0 has wire spacings 4 in. in each way with smooth wire of cross-sectional area 0.05 in^2 in each direction. Sizes and spacings for common types of welded wire fabric and cross-sectional areas of steel per foot, as well as weight per 100 ft^2, are shown in Table A.12 of Appendix A.

ASTM Specifications A185 and A497 pertain to smooth and deformed welded wire fabric, respectively, as shown in Table 2.4. Because the yield stresses shown are specified at a strain of 0.005, the ACI Code requires that f_y be taken equal to 60 ksi unless the stress at a strain of 0.0035 is used.

2.16 PRESTRESSING STEELS

Prestressing steel is used in three forms: round wires, stranded cable, and alloy steel bars. Prestressing wire ranges in diameter from 0.192 to 0.276 in. It is made by cold-drawing high-carbon steel after which the wire is stress-relieved by heat treatment to produce the prescribed mechanical properties. Wires are normally bundled in groups of up to about 50 individual wires to produce prestressing tendons of the required strength. Stranded cable, more common than wire in U.S. practice, is fabricated with six wires wound around a seventh of slightly larger diameter. The pitch of the spiral winding is between 12 and 16 times the nominal diameter of the strand. Strand diameters range from 0.250 to 0.700 in. Alloy steel bars for prestressing are available in diameters from 0.750 to 1.375 in. as plain round bars and from 0.625 to 2.50 in. as deformed bars. Specific requirements for prestressing steels are found in ASTM A421, "Standard Specification for Uncoated Stress-Relieved Steel Wire for Prestressed Concrete"; ASTM A416, "Standard Specification for Steel Strand, Uncoated Seven-Wire Stress-Relieved for Prestressed Concrete"; and ASTM A722, "Standard Specification for Uncoated High-Strength Steel Bar for Prestressing Concrete." Table A.15 of Appendix A provides design information for U.S. prestressing steels.

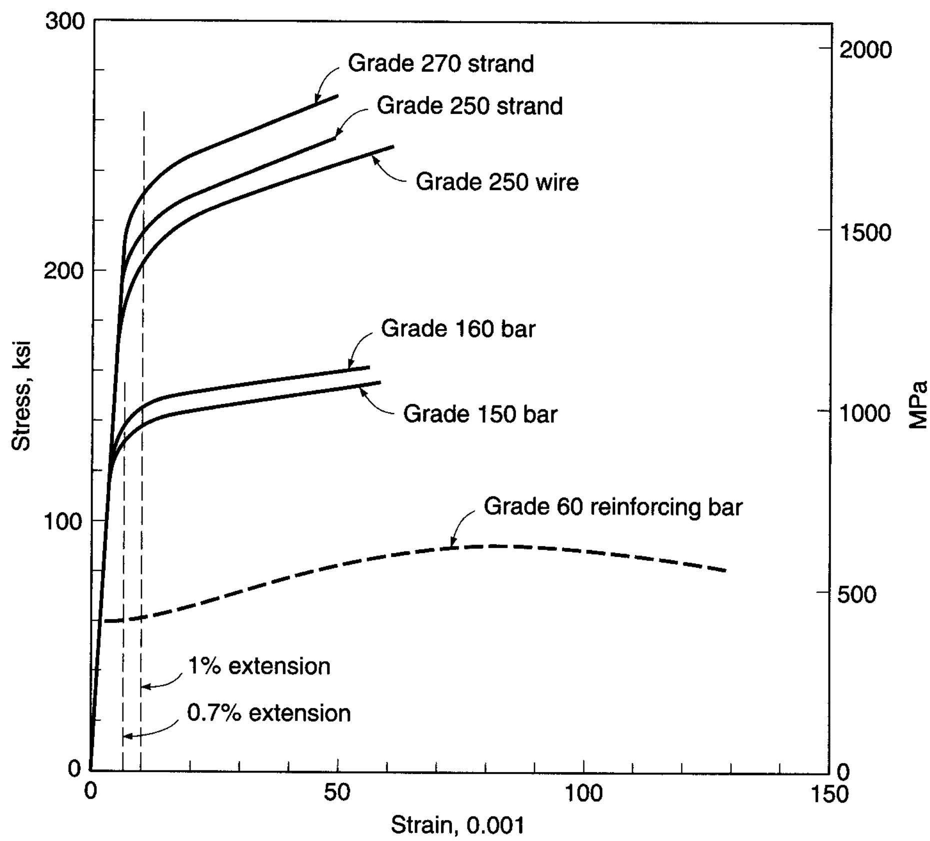
a. Grades and Strengths

The tensile strengths of prestressing steels range from about 2.5 to 6 times the yield strengths of commonly used reinforcing bars. The grade designations correspond to the minimum specified tensile strength in ksi. For the widely used seven-wire strand, three grades are available: Grade 250 (f_{pu} = 250 ksi), Grade 270, and Grade 300, although the last is not yet recognized in ASTM A421. Grade 270 strand is used most often. For alloy steel bars, two grades are used: the regular Grade 150 is most common, but special Grade 160 bars may be ordered. Round wires may be obtained in Grades 235, 240, and 250, depending on diameter.

b. Stress-Strain Curves

Figure 2.16 shows stress-strain curves for prestressing wires, strand, and alloy bars of various grades. For comparison, the stress-strain curve for a Grade 60 reinforcing bar is also shown. It is seen that, in contrast to reinforcing bars, prestressing steels do not show a sharp yield point or yield plateau; i.e., they do not yield at constant or nearly constant stress. Yielding develops gradually, and in the inelastic range the curve continues to rise smoothly until the tensile strength is reached. Because well-defined yielding is not observed in these steels, the yield strength is somewhat arbitrarily defined as the stress at a total elongation of 1 percent for strand and wire and at 0.7 percent for alloy steel bars. Figure 2.16 shows that the yield strengths so defined represent a good limit below which stress and strain are fairly proportional, and above which strain increases much more rapidly with increasing stress. It is also seen that the spread between tensile strength and yield strength is smaller in prestressing steels than in reinforcing steels. It may further be noted that prestressing steels have significantly less ductility.

FIGURE 2.16
Typical stress-strain curves for prestressing steels.



While the modulus of elasticity E_s for bar reinforcement is taken as 29,000,000 psi, the effective modulus of prestressing steel varies, depending on the type of steel (e.g., strand vs. wire or bars) and type of use, and is best determined by test or supplied by the manufacturer. For unbonded strand (i.e., strand not embedded in concrete), the modulus may be as low as 26,000,000 psi. For bonded strand, E_s is usually about 27,000,000 psi, while for smooth round wires E_s is about 29,000,000 psi, the same as for reinforcing bars. The elastic modulus of alloy steel bars is usually taken as $E_s = 27{,}000{,}000$ psi.

c. Relaxation

When prestressing steel is stressed to the levels that are customary during initial tensioning and at service loads, it exhibits a property known as *relaxation*. Relaxation is defined as the loss of stress in stressed material held at constant length. (The same basic phenomenon is known as creep when defined in terms of change in strain of a material under constant stress.) To be specific, if a length of prestressing steel is stressed to a sizable fraction of its yield strength f_{py} (say, 80 to 90 percent) and held at a constant strain between fixed points such as the ends of a beam, the steel stress f_p will gradually decrease from its initial value f_{pi}. In prestressed concrete members this stress relaxation is important because it modifies the internal stresses in the concrete and changes the deflections of the beam some time after initial prestress was applied.

The amount of relaxation varies, depending on the type and grade of steel, the time under load, and the initial stress level. A satisfactory estimate for ordinary stress-relieved strand and wires can be obtained from Eq. (2.11), which was derived from more than 400 relaxation tests of up to 9 years' duration:

$$\frac{f_p}{f_{pi}} = 1 - \frac{\log t}{10}\left(\frac{f_{pi}}{f_{py}} - 0.55\right) \tag{2.11}$$

where f_p is the final stress after t hours, f_{pi} is the initial stress, and f_{py} is the nominal yield stress (Ref. 2.58). In Eq. (2.11), $\log t$ is to the base 10, and f_{pi}/f_{py} not less than 0.55; below that value essentially no relaxation occurs.

The tests on which Eq. (2.11) is based were carried out on round, stress-relieved wires and are equally applicable to stress-relieved strand. In the absence of other information, results may be applied to alloy steel bars as well.

Low-relaxation strand has replaced stress-relieved strand as the industry standard. According to ASTM A416, such steel must exhibit relaxation after 1000 hours of not more than 2.5 percent when initially stressed to 70 percent of specified tensile strength and not more than 3.5 percent when loaded to 80 percent of tensile strength. For low-relaxation strand, Eq. (2.11) is replaced by

$$\frac{f_p}{f_{pi}} = 1 - \frac{\log t}{45}\left(\frac{f_{pi}}{f_{py}} - 0.55\right) \tag{2.12}$$

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PROBLEMS

2.1. The specified concrete strength f_c' for a new building is 6000 psi. Calculate the required average strength f_c' for the concrete (*a*) if there are no prior test results for concrete with a compressive strength within 1000 psi of f_c' made with similar materials, (*b*) if 20 test results for concrete with $f_c' = 5000$ psi made with similar materials produce a sample standard deviation s_s of 580 psi, and (*c*) if 30 tests with $f_c' = 5500$ psi made with similar materials produce a sample standard deviation s_s of 590 psi.

2.2. Ten consecutive strength tests are available for a new concrete mixture with $f_c' = 4000$ psi: 4590, 4750, 5280, 4210, 4460, 4170, 3750, 5110, 4640, and 4170 psi.

(*a*) Do the strength results represent concrete of satisfactory quality? Explain your reasoning.
(*b*) If f'_{cr} has been selected based on 30 consecutive test results from an earlier project with a sample standard deviation s_s of 510 psi, must the mixture proportions be adjusted? Explain.

3 Flexural Analysis and Design of Beams

3.1 INTRODUCTION

The fundamental assumptions upon which the analysis and design of reinforced concrete members are based were introduced in Section 1.8, and the application of those assumptions to the simple case of axial loading was developed in Section 1.9. The student should review Sections 1.8 and 1.9 at this time. In developing methods for the analysis and design of beams in this chapter, the same assumptions apply, and identical concepts will be used. This chapter will include analysis and design for flexure, including the dimensioning of the concrete cross section and the selection and placement of reinforcing steel. Other important aspects of beam design including shear reinforcement, bond, and anchorage of reinforcing bars, and the important questions of serviceability (e.g., limiting deflections and controlling concrete cracking) will be treated in Chapters 4, 5, and 6.

3.2 BENDING OF HOMOGENEOUS BEAMS

Reinforced concrete beams are nonhomogeneous in that they are made of two entirely different materials. The methods used in the analysis of reinforced concrete beams are therefore different from those used in the design or investigation of beams composed entirely of steel, wood, or any other structural material. The fundamental principles involved are, however, essentially the same. Briefly, these principles are as follows.

At any cross section there exist internal forces that can be resolved into components normal and tangential to the section. Those components that are normal to the section are the *bending* stresses (tension on one side of the neutral axis and compression on the other). Their function is to resist the bending moment at the section. The tangential components are known as the *shear* stresses, and they resist the transverse or shear forces.

Fundamental assumptions relating to flexure and flexural shear are as follows:

1. A cross section that was plane before loading remains plane under load. This means that the unit strains in a beam above and below the neutral axis are proportional to the distance from that axis.
2. The bending stress f at any point depends on the strain at that point in a manner given by the stress-strain diagram of the material. If the beam is made of a homogeneous material whose stress-strain diagram in tension and compression is that of Fig. 3.1a, the following holds. If the maximum strain at the outer fibers is smaller than the strain ϵ_p up to which stress and strain are proportional for the

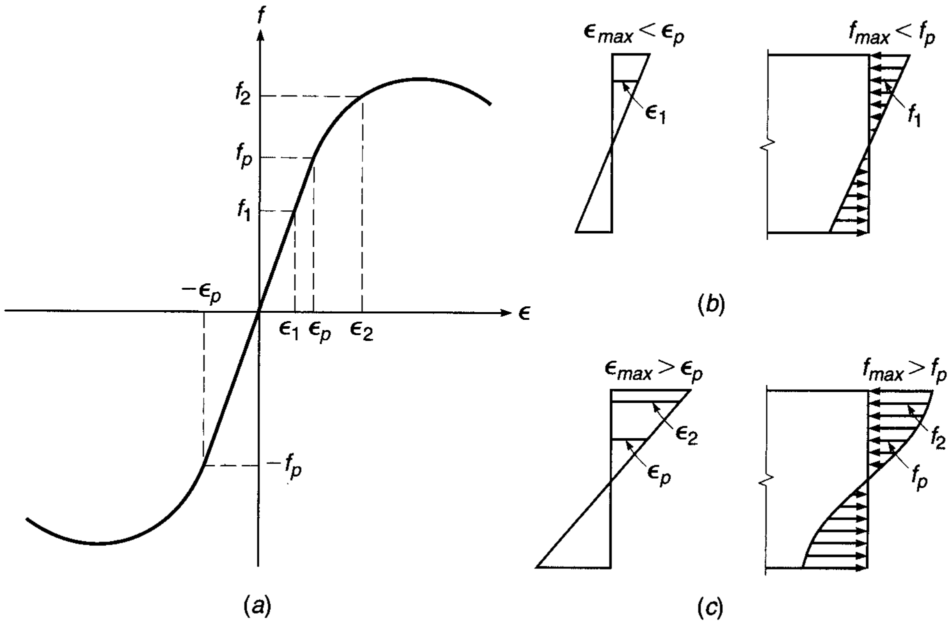


FIGURE 3.1
Elastic and inelastic stress distributions in homogeneous beams.

given material, then the compression and tension stresses on either side of the axis are proportional to the distance from the axis, as shown in Fig. 3.1*b*. However, if the maximum strain at the outer fibers is larger than ϵ_p, this is no longer true. The situation that then occurs is shown in Fig. 3.1*c*; i.e., in the outer portions of the beam, where $\epsilon > \epsilon_p$, stresses and strains are no longer proportional. In these regions, the magnitude of stress at any level, such as f_2 in Fig. 3.1*c*, depends on the strain ϵ_2 at that level in the manner given by the stress-strain diagram of the material. In other words, for a given strain in the beam, the stress at a point is the same as that given by the stress-strain diagram for the same strain.

3. The distribution of the shear stresses v over the depth of the section depends on the shape of the cross section and of the stress-strain diagram. These shear stresses are largest at the neutral axis and equal to zero at the outer fibers. The shear stresses on horizontal and vertical planes through any point are equal.
4. Owing to the combined action of shear stresses (horizontal and vertical) and flexure stresses, at any point in a beam there are inclined stresses of tension and compression, the largest of which form an angle of 90° with each other. The intensity of the inclined maximum or principal stress at any point is given by

$$t = \frac{f}{2} \pm \sqrt{\frac{f^2}{4} + v^2} \tag{3.1}$$

where f = intensity of normal fiber stress
v = intensity of tangential shearing stress

The inclined stress makes an angle α with the horizontal such that $\tan 2\alpha = 2v/f$.

5. Since the horizontal and vertical shearing stresses are equal and the flexural stresses are zero at the neutral plane, the inclined tensile and compressive stresses at any point in that plane form an angle of 45° with the horizontal, the intensity of each being equal to the unit shear at the point.

6. When the stresses in the outer fibers are smaller than the proportional limit f_p, the beam behaves *elastically*, as shown in Fig. 3.1*b*. In this case the following pertains:
   (a) The neutral axis passes through the center of gravity of the cross section.
   (b) The intensity of the bending stress normal to the section increases directly with the distance from the neutral axis and is a maximum at the extreme fibers. The stress at any given point in the cross section is represented by the equation

$$f = \frac{My}{I} \tag{3.2}$$

where f = bending stress at a distance y from neutral axis
M = external bending moment at section
I = moment of inertia of cross section about neutral axis

The maximum bending stress occurs at the outer fibers and is equal to

$$f_{\max} = \frac{Mc}{I} = \frac{M}{S} \tag{3.3}$$

where c = distance from neutral axis to outer fiber
$S = I/c$ = section modulus of cross section

   (c) The shear stress (horizontal equals vertical) v at any point in the cross section is given by

$$v = \frac{VQ}{Ib} \tag{3.4}$$

where V = total shear at section
Q = statical moment about neutral axis of that portion of cross section lying between a line through point in question parallel to neutral axis and nearest face (upper or lower) of beam
I = moment of inertia of cross section about neutral axis
b = width of beam at a given point

   (d) The intensity of shear along a vertical cross section in a rectangular beam varies as the ordinates of a parabola, the intensity being zero at the outer fibers of the beam and a maximum at the neutral axis. For a total depth h, the maximum is $\frac{3}{2}V/bh$, since at the neutral axis $Q = bh^2/8$ and $I = bh^3/12$ in Eq. (3.4).

The remainder of this chapter deals only with bending stresses and their effects on reinforced concrete beams. Shear stresses and their effects are discussed separately in Chapter 4.

3.3 REINFORCED CONCRETE BEAM BEHAVIOR

Plain concrete beams are inefficient as flexural members because the tensile strength in bending (modulus of rupture, see Section 2.9) is a small fraction of the compressive strength. As a consequence, such beams fail on the tension side at low loads long before the strength of the concrete on the compression side has been fully utilized. For this reason, steel reinforcing bars are placed on the tension side as close to the extreme tension fiber as is compatible with proper fire and corrosion protection of the steel. In such a reinforced concrete beam, the tension caused by the bending moments is chiefly

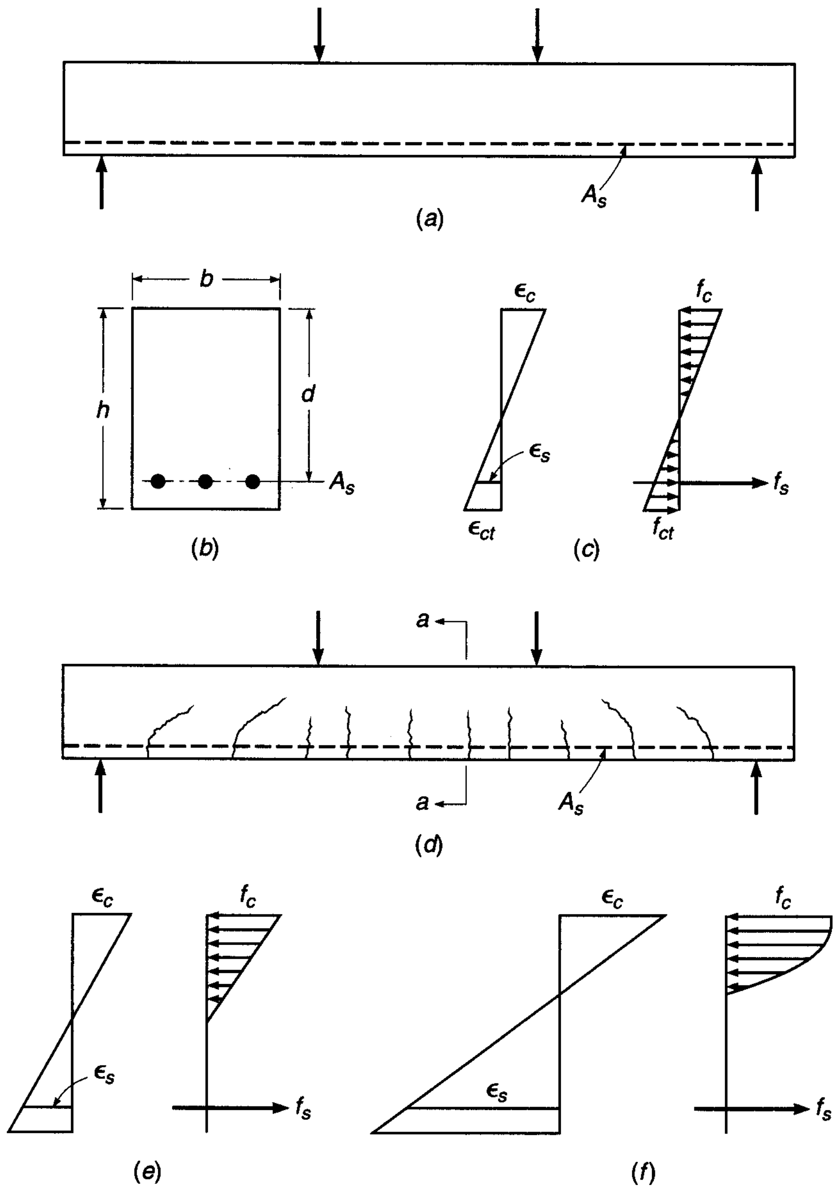


FIGURE 3.2
Behavior of reinforced concrete beam under increasing load.

resisted by the steel reinforcement, while the concrete alone is usually capable of resisting the corresponding compression. Such joint action of the two materials is ensured if relative slip is prevented. This is achieved by using deformed bars with their high bond strength at the steel-concrete interface (see Section 2.14) and, if necessary, by special anchorage of the ends of the bars. A simple example of such a beam, with the customary designations for the cross-sectional dimensions, is shown in Fig. 3.2. For simplicity, the discussion that follows will deal with beams of rectangular cross section, even though members of other shapes are very common in most concrete structures.

When the load on such a beam is gradually increased from zero to the magnitude that will cause the beam to fail, several different stages of behavior can be clearly distinguished. At low loads, as long as the maximum tensile stress in the concrete is smaller than the modulus of rupture, the entire concrete is effective in resisting stress, in compression on one side and in tension on the other side of the neutral axis. In addition, the reinforcement, deforming the same amount as the adjacent concrete, is also subject to tensile stresses. At this stage, all stresses in the concrete are of small magnitude and are proportional to strains. The distribution of strains and stresses in concrete and steel over the depth of the section is shown in Fig. 3.2*c*.

When the load is further increased, the tensile strength of the concrete is soon reached, and at this stage tension cracks develop. These propagate quickly upward to or close to the level of the neutral plane, which in turn shifts upward with progressive cracking. The general shape and distribution of these tension cracks is shown in Fig. 3.2*d*. In well-designed beams, the width of these cracks is so small (hairline cracks) that they are not objectionable from the viewpoint of either corrosion protection or appearance. Their presence, however, profoundly affects the behavior of the beam under load. Evidently, in a cracked section, i.e., in a cross section located at a crack such as *a-a* in Fig. 3.2*d*, the concrete does not transmit any tensile stresses. Hence, just as in tension members (Section 1.9b), the steel is called upon to resist the entire tension. At moderate loads, if the concrete stresses do not exceed approximately $f_c'/2$, stresses and strains continue to be closely proportional (see Fig. 1.16). The distribution of strains and stresses at or near a cracked section is then that shown in Fig. 3.2*e*. When the load is still further increased, stresses and strains rise correspondingly and are no longer proportional. The ensuing nonlinear relation between stresses and strains is that given by the concrete stress-strain curve. Therefore, just as in homogeneous beams (see Fig. 3.1), the distribution of concrete stresses on the compression side of the beam is of the same shape as the stress-strain curve. Figure 3.2*f* shows the distribution of strains and stresses close to the ultimate load.

Eventually, the carrying capacity of the beam is reached. Failure can be caused in one of two ways. When relatively moderate amounts of reinforcement are employed, at some value of the load the steel will reach its yield point. At that stress, the reinforcement yields suddenly and stretches a large amount (see Fig. 2.15), and the tension cracks in the concrete widen visibly and propagate upward, with simultaneous significant deflection of the beam. When this happens, the strains in the remaining compression zone of the concrete increase to such a degree that crushing of the concrete, the *secondary compression failure*, ensues at a load only slightly larger than that which caused the steel to yield. Effectively, therefore, attainment of the yield point in the steel determines the carrying capacity of moderately reinforced beams. Such yield failure is gradual and is preceded by visible signs of distress, such as the widening and lengthening of cracks and the marked increase in deflection.

On the other hand, if large amounts of reinforcement or normal amounts of steel of very high strength are employed, the compressive strength of the concrete may be exhausted before the steel starts yielding. Concrete fails by crushing when strains become so large that they disrupt the integrity of the concrete. Exact criteria for this occurrence have yet to be established, but it has been observed that rectangular beams fail in compression when the concrete strains reach values of about 0.003 to 0.004. Compression failure through crushing of the concrete is sudden, of an almost explosive nature, and occurs without warning. For this reason it is good practice to dimension beams in such a manner that should they be overloaded, failure would be initiated by yielding of the steel rather than by crushing of the concrete.

The analysis of stresses and strength in the different stages just described will be discussed in the next several sections.

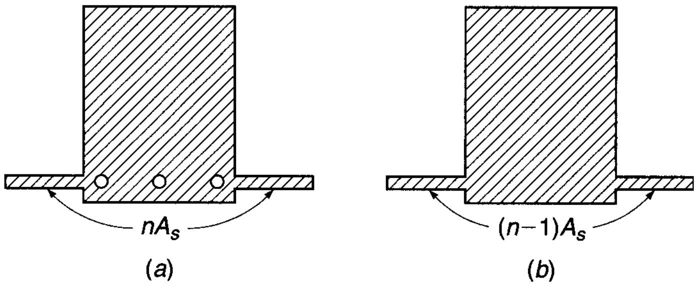


FIGURE 3.3
Uncracked transformed beam section.

a. Stresses Elastic and Section Uncracked

As long as the tensile stress in the concrete is smaller than the modulus of rupture, so that no tension cracks develop, the strain and stress distribution as shown in Fig. 3.2*c* is essentially the same as in an elastic, homogeneous beam (Fig. 3.1*b*). The only difference is the presence of another material, the steel reinforcement. As shown in Section 1.9a, in the elastic range, for any given value of strain, the stress in the steel is n times that of the concrete [Eq. (1.6)]. In the same section, it was shown that one can take account of this fact in calculations by replacing the actual steel-and-concrete cross section with a fictitious section thought of as consisting of concrete only. In this "transformed section," the actual area of the reinforcement is replaced with an equivalent concrete area equal to nA_s located at the level of the steel. The transformed, uncracked section pertaining to the beam of Fig. 3.2*b* is shown in Fig. 3.3.

Once the transformed section has been obtained, the usual methods of analysis of elastic homogeneous beams apply. That is, the section properties (location of neutral axis, moment of inertia, section modulus, etc.) are calculated in the usual manner, and, in particular, stresses are computed with Eqs. (3.2) to (3.4).

EXAMPLE 3.1 A rectangular beam has the dimensions (see Fig. 3.2*b*) $b = 10$ in., $h = 25$ in., and $d = 23$ in. and is reinforced with three No. 8 (No. 25) bars so that $A_s = 2.37$ in^2. The concrete cylinder strength f_c' is 4000 psi, and the tensile strength in bending (modulus of rupture) is 475 psi. The yield point of the steel f_y is 60,000 psi, the stress-strain curves of the materials being those of Fig. 1.16. Determine the stresses caused by a bending moment $M = 45$ ft-kips.

Solution. With a value $n = E_s/E_c = 29{,}000{,}000/3{,}600{,}000 = 8$, one has to add to the rectangular outline an area $(n-1)A_s = 7 \times 2.37 = 16.59$ in^2, disposed as shown on Fig. 3.4, to obtain the uncracked, transformed section. Conventional calculations show that the location of the neutral axis of this section is given by $\bar{y} = 13.2$ in. from the top of the section, and its moment of inertia about this axis is 14,740 in^4. For $M = 45$ ft-kips $= 540{,}000$ in-lb, the concrete compression stress at the top fiber is, from Eq. (3.3),

$$f_c = \frac{M\bar{y}}{I} = \frac{540{,}000 \times 13.2}{14{,}740} = 484 \text{ psi}$$

and, similarly, the concrete tension stress at the bottom fiber, 11.8 in. from the neutral axis, is

$$f_{ct} = \frac{540{,}000 \times 11.8}{14{,}740} = 432 \text{ psi}$$

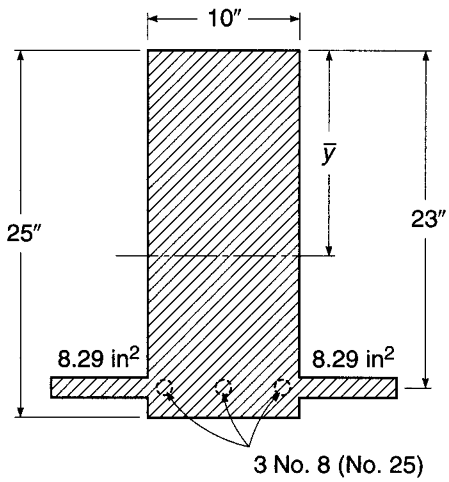


FIGURE 3.4
Transformed beam section of Example 3.1.

Since this value is below the given tensile bending strength of the concrete, 475 psi, no tension cracks will form, and calculation by the uncracked, transformed section is justified. The stress in the steel, from Eqs. (1.6) and (3.2), is

$$f_s = n\frac{My}{I} = 8\left(\frac{540{,}000 \times 9.8}{14{,}740}\right) = 2870 \text{ psi}$$

By comparing f_c and f_s with the concrete cylinder strength and the yield point, respectively, it is seen that at this stage the actual stresses are quite small compared with the available strengths of the two materials.

b. Stresses Elastic and Section Cracked

When the tensile stress f_{ct} exceeds the modulus of rupture, cracks form, as shown in Fig. 3.2*d*. If the concrete compressive stress is less than approximately $\frac{1}{2}f_c'$ and the steel stress has not reached the yield point, both materials continue to behave elastically, or very nearly so. This situation generally occurs in structures under normal service conditions and loads, since at these loads the stresses are generally of the order of magnitude just discussed. At this stage, for simplicity and with little if any error, it is assumed that tension cracks have progressed all the way to the neutral axis and that sections plane before bending are plane in the deformed member. The situation with regard to strain and stress distribution is that shown in Fig. 3.2*e*.

To compute stresses, and strains if desired, the device of the transformed section can still be used. One need only take account of the fact that all of the concrete that is stressed in tension is assumed cracked, and therefore effectively absent. As shown in Fig. 3.5*a*, the transformed section then consists of the concrete in compression on one side of the axis and n times the steel area on the other. The distance to the neutral axis, in this stage, is conventionally expressed as a fraction kd of the effective depth d. (Once the concrete is cracked, any material located below the steel is ineffective, which is why d is the effective depth of the beam.) To determine the location of the

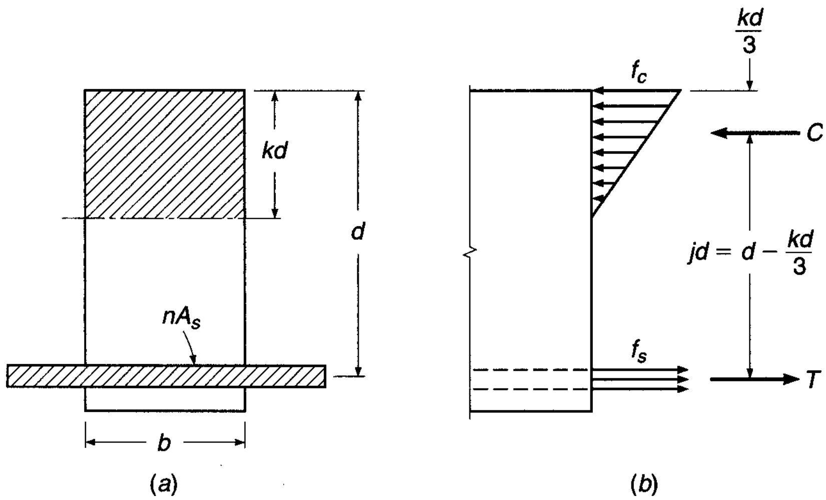


FIGURE 3.5
Cracked transformed section.

neutral axis, the moment of the tension area about the axis is set equal to the moment of the compression area, which gives

$$b\frac{(kd)^2}{2} - nA_s(d - kd) = 0 \tag{3.5}$$

Having obtained kd by solving this quadratic equation, one can determine the moment of inertia and other properties of the transformed section as in the preceding case. Alternatively, one can proceed from basic principles by accounting directly for the forces that act on the cross section. These are shown in Fig. 3.5*b*. The concrete stress, with maximum value f_c at the outer edge, is distributed linearly as shown. The entire steel area A_s is subject to the stress f_s. Correspondingly, the total compression force C and the total tension force T are

$$C = \frac{f_c}{2}bkd \qquad \text{and} \qquad T = A_sf_s \tag{3.6}$$

The requirement that these two forces be equal numerically has been taken care of by the manner in which the location of the neutral axis has been determined.

Equilibrium requires that the couple constituted by the two forces C and T be equal numerically to the external bending moment M. Hence, taking moments about C gives

$$M = Tjd = A_sf_sjd \tag{3.7}$$

where jd is the internal lever arm between C and T. From Eq. (3.7), the steel stress is

$$f_s = \frac{M}{A_sjd} \tag{3.8}$$

Conversely, taking moments about T gives

$$M = Cjd = \frac{f_c}{2}bkdjd = \frac{f_c}{2}kjbd^2 \tag{3.9}$$

from which the concrete stress is

$$f_c = \frac{2M}{kjbd^2} \tag{3.10}$$

In using Eqs. (3.6) through (3.10), it is convenient to have equations by which k and j may be found directly, to establish the neutral axis distance kd and the internal lever arm jd. First defining the *reinforcement ratio* as

$$\rho = \frac{A_s}{bd} \tag{3.11}$$

then substituting $A_s = \rho bd$ into Eq. (3.5) and solving for k, one obtains

$$k = \sqrt{(\rho n)^2 + 2\rho n} - \rho n \tag{3.12}$$

From Fig. 3.5*b* it is seen that $jd = d - kd/3$, or

$$j = 1 - \frac{k}{3} \tag{3.13}$$

Values of k and j for elastic cracked section analysis, for common reinforcement ratios and modular ratios, are found in Table A.6 of Appendix A.

EXAMPLE 3.2 The beam of Example 3.1 is subject to a bending moment M = 90 ft-kips (rather than 45 ft-kips as previously). Calculate the relevant properties and stresses.

Solution. If the section were to remain uncracked, the tensile stress in the concrete would now be twice its previous value, that is, 864 psi. Since this exceeds by far the modulus of rupture of the given concrete (475 psi), cracks will have formed and the analysis must be adapted consistent with Fig. 3.5. Equation (3.5), with the known quantities b, n, and A_s inserted, gives the distance to the neutral axis kd = 7.6 in., or k = 7.6/23 = 0.33. From Eq. (3.13), $j = 1 - 0.33/3 = 0.89$. With these values the steel stress is obtained from Eq. (3.8) as f_s = 22,300 psi, and the maximum concrete stress from Eq. (3.10) as f_c = 1390 psi.

Comparing the results with the pertinent values for the same beam when subject to one-half the moment, as previously calculated, one notices that (1) the neutral plane has migrated upward so that its distance from the top fiber has changed from 13.2 to 7.6 in.; (2) even though the bending moment has only been doubled, the steel stress has increased from 2870 to 22,300 psi, or about 7.8 times, and the concrete compression stress has increased from 484 to 1390 psi, or 2.9 times; (3) the moment of inertia of the cracked transformed section is easily computed to be 5910 in^4, compared with 14,740 in^4 for the uncracked section. This affects the magnitude of the deflection, as discussed in Chapter 6. Thus, it is seen how radical is the influence of the formation of tension cracks on the behavior of reinforced concrete beams.

c. Flexural Strength

It is of interest in structural practice to calculate those stresses and deformations that occur in a structure in service under design load. For reinforced concrete beams, this can be done by the methods just presented, which assume elastic behavior of both materials. It is equally, if not more, important that the structural engineer be able to predict with satisfactory accuracy the strength of a structure or structural member. By making this strength larger by an appropriate amount than the largest loads that can be expected during the lifetime of the structure, an adequate margin of safety is ensured. In the past, methods based on elastic analysis, like those just presented or variations thereof, have been used for this purpose. It is clear, however, that at or near the ultimate load, stresses are no longer proportional to strains. In regard to axial compression, this has been discussed in detail in Section 1.9, and in regard to bending, it has been pointed out that at high loads, close to failure, the distribution of stresses and strains is that of Fig. 3.2*f* rather than the elastic distribution of Fig. 3.2*e*. More realistic methods of analysis, based on actual inelastic rather than assumed elastic behavior of the materials and on results of extremely extensive experimental research, have been developed to predict the member strength. They are now used almost exclusively in structural design practice.

If the distribution of concrete compressive stresses at or near ultimate load (Fig. 3.2*f*) had a well-defined and invariable shape—parabolic, trapezoidal, or otherwise—it would be possible to derive a completely rational theory of bending strength, just as the theory of elastic bending with its known triangular shape of stress distribution (Figs. 3.1*b* and 3.2*c* and *e*) is straightforward and rational. Actually, inspection of Figs. 2.3, 2.4, and 2.6, and of many more concrete stress-strain curves that have been published, shows that the geometric shape of the stress distribution is quite varied and depends on a number of factors, such as the cylinder strength and the rate and duration of loading. For this and other reasons, a wholly rational flexural theory for reinforced concrete has not yet been developed (Refs. 3.1 to 3.3). Present methods of analysis, therefore, are based in part on known laws of mechanics and are supplemented, where needed, by extensive test information.

Let Fig. 3.6 represent the distribution of internal stresses and strains when the beam is about to fail. One desires a method to calculate that moment M_n (nominal moment) at which the beam will fail either by tension yielding of the steel or by crushing of the concrete in the outer compression fiber. For the first mode of failure, the criterion is that the steel stress equal the yield point, $f_s = f_y$. It has been mentioned before that an exact criterion for concrete compression failure is not yet known, but that for rectangular beams, strains of 0.003 to 0.004 have been measured immediately preceding failure. If one assumes, usually slightly conservatively, that the concrete is about to crush when the maximum strain reaches $\epsilon_u = 0.003$, comparison with a great many tests of beams and columns of a considerable variety of shapes and conditions of loading shows that a satisfactorily accurate and safe strength prediction can be

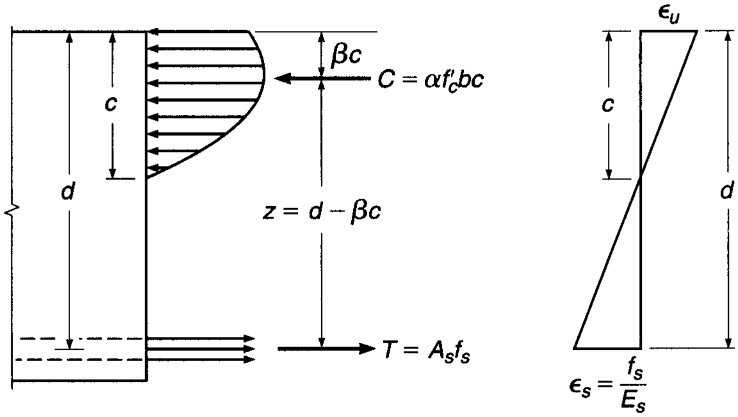


FIGURE 3.6
Stress distribution at ultimate load.

made (Ref. 3.4). In addition to these two criteria (yielding of the steel at a stress of f_y and crushing of the concrete at a strain of 0.003), it is not really necessary to know the exact shape of the concrete stress distribution in Fig. 3.6. What is necessary is to know, for a given distance c of the neutral axis, (1) the total resultant compression force C in the concrete and (2) its vertical location, i.e., its distance from the outer compression fiber.

In a rectangular beam, the area that is in compression is bc, and the total compression force on this area can be expressed as $C = f_{av}bc$, where f_{av} is the average compression stress on the area bc. Evidently, the average compressive stress that can be developed before failure occurs becomes larger, the higher the cylinder strength f_c' of the particular concrete. Let

$$\alpha = \frac{f_{av}}{f_c'} \tag{3.14}$$

Then

$$C = \alpha f_c' bc \tag{3.15}$$

For a given distance c to the neutral axis, the location of C can be defined as some fraction β of this distance. Thus, as indicated in Fig. 3.6, for a concrete of given strength it is necessary to know only α and β to completely define the effect of the concrete compressive stresses.

Extensive direct measurements, as well as indirect evaluations of numerous beam tests, have shown that the following values for α and β are satisfactorily accurate (see Ref. 3.5, where α is designated as k_1k_3 and β as k_2):

α equals 0.72 for $f_c' \leq 4000$ psi and decreases by 0.04 for every 1000 psi above 4000 up to 8000 psi. For $f_c' > 8000$ psi, $\alpha = 0.56$.

β equals 0.425 for $f_c' \leq 4000$ psi and decreases by 0.025 for every 1000 psi above 4000 up to 8000 psi. For $f_c' > 8000$ psi, $\beta = 0.325$.

The decrease in α and β for high-strength concretes is related to the fact that such concretes are more brittle; i.e., they show a more sharply curved stress-strain plot with a smaller near-horizontal portion (see Figs. 2.3 and 2.4). Figure 3.7 shows these simple relations.

If this experimental information is accepted, the maximum moment can be calculated from the laws of equilibrium and from the assumption that plane cross sections remain plane. Equilibrium requires that

$$C = T \qquad \text{or} \qquad \alpha f_c' bc = A_s f_s \tag{3.16}$$

Also, the bending moment, being the couple of the forces C and T, can be written as either

$$M = Tz = A_s f_s(d - \beta c) \tag{3.17}$$

or

$$M = Cz = \alpha f_c' bc(d - \beta c) \tag{3.18}$$

For failure initiated by yielding of the tension steel, $f_s = f_y$. Substituting this value in Eq. (3.16), one obtains the distance to the neutral axis

$$c = \frac{A_s f_y}{\alpha f_c' b} \tag{3.19a}$$

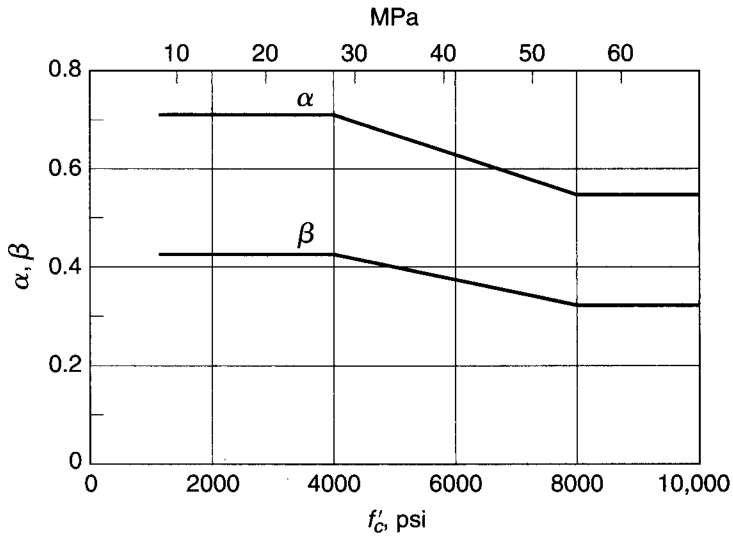


FIGURE 3.7
Variation of α and β with concrete strength f_c'.

Alternatively, using $A_s = \rho bd$, the neutral axis distance is

$$c = \frac{\rho f_y d}{\alpha f_c'} \tag{3.19b}$$

giving the distance to the neutral axis when tension failure occurs. The nominal moment M_n is then obtained from Eq. (3.17) with the value for c just determined, and $f_s = f_y$; that is,

$$M_n = \rho f_y bd^2\left(1 - \frac{\beta f_y \rho}{\alpha f_c'}\right) \tag{3.20a}$$

With the specific, experimentally obtained values for α and β given previously, this becomes

$$M_n = \rho f_y bd^2\left(1 - 0.59\frac{\rho f_y}{f_c'}\right) \tag{3.20b}$$

If, for larger reinforcement ratios, the steel does not reach yield at failure, then the strain in the concrete becomes $\epsilon_u = 0.003$, as previously discussed. The steel stress f_s, not having reached the yield point, is proportional to the steel strain ϵ_s; i.e., according to Hooke's law,

$$f_s = \epsilon_s E_s$$

From the strain distribution of Fig. 3.6, the steel strain ϵ_s can be expressed in terms of the distance c by evaluating similar triangles, after which it is seen that

$$f_s = \epsilon_u E_s \frac{d - c}{c} \tag{3.21}$$

Then, from Eq. (3.16),

$$\alpha f_c' bc = A_s \epsilon_u E_s \frac{d - c}{c} \tag{3.22}$$

and this quadratic may be solved for c, the only unknown for the given beam. With both c and f_s known, the nominal moment of the beam, so heavily reinforced that failure occurs by crushing of the concrete, may be found from either Eq. (3.17) or Eq. (3.18).

Whether or not the steel has yielded at failure can be determined by comparing the actual reinforcement ratio with the *balanced reinforcement ratio* ρ_b, representing that amount of reinforcement necessary for the beam to fail by crushing of the concrete at the same load that causes the steel to yield. This means that the neutral axis must be so located that at the load at which the steel starts yielding, the concrete reaches its compressive strain limit ϵ_u. Correspondingly, setting $f_s = f_y$ in Eq. (3.21) and substituting the yield strain ϵ_y for f_y/E_s, one obtains the value of c defining the unique position of the neutral axis corresponding to simultaneous crushing of the concrete and initiation of yielding in the steel

$$c = \frac{\epsilon_u}{\epsilon_u + \epsilon_y} d \tag{3.23}$$

Substituting that value of c into Eq. (3.16), with $A_s f_s = \rho b d f_y$, one obtains for the balanced reinforcement ratio

$$\rho_b = \frac{\alpha f'_c}{f_y} \frac{\epsilon_u}{\epsilon_u + \epsilon_y} \tag{3.24}$$

EXAMPLE 3.3 Determine the nominal moment M_n at which the beam of Examples 3.1 and 3.2 will fail.

Solution. For this beam the reinforcement ratio $\rho = A_s/(bd) = 2.37/(10 \times 23) = 0.0103$. The balanced reinforcement ratio is found from Eq. (3.24) to be 0.0284. Since the amount of steel in the beam is less than that which would cause failure by crushing of the concrete, the beam will fail in tension by yielding of the steel. Its nominal moment, from Eq. (3.20*b*), is

$$M_n = 0.0103 \times 60{,}000 \times 10 \times 23^2 \left(1 - 0.59 \frac{0.0103 \times 60{,}000}{4000}\right)$$

$$= 2{,}970{,}000 \text{ in-lb} = 248 \text{ ft-kips}$$

When the beam reaches M_n, the distance to its neutral axis, from Eq. (3.19*b*), is

$$c = \frac{0.0103 \times 60{,}000 \times 23}{0.72 \times 4000} = 4.94$$

It is informative to compare this result with those of Examples 3.1 and 3.2. In the previous calculations, it was found that at low loads, when the concrete had not yet cracked in tension, the neutral axis was located at a distance of 13.2 in. from the compression edge; at higher loads, when the tension concrete was cracked but stresses were still sufficiently small to be elastic, this distance was 7.6 in. Immediately before the beam fails, as has just been shown, this distance has further decreased to 4.9 in. For these same stages of loading, the stress in the steel increased from 2870 psi in the uncracked section, to 22,300 psi in the cracked elastic section, and to 60,000 psi at the nominal moment capacity. This migration of the neutral axis toward the compression edge and the increase in steel stress as load is increased is a graphic illustration of the differences between the various stages of behavior through which a reinforced concrete beam passes as its load is increased from zero to the value that causes it to fail. The examples also illustrate the fact that nominal moments cannot be determined accurately by elastic calculations.

3.4 DESIGN OF TENSION-REINFORCED RECTANGULAR BEAMS

For reasons that were explained in Chapter 1, the present design of reinforced concrete structures is based on the concept of providing sufficient strength to resist hypothetical overloads. The *nominal strength* of a proposed member is calculated based on the best current knowledge of member and material behavior. That nominal strength is modified by a *strength reduction factor* ϕ, less than unity, to obtain the *design strength*. The *required strength*, should the hypothetical overload stage actually be realized, is found by applying *load factors* γ, greater than unity, to the loads actually expected. These expected *service loads* include the calculated dead load, the calculated or legally specified live load, and environmental loads such as those due to wind, seismic action, or temperature. Thus reinforced concrete members are proportioned so that, as shown in Eq. (1.5),

$$M_u \le \phi M_n$$

$$P_u \le \phi P_n$$

$$V_u \le \phi V_n$$

where the subscripts n denote the nominal strengths in flexure, thrust, and shear, respectively, and the subscripts u denote the factored load moment, thrust, and shear. The strength reduction factors ϕ normally differ, depending upon the type of strength to be calculated, the importance of the member in the structure, and other considerations discussed in detail in Chapter 1.

A member proportioned on the basis of adequate strength at a hypothetical overload stage must also perform in a satisfactory way under normal service load conditions. In specific terms, the deflection must be limited to an acceptable value, and concrete tensile cracks, which inevitably occur, must be of narrow width and well distributed throughout the tensile zone. Therefore, after proportioning for adequate strength, deflections are calculated and compared against limiting values (or otherwise controlled), and crack widths limited by specific means. This approach to design, referred to in Europe, and to some extent in U.S. practice, as *limit states design*, is the basis of the 2008 ACI Code, and it is the approach that will be followed in this and later chapters.

a. Equivalent Rectangular Stress Distribution

The method presented in Section 3.3c for calculating the flexural strength of reinforced concrete beams, derived from basic concepts of structural mechanics and pertinent experimental research information, also applies to situations other than the case of rectangular beams reinforced on the tension side. It can be used and gives valid answers for beams of other cross-sectional shapes, reinforced in other manners, and for members subject not only to simple bending but also to the simultaneous action of bending and axial force (compression or tension). However, the pertinent equations for these more complex cases become increasingly cumbersome and lengthy. What is more important, it becomes increasingly difficult for the designer to visualize the physical basis for the design methods and formulas; this could lead to a blind reliance on formulas, with a resulting lack of actual understanding. This is not only undesirable on general grounds but also, practically, is more likely to lead to numerical errors in design work than when the designer at all times has a clear picture of the physical situation in the member being dimensioned or analyzed. Fortunately, it is possible, essentially by a conceptual trick, to formulate the strength analysis of reinforced concrete members in a different manner, which gives the same answers as the general analysis just developed but which is much more easily visualized and much more easily applied to cases of greater complexity than that of the simple rectangular beam. Its consistency is shown, and its application to more complex cases has been checked against the results of a vast number of tests on a great variety of types of members and conditions of loading (Ref. 3.4).

It was noted in the preceding section that the actual geometric shape of the concrete compressive stress distribution varies considerably and that, in fact, one need not know this shape exactly, provided one does know two things: (1) the magnitude C of the resultant of the concrete compressive stresses and (2) the location of this resultant. Information on these two quantities was obtained from the results of experimental research and expressed in the two parameters α and β.

Evidently, then, one can think of the actual complex stress distribution as replaced by a fictitious one of some simple geometric shape, provided that this fictitious distribution results in the same total compression force C applied at the same location as in the actual member when it is on the point of failure. Historically, a number of simplified, fictitious equivalent stress distributions have been proposed by investigators in various countries. The one generally accepted in this country, and increasingly abroad, was first proposed by C. S. Whitney (Ref. 3.4) and was subsequently elaborated and checked experimentally by others (see, for example, Refs. 3.5 and 3.6). The actual stress distribution immediately before failure and the fictitious equivalent distribution are shown in Fig. 3.8.

It is seen that the actual stress distribution is replaced by an equivalent one of simple rectangular outline. The intensity $\gamma f_c'$ of this equivalent constant stress and its depth $a = \beta_1 c$ are easily calculated from the two conditions that (1) the total compression force C and (2) its location, i.e., distance from the top fiber, must be the same in the equivalent rectangular as in the actual stress distribution. From Fig. 3.8*a* and *b* the first condition gives

$$C = \alpha f_c' cb = \gamma f_c' ab \qquad \text{from which} \qquad \gamma = \alpha \frac{c}{a}$$

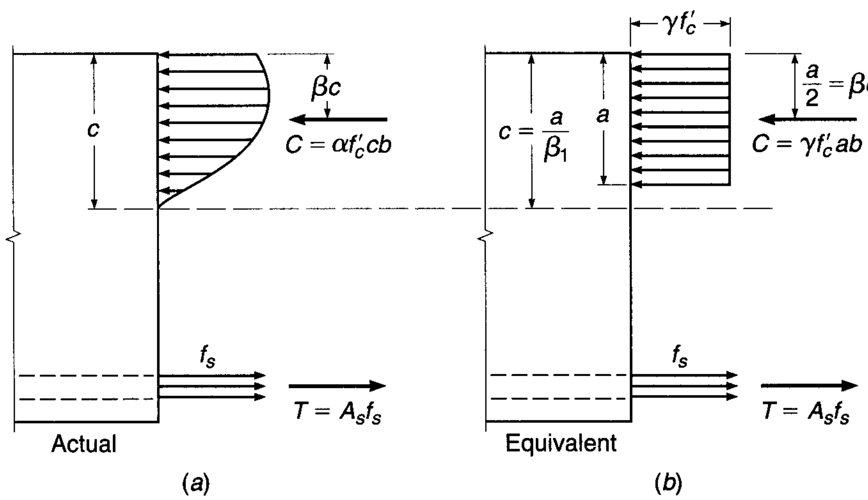


FIGURE 3.8
Actual and equivalent rectangular stress distributions at ultimate load.

TABLE 3.1
Concrete stress block parameters

	f'_c, psi				
	≤4000	5000	6000	7000	≥8000
α	0.72	0.68	0.64	0.60	0.56
β	0.425	0.400	0.375	0.350	0.325
$\beta_1 = 2\beta$	0.85	0.80	0.75	0.70	0.65
$\gamma = \alpha/\beta_1$	0.85	0.85	0.85	0.86	0.86

With $a = \beta_1 c$, this gives $\gamma = \alpha/\beta_1$. The second condition simply requires that in the equivalent rectangular stress block, the force C be located at the same distance βc from the top fiber as in the actual distribution. It follows that $\beta_1 = 2\beta$.

To supply the details, the upper two lines of Table 3.1 present the experimental evidence of Fig. 3.7 in tabular form. The lower two lines give the just-derived parameters β_1 and γ for the rectangular stress block. It is seen that the stress intensity factor γ is essentially independent of f'_c and can be taken as 0.85 throughout. Hence, regardless of f'_c, the concrete compression force at failure in a rectangular beam of width b is

$$C = 0.85 f'_c ab \tag{3.25}$$

Also, for the common concretes with $f'_c \leq 4000$ psi, the depth of the rectangular stress block is $a = 0.85c$, with c being the distance to the neutral axis. For higher-strength concretes, this distance is $a = \beta_1 c$, with the β_1 values shown in Table 3.1. This is expressed in ACI Code 10.2.7.3 as follows: For f'_c between 2500 and 4000 psi, β_1 shall be taken as 0.85; for f'_c above 4000 psi, β_1 shall be reduced linearly at a rate of 0.05 for each 1000 psi of strength in excess of 4000 psi, but β_1 shall not be taken as less than 0.65. In mathematical terms, the relationship between β_1 and f'_c can be expressed as

$$\beta_1 = 0.85 - 0.05\frac{f'_c - 4000}{1000} \qquad \text{and} \qquad 0.65 \leq \beta_1 \leq 0.85 \tag{3.26}$$

The equivalent rectangular stress distribution can be used for deriving the equations that have been developed in Section 3.3c. The failure criteria, of course, are the same as before: yielding of the steel at $f_s = f_y$ or crushing of the concrete at $\epsilon_u = 0.003$. Because the rectangular stress block is easily visualized and its geometric properties are extremely simple, many calculations are carried out directly without reference to formally derived equations, as will be seen in the following sections.

b. Balanced Strain Condition

A reinforcement ratio ρ_b producing balanced strain conditions can be established based on the condition that, at balanced failure, the steel strain is exactly equal to ϵ_y when the strain in the concrete simultaneously reaches the crushing strain of $\epsilon_u = 0.003$. Referring to Fig. 3.6,

$$c = \frac{\epsilon_u}{\epsilon_u + \epsilon_y} d \tag{3.27}$$

which is seen to be identical to Eq. (3.23). Then from the equilibrium requirement that $C = T$

$$\rho_b f_y bd = 0.85 f'_c ab = 0.85 \beta_1 f'_c bc$$

from which

$$\rho_b = 0.85 \beta_1 \frac{f'_c}{f_y} \frac{\epsilon_u}{\epsilon_u + \epsilon_y} \tag{3.28}$$

This is easily shown to be equivalent to Eq. (3.24).

c. Underreinforced Beams

A compression failure in flexure, should it occur, gives little if any warning of distress, while a tension failure, initiated by yielding of the steel, typically is gradual. Distress is obvious from observing the large deflections and widening of concrete cracks associated with yielding of the steel reinforcement, and measures can be taken to avoid total collapse. In addition, most beams for which failure initiates by yielding possess substantial strength based on strain-hardening of the reinforcing steel, which is not accounted for in the calculations of M_n.

Because of these differences in behavior, it is prudent to require that beams be designed such that failure, if it occurs, will be by yielding of the steel, not by crushing of the concrete. This can be done, theoretically, by requiring that the reinforcement ratio ρ be less than the balance ratio ρ_b given by Eq. (3.28).

In actual practice, the upper limit on ρ should be below ρ_b for the following reasons: (1) for a beam with ρ exactly equal to ρ_b, the compressive strain limit of the concrete would be reached, theoretically, at precisely the same moment that the steel reaches its yield stress, without significant yielding before failure; (2) material properties are never known precisely; (3) strain-hardening of the reinforcing steel, not accounted for in design, may lead to a brittle concrete compression failure even though ρ may be somewhat less than ρ_b; (4) the actual steel area provided, considering standard reinforcing bar sizes, will always be equal to or larger than required, based on selected reinforcement ratio ρ, tending toward overreinforcement; and (5) the extra ductility provided by beams with lower values of ρ increases the deflection capability substantially and thus provides warning prior to failure.

d. ACI Code Provisions for Underreinforced Beams

While the nominal strength of a member may be computed based on principles of mechanics, the mechanics alone cannot establish safe limits for maximum reinforcement ratios. These limits are defined by the ACI Code. The limitations take two forms. First, the Code addresses the minimum tensile reinforcement strain allowed at nominal strength in the design of beams. Second, the Code defines strength reduction factors that may depend on the tensile strain at nominal strength. Both limitations are based on the *net tensile strain* ϵ_t of the reinforcement farthest from the compression face of the concrete at the depth d_t. The net tensile strain is exclusive of prestress, temperature, and shrinkage effects. For beams with a single layer of reinforcement, the depth to the centroid of the steel d is the same as d_t. For beams with multiple layers of reinforcement, d_t is greater than the depth to the centroid of the reinforcement d. Substituting d_t for d and ϵ_t for ϵ_y in Eq. (3.27), the net tensile strain may be represented as

$$\epsilon_t = \epsilon_u \frac{d_t - c}{c} \tag{3.29}$$

Then based on Eq. (3.28), the reinforcement ratio to produce a selected value of net tensile strain is

$$\rho = 0.85\beta_1 \frac{f'_c}{f_y} \frac{d_t}{d} \frac{\epsilon_u}{\epsilon_u + \epsilon_t} \tag{3.30a}$$

or somewhat conservatively

$$\rho = 0.85\beta_1 \frac{f'_c}{f_y} \frac{\epsilon_u}{\epsilon_u + \epsilon_t} \tag{3.30b}$$

To ensure underreinforced behavior, ACI Code 10.3.5 establishes a minimum net tensile strain ϵ_t at the nominal member strength of 0.004 for members subjected to axial loads less than $0.10f'_cA_g$, where A_g is the gross area of the cross section. By way of comparison ϵ_y, the steel strain at the balanced condition, is 0.00207 for f_y = 60,000 psi and 0.00259 for f_y = 75,000 psi.

Using $\epsilon_t = 0.004$ in Eq. (3.30*b*) provides the maximum reinforcement ratio allowed by the ACI Code for beams

$$\rho_{\max} = 0.85\beta_1 \frac{f'_c}{f_y} \frac{\epsilon_u}{\epsilon_u + 0.004} \tag{3.30c}$$

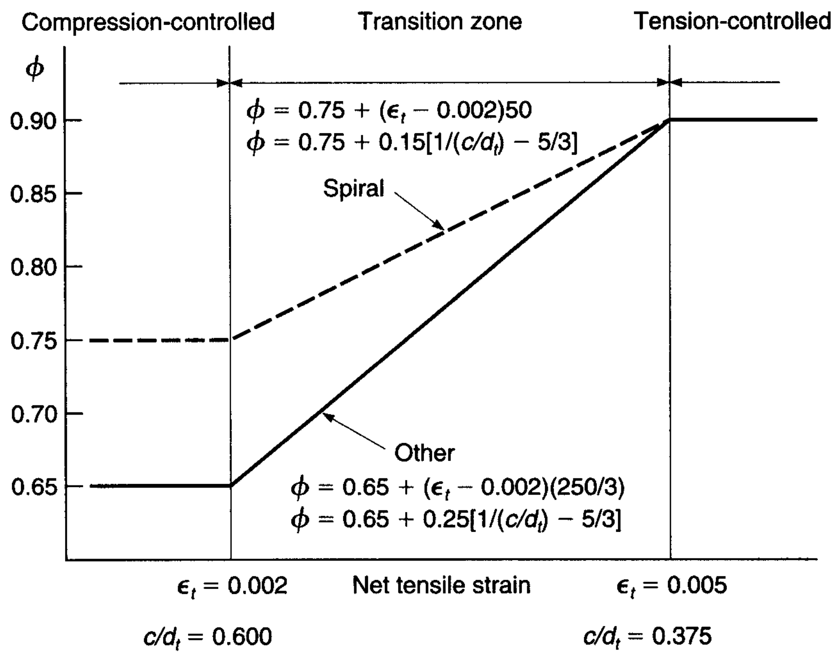
The ACI Code further encourages the use of lower reinforcement ratios by allowing higher strength reduction factors in such beams. The Code defines a *tension-controlled member* as one with a net tensile strain greater than or equal to 0.005. The corresponding strength reduction factor is $\phi = 0.9$.† The Code additionally defines a *compression-controlled member* as having a net tensile strain of less than 0.002. The strength reduction factor for compression-controlled members is 0.65. A value of 0.75 may be used if the members are spirally reinforced. A value of $\epsilon_t = 0.002$ corresponds approximately to the yield strain for steel with f_y = 60,000 psi yield strength. Between net tensile strains of 0.002 and 0.005, the strength reduction factor varies linearly, and the ACI Code allows a linear interpolation of ϕ based on ϵ_t, as shown in Fig. 3.9. Based on Eq. (3.30*b*), the maximum reinforcement ratio for a tension-controlled beam is

$$\rho_{0.005} = 0.85\beta_1 \frac{f'_c}{f_y} \frac{\epsilon_u}{\epsilon_u + 0.005} \tag{3.30d}$$

A comparison of Eqs. (3.30*c*) and (3.30*d*) shows that, for a given concrete cross section, using $\epsilon_t = 0.004$ will result in a higher reinforcement ratio, and thus a higher nominal flexural strength, than using $\epsilon_t = 0.005$. This higher strength, however, cannot be used to full advantage in design because the increase in flexural strength is canceled by the drop in ϕ as ϵ_t decreases from 0.005 to 0.004. As a result, the maximum practical reinforcement ratio for beams is attained at a net tensile strain of 0.005. Values of ϵ_t below 0.005 are not recommended for the design of members with low axial loads.

† The selection of a net tensile strain of 0.005 is intended to encompass the yield strain of all reinforcing steel including high-strength bars and prestressing tendons.

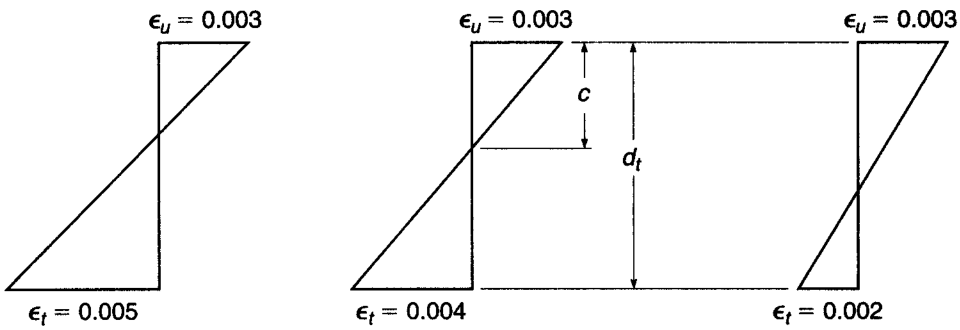
FIGURE 3.9
Variation of strength reduction factor with net tensile strain in the steel.



Calculation of the nominal moment capacity frequently involves determination of the depth of the equivalent rectangular stress block a. Since $c = a/\beta_1$, it is sometimes more convenient to compute c/d_t ratios than either ρ or the net tensile strain. The assumption that plane sections remain plane ensures a direct correlation between net tensile strain and the c/d_t ratio, as shown in Fig. 3.10. The maximum value of c/d_t for $\epsilon_t \geq 0.005$ is 0.375.

Comparing Eqs. (3.30*a*) and (3.30*b*), it can be seen that the maximum reinforcement ratios in Eqs. (3.30*c*) and (3.30*d*) are exact for beams with a single layer of reinforcement and slightly conservative for beams with multiple layers of reinforcement, where d_t is greater than d. Because $\epsilon_t \geq 0.004$ (better yet $\epsilon_t \geq 0.005$) ensures

FIGURE 3.10
Net tensile strain and c/d_t ratios.



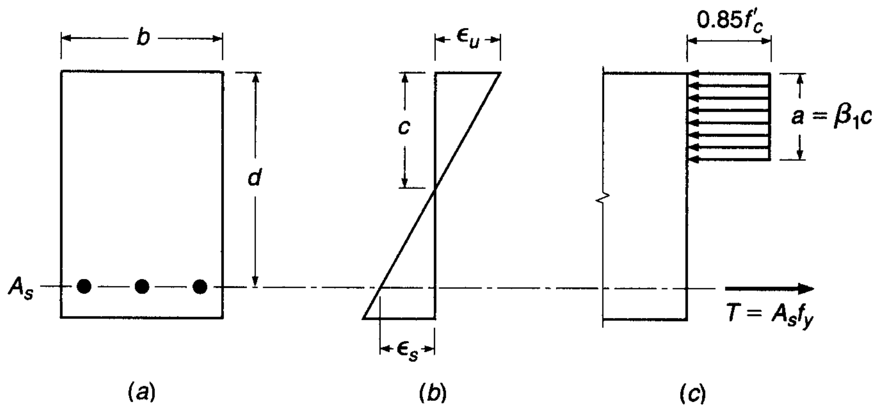
$$\frac{c}{d_t} = \frac{0.003}{0.003 + 0.005} = 0.375 \qquad \frac{c}{d_t} = \frac{0.003}{0.003 + 0.004} = 0.429 \qquad \frac{c}{d_t} = \frac{0.003}{0.003 + 0.002} = 0.600$$

(*a*) Tension-controlled member

(*b*) Minimum net tensile strain for flexural member

(*c*) Compression-controlled member

FIGURE 3.11
Singly reinforced rectangular beam.



that steel is yielding in tension, $f_s = f_y$ at failure, and the nominal flexural strength (referring to Fig. 3.11) is given by

$$M_n = A_s f_y \left(d - \frac{a}{2} \right) \tag{3.31}$$

where

$$a = \frac{A_s f_y}{0.85 f'_c b} \tag{3.32}$$

EXAMPLE 3.4 Using the equivalent rectangular stress distribution, directly calculate the nominal strength of the beam previously analyzed in Example 3.3. Recall that $b = 10$ in., $d = 23$ in., $A_s = 2.37$ in^2., $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, and $\beta_1 = 0.85$.

Solution. The distribution of stresses, internal forces, and strains is shown in Fig. 3.11. The maximum practical reinforcement ratio is calculated from Eq. (3.30*d*) as

$$\rho_{0.005} = 0.85 \times 0.85 \frac{4000}{60{,}000} \frac{0.003}{0.003 + 0.005} = 0.0181$$

and comparison with the actual reinforcement ratio of 0.0103 confirms that the member is underreinforced and will fail by yielding of the steel. Alternatively, recalling that $c = 4.94$ in.,

$$\frac{c}{d_t} = \frac{c}{d} = \frac{4.94}{23} = 0.215$$

which is less than 0.375, the value of c/d_t corresponding to $\epsilon_t = 0.005$, also confirming that the member is underreinforced. The depth of the equivalent stress block is found from the equilibrium condition that $C = T$. Hence $0.85 f'_c ab = A_s f_y$, or $a = 2.37 \times 60{,}000/(0.85 \times 4000 \times 10) = 4.18$. The nominal moment is

$$M_n = A_s f_y \left(d - \frac{a}{2} \right) = 2.37 \times 60{,}000(23 - 2.09) = 2{,}970{,}000 \text{ in-lb} = 248 \text{ ft-kips}$$

The results of this simple and direct numerical analysis, based on the equivalent rectangular stress distribution, are identical with those previously determined from the general strength analysis described in Section 3.3c.

It is convenient for everyday design to combine Eqs. (3.31) and (3.32) as follows. Noting that $A_s = \rho bd$, Eq. (3.32) can be rewritten as

$$a = \frac{\rho f_y d}{0.85 f_c'} \tag{3.33}$$

This is then substituted into Eq. (3.31) to obtain

$$M_n = \rho f_y bd^2 \left(1 - 0.59 \frac{\rho f_y}{f_c'}\right) \tag{3.34}$$

which is identical to Eq. (3.20*b*) derived in Section 3.3c. This basic equation can be simplified further as follows:

$$M_n = Rbd^2 \tag{3.35}$$

in which

$$R = \rho f_y \left(1 - 0.59 \frac{\rho f_y}{f_c'}\right) \tag{3.36}$$

The *flexural resistance factor R* depends only on the reinforcement ratio and the strengths of the materials and is easily tabulated. Tables A.5a and A.5b of Appendix A give R values for ordinary combinations of steel and concrete and the full practical range of reinforcement ratios.

In accordance with the safety provisions of the ACI Code, the nominal flexural strength M_n is reduced by imposing the strength reduction factor ϕ to obtain the *design strength*

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) \tag{3.37}$$

or, alternatively,

$$\phi M_n = \phi \rho f_y bd^2 \left(1 - 0.59 \frac{\rho f_y}{f_c'}\right) \tag{3.38}$$

or

$$\phi M_n = \phi Rbd^2 \tag{3.39}$$

EXAMPLE 3.4 (*continued*) Calculate the design moment capacity ϕM_n for the beam analyzed earlier in Example 3.4.

Solution. Comparing ρ with $\rho_{0.005}$ or c/d_t for the beam with the value of c/d_t corresponding to $\epsilon_t = 0.005$ demonstrates that $\epsilon_t > 0.005$. Therefore, $\phi = 0.90$ and the design capacity is

$$\phi M_n = 0.9 \times 248 = 223 \text{ ft-kips}$$

---

e. Minimum Reinforcement Ratio

Another mode of failure may occur in very lightly reinforced beams. If the flexural strength of the cracked section is less than the moment that produced cracking of the previously uncracked section, the beam will fail immediately and without warning of distress upon formation of the first flexural crack. To ensure against this type of failure, a *lower limit* can be established for the reinforcement ratio by equating the cracking moment, computed from the concrete modulus of rupture (Section 2.9), to the strength of the cracked section.

For a rectangular section having width b, total depth h, and effective depth d (see Fig. 3.2*b*), the section modulus with respect to the tension fiber is $bh^2/6$. For typical cross sections, it is satisfactory to assume that $h/d = 1.1$ and that the internal lever arm at flexural failure is $0.95d$. If the modulus of rupture is taken as $f_r = 7.5\sqrt{f_c'}$, as usual, then an analysis equating the cracking moment to the flexural strength results in

$$A_{s,\min} = \frac{1.6\sqrt{f_c'}}{f_y} bd \tag{3.40a}$$

This development can be generalized to apply to beams having a T cross section (see Section 3.8 and Fig. 3.16). The corresponding equations depend on the proportions of the cross section and on whether the beam is bent with the flange (slab) in tension or in compression. For T beams of typical proportions that are bent with the flange in compression, analysis will confirm that the minimum steel area should be

$$A_{s,\min} = \frac{2.7\sqrt{f_c'}}{f_y} b_w d \tag{3.40b}$$

where b_w is the width of the web, or stem, projecting below the slab. For T beams that are bent with the flange in tension, from a similar analysis, the minimum steel area is

$$A_{s,\min} = \frac{6.2\sqrt{f_c'}}{f_y} b_w d \tag{3.40c}$$

The ACI Code requirements for minimum steel area are based on the results just discussed, but there are some differences. According to ACI Code 10.5, at any section where tensile reinforcement is required by analysis, with some exceptions as noted below, the area A_s provided must not be less than

$$A_{s,\min} = \frac{3\sqrt{f_c'}}{f_y} b_w d \geq \frac{200 b_w d}{f_y} \tag{3.41}$$

This applies to both positive and negative bending sections. The inclusion of the additional limit of $200b_w d/f_y$ is merely for historical reasons; it happens to give the same minimum reinforcement ratio of 0.005 that was imposed in earlier codes for then-common material strengths. Note that in Eq. (3.41) the section width b_w is used; it is understood that for rectangular sections $b_w = b$. Note further that the ACI coefficient of 3 is a conservatively rounded value compared with 2.7 in Eq. (3.40*b*) for T beams with the flange in compression, and is very conservative when applied to rectangular beam sections, for which a rational analysis gives 1.6 in Eq. (3.40*a*). This probably reflects the view that the minimum steel for the negative bending sections of a continuous T beam (which are, in effect, rectangular sections, as discussed in Section 3.8c) should be no less than for the positive bending sections, where the moment is generally smaller.

ACI Code 10.5 treats *statically determinate* T beams with the flange in *tension* as a special case, for which the minimum steel area is equal to or greater than the value given by Eq. (3.41) with b_w replaced by either $2b_w$ or the width of the *flange*, whichever is smaller.

Note that ACI Code Eq. (3.41) is conveniently expressed in terms of a *minimum tensile reinforcement ratio* ρ_{min} by dividing both sides by $b_w d$.

According to ACI Code 10.5, the requirements of Eq. (3.41) need not be imposed if, at every section, the area of tensile reinforcement provided is at least one-third greater than that required by analysis. This provides sufficient reinforcement for large members such as grade beams, where the usual equations would require excessive amounts of steel.

For structural slabs and footings of uniform thickness, the minimum area of tensile reinforcement in the direction of the span is that required for shrinkage and temperature steel (see Section 13.3 and Table 13.2), and the above minimums need not be imposed. The maximum spacing of such steel is the smaller of 3 times the total slab thickness or 18 in.

f. Examples of Rectangular Beam Analysis and Design

Flexural problems can be classified broadly as *analysis problems* or *design problems.* In analysis problems, the section dimensions, reinforcement, and material strengths are known, and the moment capacity is required. In the case of design problems, the required moment capacity is given, as are the material strengths, and it is required to find the section dimensions and reinforcement. Examples 3.5 and 3.6 illustrate analysis and design, respectively.

EXAMPLE 3.5 **Flexural strength of a given member.** A rectangular beam has width 12 in. and effective depth 17.5 in. It is reinforced with four No. 9 (No. 29) bars in one row. If $f_y = 60{,}000$ psi and $f'_c = 4000$ psi, what is the nominal flexural strength, and what is the maximum moment that can be utilized in design, according to the ACI Code?

SOLUTION. From Table A.2 of Appendix A, the area of four No. 9 (No. 29) bars is 4.00 in^2. Assuming that the beam is underreinforced and using Eq. (3.32),

$$a = \frac{4.00 \times 60}{0.85 \times 4 \times 12} = 5.88 \text{ in.}$$

The depth of the neutral axis is $c = a/\beta_1 = 5.88/0.85 = 6.92$, giving

$$\frac{c}{d_t} = \frac{6.92}{17.5} = 0.395$$

which is between 0.429 and 0.375, the values corresponding, respectively, to $\epsilon_t = 0.004$ and $\epsilon_t = 0.005$, as shown in Fig. 3.10. Thus, the beam is, as assumed, underreinforced, and from Eq. (3.31)

$$M_n = 4.00 \times 60\left(17.5 - \frac{5.88}{2}\right) = 3490 \text{ in-kips}$$

The fact that the beam is unreinforced could also have been established by calculating $\rho = 4.00/(12 \times 17.5) = 0.190$, which just exceeds $\rho_{0.005}$, which is calculated using Eq. (3.30*d*).

$$\rho_{0.005} = 0.85 \times 0.85\left(\frac{4}{60}\right)\left(\frac{0.003}{0.003 + 0.005}\right) = 0.0181$$

Because the net tensile strain ϵ_t is between 0.004 and 0.005, ϕ must be calculated: $\epsilon_t = \epsilon_u(d - c)/c = 0.003 \times 17.5 - 6.92/6.92 = 0.00458$. Using linear interpolation from Fig. 3.9, $\phi = 0.87$, and the design strength is taken as

$$\phi M_n = 0.87 \times 3490 = 3040 \text{ in-kips}$$

The ACI Code limits on the reinforcement ratio

$$\rho_{max} = 0.0206$$

$$\rho_{min} = \frac{3\sqrt{4000}}{60{,}000} \geq \frac{200}{60{,}000} = 0.0033$$

are satisfied for this beam.

EXAMPLE 3.6 **Concrete dimensions and steel area to resist a given moment.** Find the concrete cross section and the steel area required for a simply supported rectangular beam with a span of 15 ft that is to carry a computed dead load of 1.27 kips/ft and a service live load of 2.15 kips/ft, as shown in Fig. 3.12. Material strengths are f'_c = 4000 psi and f_y = 60,000 psi.

Solution. Load factors are first applied to the given service loads to obtain the factored load for which the beam is to be designed, and the corresponding moment:

$$w_u = 1.2 \times 1.27 + 1.6 \times 2.15 = 4.96 \text{ kips/ft}$$

$$M_u = \frac{1}{8} \times 4.96 \times 15^2 \times 12 = 1670 \text{ in-kips}$$

The concrete dimensions will depend on the designer's choice of reinforcement ratio. To minimize the concrete section, it is desirable to select the maximum permissible reinforcement ratio. To maintain $\phi = 0.9$, the maximum reinforcement ratio corresponding to a net tensile strain of 0.005 will be selected (see Fig. 3.9). Then, from Eq. (3.30*d*)

$$\rho_{0.005} = 0.85\beta_1 \frac{f'_c}{f_y} \frac{\epsilon_u}{\epsilon_u + 0.005} = 0.85 \times 0.85 \left(\frac{4}{60}\right)\left(\frac{0.003}{0.003 + 0.005}\right) = 0.0181$$

Using Eq. (3.30*c*) gives $\rho_{max} = 0.0206$, but would require a lower strength reduction factor. Setting the required flexural strength equal to the design strength from Eq. (3.38), and substituting the selected values for ρ and material strengths,

$$M_u = \phi M_n$$

$$1670 = 0.90 \times 0.0181 \times 60bd^2\left(1 - 0.59\,\frac{0.0181 \times 60}{4}\right)$$

from which

$$bd^2 = 2040 \text{ in}^3$$

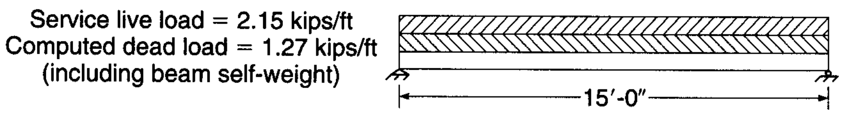
A beam with width b = 10 in. and d = 14.3 in. will satisfy this requirement. The required steel area is found by applying the chosen reinforcement ratio to the required concrete dimensions:

$$A_s = 0.0181 \times 10 \times 14.3 = 2.59 \text{ in}^2$$

Two No. 10 (No. 32) bars provide 2.54 in^2, which is very close to the required area.

Assuming 2.5 in. concrete cover from the centroid of the bars, the required total depth is h = 16.8 in. In actual practice, however, the concrete dimensions b and h are always rounded up to the nearest inch, and often to the nearest multiple of 2 in. (see Section 3.5). The

FIGURE 3.12
Structural loads for Example 3.6.



actual d is then found by subtracting the required concrete cover dimension from h. For the present example, $b = 10$ in. and $h = 18$ in. will be selected, resulting in effective depth $d = 15.5$ in. Improved economy then may be possible, refining the steel area based on the actual, larger, effective depth. One can obtain the revised steel requirement directly by solving Eq. (3.38) for ρ, with $\phi M_n = M_u$. A quicker solution can be obtained by iteration. First a reasonable value of a is assumed, and A_s is found from Eq. (3.37). From Eq. (3.32) a revised estimate of a is obtained, and A_s is revised. This method converges very rapidly. For example, assume $a = 5$ in. Then

$$A_s = \frac{1670}{0.90 \times 60(15.5 - 2.5)} = 2.38 \text{ in}^2$$

Checking the assumed a gives

$$a = \frac{2.38 \times 60}{0.85 \times 4 \times 10} = 4.20 \text{ in.}$$

This is close enough to the assumed value that no further calculation is required. The required steel area of 2.38 in^2 could be provided using three No. 8 (No. 25) bars, but for simplicity of construction, two No. 10 (No. 32) bars will be used as before.

A somewhat larger beam cross section using less steel may be more economical, and will tend to reduce deflections. As an alternative solution, the beam will be redesigned with a lower reinforcement ratio of $\rho = 0.60\rho_{\max} = 0.60 \times 0.0206 = 0.0124$. Setting the required strength equal to the design strength [Eq. (3.38)] as before,

$$1670 = 0.90 \times 0.0124 \times 60bd^2\left(1 - 0.59\,\frac{0.0124 \times 60}{4}\right)$$

and

$$bd^2 = 2800 \text{ in}^3$$

A beam with $b = 10$ in. and $d = 16.7$ in. will meet the requirement, for which

$$A_s = 0.0124 \times 10 \times 16.7 = 2.07 \text{ in}^2$$

Two No. 9 (No. 29) bars are almost sufficient, providing an area of 2.00 in^2. If the total concrete height is rounded up to 20 in., a 17.5 in. effective depth results, reducing the required steel area to 1.96 in^2. Two No. 9 (No. 29) bars remain the best choice.

It is apparent that an infinite number of solutions to the stated problem are possible, depending upon the reinforcement ratio selected. That ratio may vary from an upper limit of $\rho_{\max}$ to a lower limit of $3\sqrt{f_c'}/f_y \geq 200/f_y$ for beams, according to the ACI Code. To compare the two solutions (using the theoretical dimensions, unrounded for the comparison, and assuming h is 2.5 in. greater than d in each case), increasing the concrete section area by 14 percent achieves a steel saving of 20 percent. The second solution would certainly be more economical and would be preferred, unless beam dimensions must be minimized for architectural or functional reasons. Economical designs will typically have reinforcement ratios between $0.50\rho_{0.005}$ and $0.75\rho_{0.005}$.

There is a type of problem, occurring frequently, that does not fall strictly into either the analysis or the design category. The concrete dimensions are given and are known to be adequate to carry the required moment, and it is necessary only to find the steel area. Typically, this is the situation at critical design sections of continuous beams, in which the concrete dimensions are often kept constant, although the steel reinforcement varies along the span according to the required flexural resistance. Dimensions b, d, and h are determined at the maximum moment section, usually at one of the supports. At other supports, and at midspan locations, where moments are usually smaller, the concrete dimensions are known to be adequate and only the tensile steel remains to be found. An identical situation was encountered in the design problem of Example 3.6, in which concrete dimensions were rounded up from the minimum required values, and the required steel area was to be found. In either case, the iterative approach demonstrated in Example 3.6 is convenient.

EXAMPLE 3.7 **Determination of steel area.** Using the same concrete dimensions as were used for the second solution of Example 3.6 (b = 10 in., d = 17.5 in., and h = 20 in.) and the same material strengths, find the steel area required to resist a moment M_u of 1300 in-kips.

Solution. Assume a = 4.0 in. Then

$$A_s = \frac{1300}{0.90 \times 60(17.5 - 2.0)} = 1.55 \text{ in}^2$$

Checking the assumed a gives

$$a = \frac{1.55 \times 60}{0.85 \times 4 \times 10} = 2.74 \text{ in.}$$

Next assume a = 2.6 in. and recalculate A_s:

$$A_s = \frac{1300}{0.90 \times 60(17.5 - 1.3)} = 1.49 \text{ in}^2$$

No further iteration is required. Use A_s = 1.49 in^2. Two No. 8 (No. 25) bars, A_s = 1.58 in.^{2}, will be used. A check of the reinforcement ratio shows $\rho < \rho_{0.005}$ and $\phi = 0.9$.

As seen in Example 3.5, the strength reduction factor becomes a variable at high reinforcement ratios. Example 3.8 demonstrates how the variation in strength reduction factor affects the design process.

EXAMPLE 3.8 **Determination of steel area and variable strength reduction factor.** Architectural considerations limit the height of a 20 ft long simple span beam to 16 in. and the width to 12 in. The following loads and material properties are given: w_d = 0.79 kips/ft, w_l = 1.65 kips/ft, f'_c = 5000 psi, and f_y = 60,000 psi. Determine the reinforcement for the beam.

Solution. Calculating the factored loads gives

$$w_u = 1.2 \times 0.79 + 1.6 \times 1.65 = 3.59 \text{ kips/ft}$$

$$M_u = 3.59 \times \frac{20^2}{8} = 179 \text{ ft-kips} = 2150 \text{ in-kips}$$

Assume a = 4.0 in. and ϕ = 0.90. The structural depth is (16 − 2.5) in. = 13.5 in. Calculating A_s gives

$$A_s = \frac{M_u/\phi}{f_y(d - a/2)} = \frac{2150/0.90}{60(13.5 - 2.0)} = 3.46 \text{ in}^2$$

Try two No. 10 (No. 32) and one No. 9 (No. 29) bar, A_s = 3.54 in^2.

Check $a = 3.54 \times 60/(0.85 \times 5 \times 12) = 4.16$ in. from Eq. (3.32). This is more than assumed; therefore, continue to check the moment capacity.

$$M_n = 3.54 \times 60(13.5 - 4.16/2) = 2426 \text{ in-kips}$$

Using a ϕ of 0.90 gives $\phi M_n = 2183$ in-kips, which is adequate; however, the net tensile strain must be checked to validate the selection of $\phi = 0.9$. In this case $c = a/\beta_1 = 4.16/0.80 = 5.20$ in. The c/d ratio is $0.385 > 0.375$, so $\epsilon_t > 0.005$ is not satisfied. The corresponding net tensile strain is

$$\epsilon_t = 0.003 \frac{13.5 - 5.2}{5.2} = 0.00479$$

A value of $\epsilon_t = 0.00479$ is allowed by the ACI Code, but only if the strength reduction factor is adjusted. A linear interpolation from Fig. 3.9 gives $\phi = 0.88$ and $M_u = \phi M_n = 2140$ in-kips, which is less than the required capacity. Try increasing the reinforcement to three No. 10 (No. 32) bars, $A_s = 3.81$ in^2. Repeating the calculations,

$$a = \frac{3.81 \times 60}{0.85 \times 5 \times 12} = 4.48 \text{ in.}$$

$$c = \frac{4.48}{0.80} = 5.60 \text{ in.}$$

$$M_n = 3.81 \times 60\left(13.5 - \frac{4.48}{2}\right) = 2574 \text{ in-kips}$$

$$\epsilon_t = \frac{0.003(13.5 - 5.60)}{5.60} = 0.00423$$

$$\phi = 0.483 + 83.3 \times 0.00423 = 0.835$$

$$M_u = \phi M_n = 0.835 \times 2574 = 2150 \text{ in-kips}$$

which meets the design requirements.

In actuality, the first solution deviates less than 1 percent from the desired value and would likely be acceptable. The remaining portion of the example demonstrates the design implications of requiring a variable strength reduction factor when the net tensile strain falls between 0.005 and 0.004. In this example, the reinforcement increased nearly 8 percent, yet the design moment capacity ϕM_n only increased 0.5 percent due to the decreasing strength reduction factor. For this reason, designs with $\rho < \rho_{0.005}$ are desirable.

In solving these examples, the basic equations have been used to develop familiarity with them. In actual practice, however, design aids such as Table A.4 of Appendix A, giving values of maximum and minimum reinforcement ratios, and Table A.5, providing values of flexural resistance factor R, are more convenient. The example problems will be repeated in Section 3.5 to demonstrate use of these aids.

g. Overreinforced Beams

According to the ACI Code, all beams are to be designed for yielding of the tension steel with ϵ_t not less than 0.004 and thus $\rho \leq \rho_{max}$. Occasionally, however, such as when analyzing the capacity of existing construction, it may be necessary to calculate the flexural strength of an overreinforced compression-controlled member, for which f_s is less than f_y at flexural failure.

In this case, the steel strain, in Fig. 3.11*b*, will be less than the yield strain, but can be expressed in terms of the concrete strain ϵ_u and the still-unknown distance c to the neutral axis:

$$\epsilon_s = \epsilon_u \frac{d - c}{c} \tag{3.42}$$

From the equilibrium requirement that $C = T$, one can write

$$0.85\beta_1 f'_c bc = \rho \epsilon_s E_s bd$$

Substituting the steel strain from Eq. (3.42) in the last equation, and defining $k_u = c/d$, one obtains a quadratic equation in k_u as follows:

$$k_u^2 + m\rho k_u - m\rho = 0$$

Here, $\rho = A_s/bd$ as usual, and m is a material parameter given by

$$m = \frac{E_s \epsilon_u}{0.85\beta_1 f'_c} \tag{3.43}$$

Solving the quadratic equation for k_u,

$$k_u = \sqrt{m\rho + \left(\frac{m\rho}{2}\right)^2} - \frac{m\rho}{2} \tag{3.44}$$

The neutral axis depth for the overreinforced beam can then easily be found from $c = k_u d$, after which the stress-block depth $a = \beta_1 c$. With steel strain ϵ_s then computed from Eq. (3.42), and with $f_s = E_s\epsilon_s$, the nominal flexural strength is

$$M_n = A_s f_s \left(d - \frac{a}{2}\right) \tag{3.45}$$

The strength reduction factor ϕ will equal 0.65 for beams in this range.

3.5 DESIGN AIDS

Basic equations were developed in Section 3.4 for the analysis and design of reinforced concrete beams, and these were used directly in the examples. In practice, the design of beams and other reinforced concrete members is greatly facilitated by the use of aids such as those in Appendix A of this text and in Refs. 3.7 through 3.9. Tables A.1, A.2, A.4 through A.7, and Graph A.1 of Appendix A relate directly to this chapter, and the student can scan this material to become familiar with the coverage. Other aids will be discussed, and their use demonstrated, in later chapters.

Equation (3.39) gives the flexural design strength ϕM_n of an underreinforced rectangular beam with a reinforcement ratio at or below ρ_{max}. The flexural resistance factor R, from Eq. (3.36), is given in Table A.5a for lower reinforcement ratios or Table A.5b for higher reinforcement ratios. Alternatively, R can be obtained from Graph A.1. For *analysis* of the capacity of a section with known concrete dimensions b and d, having known reinforcement ratio ρ, and with known materials strengths, the design strength ϕM_n can be obtained directly by Eq. (3.39).

For *design* purposes, where concrete dimensions and reinforcement are to be found and the factored load moment M_u is to be resisted, there are two possible approaches. One starts with selecting the optimum reinforcement ratio and then calculating concrete dimensions, as follows:

1. Set the required strength M_u equal to the design strength ϕM_n from Eq. (3.39):

$$M_u = \phi R b d^2$$

2. With the aid of Table A.4, select an appropriate reinforcement ratio between ρ_{max} and ρ_{min}. Often a ratio of about $0.60\rho_{max}$ will be an economical and practical choice. Selection of $\rho \leq \rho_{0.005}$, ($\epsilon_t \geq 0.005$) ensures that ϕ will remain equal to 0.90. For $\rho_{0.005} < \rho < \rho_{max}$, an iterative solution will be necessary.
3. From Table A.5, for the specified material strengths and selected reinforcement ratio, find the flexural resistance factor R. Then

$$bd^2 = \frac{M_u}{\phi R}$$

4. Choose b and d to meet that requirement. Unless construction depth must be limited or other constraints exist (see Section 12.6), an effective depth about 2 to 3 times the width is often appropriate.
5. Calculate the required steel area

$$A_s = \rho bd$$

Then, referring to Table A.2, choose the size and number of bars, giving preference to the larger bar sizes to minimize placement costs.
6. Refer to Table A.7 to ensure that the selected beam width will provide room for the bars chosen, with adequate concrete cover and spacing. (These points will be discussed further in Section 3.6.)

The alternative approach starts with selecting concrete dimensions (see Section 12.6 for practical guidelines), after which the required reinforcement is found, as follows:

1. Select beam width b and effective depth d. Then calculate the required R:

$$R = \frac{M_u}{\phi bd^2}$$

2. Using Table A.5 for specified material strengths, find the reinforcement ratio $\rho < \rho_{max}$ that will provide the required value of R and verify the selected value of ϕ.
3. Calculate the required steel area

$$A_s = \rho bd$$

and from Table A.2 select the size and number of bars.
4. Using Table A.7, confirm that the beam width is sufficient to contain the selected reinforcement.

Use of design aids to solve the example problems of Section 3.4 will be illustrated as follows.

EXAMPLE 3.9 **Flexural strength of a given member.** Find the nominal flexural strength and design strength of the beam in Example 3.5, which has b = 12 in. and d = 17.5 in. and is reinforced with four No. 9 (No. 29) bars. Make use of the design aids of Appendix A. Material strengths are f'_c = 4000 psi and f_y = 60,000 psi.

Solution. From Table A.2, four No. 9 (No. 29) bars provide A_s = 4.00 in^2, and with b = 12 in. and d = 17.5 in., the reinforcement ratio is $\rho = 4.00/(12 \times 17.5) = 0.0190$. According to Table A.4, this is below $\rho_{max} = 0.0206$ and above $\rho_{min} = 0.0033$. Then from Table A.5b, with

$f_c' = 4000$ psi, $f_y = 60{,}000$ psi, and $\rho = 0.019$, the value $R = 949$ psi is found. The nominal and design strengths are (with $\phi = 0.87$ from Example 3.5), respectively,

$$M_n = Rbd^2 = 949 \times 12 \times \frac{17.5^2}{1000} = 3490 \text{ in-kips}$$

$$\phi M_n = 0.87 \times 3490 = 3040 \text{ in-kips}$$

as before.

EXAMPLE 3.10 **Concrete dimensions and steel area to resist a given moment.** Find the cross section of concrete and the area of steel required for the beam in Example 3.6, making use of the design aids of Appendix A. $M_u = 1670$ in-kips, $f_c' = 4000$ psi, and $f_y = 60{,}000$ psi. Use a reinforcement ratio of $0.60\rho_{max}$.

Solution. From Table A.4, the maximum reinforcement ratio is $\rho_{max} = 0.0206$. For economy, a value of $\rho = 0.60\rho_{max} = 0.0124$ will be used. For that value, by interpolation from Table A.5a, the required value of R is 663. Then

$$bd^2 = \frac{M_u}{\phi R} = \frac{1670 \times 1000}{0.90 \times 663} = 2800 \text{ in}^3$$

Concrete dimensions $b = 10$ in. and $d = 16.7$ in. will satisfy this, but the depth will be rounded to 17.5 in. to provide a total beam depth of 20.0 in. It follows that

$$R = \frac{M_u}{\phi bd^2} = \frac{1670 \times 1000}{0.90 \times 10 \times 17.5^2} = 606 \text{ psi}$$

and from Table A.5a, by interpolation, $\rho = 0.0112$. This leads to a steel requirement of $A_s = 0.0112 \times 10 \times 17.5 = 1.96$ in^2 as before.

EXAMPLE 3.11 **Determination of steel area.** Find the steel area required for the beam in Example 3.7, with concrete dimensions $b = 10$ in. and $d = 17.5$ in. known to be adequate to carry the factored load moment of 1300 in-lb. Material strengths are $f_c' = 4000$ psi and $f_y = 60{,}000$ psi.

Solution. Note that in cases in which the concrete dimensions are known to be adequate and only the reinforcement must be found, the iterative method used earlier is not required. The necessary flexural resistance factor is

$$R = \frac{M_u}{\phi bd^2} = \frac{1300 \times 1000}{0.90 \times 10 \times 17.5^2} = 472 \text{ psi}$$

According to Table A.5a, with the specified material strengths, this corresponds to a reinforcement ratio of $\rho = 0.0085$, giving a steel area of

$$A_s = 0.0085 \times 10 \times 17.5 = 1.49 \text{ in}^2$$

as before. Two No. 8 (No. 25) bars will be used.

The tables and graphs of Appendix A give basic information and are used extensively throughout this text for illustrative purposes. The reader should be aware, however, of the greatly expanded versions of these tables, plus many other useful aids, that are found in Refs. 3.7 through 3.9 and in commercial design software.

3.6 PRACTICAL CONSIDERATIONS IN THE DESIGN OF BEAMS

To focus attention initially on the basic aspects of flexural design, the preceding examples were carried out with only minimum regard for certain practical considerations that always influence the actual design of beams. These relate to optimal concrete proportions for beams, rounding of dimensions, standardization of dimensions, required cover for main and auxiliary reinforcement, and selection of bar combinations. Good judgment on the part of the design engineer is particularly important in translating from theoretical requirements to practical design. Several of the more important aspects are discussed here; much additional guidance is provided by the publications of ACI (Refs. 3.7 and 3.8) and CRSI (Refs. 3.9 to 3.11).

a. Concrete Protection for Reinforcement

To provide the steel with adequate concrete protection against fire and corrosion, the designer must maintain a certain minimum thickness of concrete cover outside of the outermost steel. The thickness required will vary, depending upon the type of member and conditions of exposure. According to ACI Code 7.7, for cast-in-place concrete, concrete protection at surfaces not exposed directly to the ground or weather should be not less than $\frac{3}{4}$ in. for slabs and walls and $1\frac{1}{2}$ in. for beams and columns. If the concrete surface is to be exposed to the weather or in contact with the ground, a protective covering of at least 2 in. is required [$1\frac{1}{2}$ in. for No. 5 (No. 16) and smaller bars], except that if the concrete is cast in direct contact with the ground without the use of forms, a cover of at least 3 in. must be furnished.

In general, the centers of main flexural bars in beams should be placed $2\frac{1}{2}$ to 3 in. from the top or bottom surface of the beam to furnish at least $1\frac{1}{2}$ in. of clear cover for the bars and the stirrups (see Fig. 3.13). In slabs, 1 in. to the center of the bar is ordinarily sufficient to give the required $\frac{3}{4}$ in. cover.

To simplify construction and thereby to reduce costs, the overall concrete dimensions of beams, b and h, are almost always rounded up to the nearest inch, and often to the next multiple of 2 in. As a result, the actual effective depth d, found by subtracting the sum of cover distance, stirrup diameter, and one-half the main

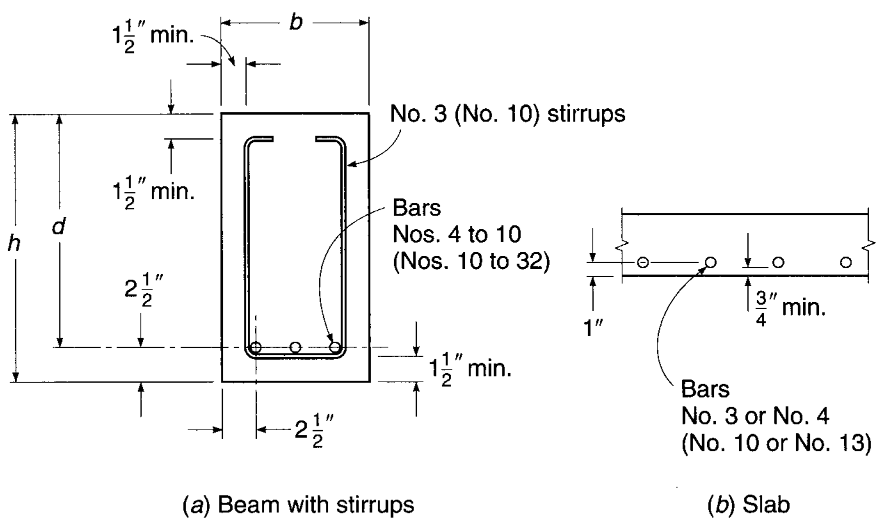


FIGURE 3.13
Requirements for concrete cover in beams and slabs.

reinforcing bar diameter from the total depth h, is seldom an even dimension. For slabs, the total depth is generally rounded up to the nearest $\frac{1}{2}$ in. up to 6 in. in depth, and to the nearest inch above that thickness. The differences between h and d shown in Fig. 3.13 are not exact, but are satisfactory for design purposes for beams with No. 3 (No. 10) stirrups and No. 10 (No. 32) longitudinal bars or smaller, and for slabs using No. 4 (No. 13) or smaller bars. If larger bars are used for the main flexural reinforcement or for the stirrups, as is frequently the case, the corresponding dimensions are easily calculated.

Recognizing the closer tolerances that can be maintained under plant-control conditions, ACI Code 7.7.3 permits some reduction in concrete protection for reinforcement in precast concrete.

b. Concrete Proportions

Reinforced concrete beams may be wide and shallow, or relatively narrow and deep. Consideration of maximum material economy often leads to proportions with effective depth d in the range from about 2 to 3 times the width b (or web width b_w for T beams). However, constraints may dictate other choices, and as will be discussed in Section 12.6, maximum material economy may not translate to maximum structural economy. For example, with one-way concrete joists supported by monolithic beams (see Chapter 18), use of beams and joists with the same total depth will permit use of a single flat-bottom form, resulting in fast, economical construction and permitting level ceilings. The beams will generally be wide and shallow, with heavier reinforcement than otherwise, but the result will be an overall saving in construction cost. In other cases, it may be necessary to limit the total depth of floor or roof construction for architectural or other reasons. An advantage of reinforced concrete is its adaptability to such special needs.

c. Selection of Bars and Bar Spacing

As noted in Section 2.14, common reinforcing bar sizes range from No. 3 to No. 11 (No. 10 to No. 36), the bar number corresponding closely to the number of eighth-inches (millimeters) of bar diameter. The two larger sizes, No. 14 (No. 43) [$1\frac{3}{4}$ in. (43 mm) diameter] and No. 18 (No. 57) [$2\frac{1}{4}$ in. (57 mm) diameter] are used mainly in columns.

It is often desirable to mix bar sizes to meet steel area requirements more closely. In general, mixed bars should be of comparable diameter, for practical as well as theoretical reasons, and generally should be arranged symmetrically about the vertical centerline. Many designers limit the variation in diameter of bars in a single layer to two bar sizes, using, say, No. 10 and No. 8 (No. 32 and No. 25) bars together, but not Nos. 11 and 6 (Nos. 36 and 19). There is some practical advantage to minimizing the number of different bar sizes used for a given structure.

Normally, it is necessary to maintain a certain minimum distance between adjacent bars to ensure proper placement of concrete around them. Air pockets below the steel are to be avoided, and full surface contact between the bars and the concrete is desirable to optimize bond strength. ACI Code 7.6 specifies that the minimum clear distance between adjacent bars not be less than the nominal diameter of the bars, or 1 in. (For columns, these requirements are increased to $1\frac{1}{2}$ bar diameters and $1\frac{1}{2}$ in.) Where beam reinforcement is placed in two or more layers, the clear distance between layers must not be less than 1 in., and the bars in the upper layer should be placed directly above those in the bottom layer.

The maximum number of bars that can be placed in a beam of given width is limited by bar diameter and spacing requirements and is also influenced by stirrup diameter, by concrete cover requirement, and by the maximum size of concrete aggregate specified. Table A.7 of Appendix A gives the maximum number of bars that can be placed in a single layer in beams, assuming $1\frac{1}{2}$ in. concrete cover and the use of No. 4 (No. 13) stirrups. When using the minimum bar spacing in conjunction with a large number of bars in a single plane of reinforcement, the designer should be aware that problems may arise in the placement and consolidation of concrete, especially when multiple layers of bars are used or when the bar spacing is smaller than the size of the vibrator head.

There are also restrictions on the *minimum* number of bars that can be placed in a single layer, based on requirements for the distribution of reinforcement to control the width of flexural cracks (see Section 6.3). Table A.8 gives the minimum number of bars that will satisfy ACI Code requirements, which will be discussed in Chapter 6.

In large girders and columns, it is sometimes advantageous to "bundle" tensile or compressive reinforcement with two, three, or four bars in contact to provide for better deposition of concrete around and between adjacent bundles. These bars may be assumed to act as a unit, with not more than four bars in any bundle, provided that stirrups or ties enclose the bundle. No more than two bars should be bundled in one plane; typical bundle shapes are triangular, square, or L-shaped patterns. Individual bars in a bundle, cut off within the span of flexural members, should terminate at different points. ACI Code 7.6.6 requires at least 40 bar diameters stagger between points of cutoff. Where spacing limitations and minimum concrete cover requirements are based on bar diameter, a unit of bundled bars is treated as a single bar with a diameter that provides the same total area.

ACI Code 7.6.6 states that bars larger than No. 11 (No. 36) shall not be bundled in beams, although the AASHTO Specifications permit bundling of No. 14 and No. 18 (No. 43 and No. 57) bars in highway bridges.

3.7 RECTANGULAR BEAMS WITH TENSION AND COMPRESSION REINFORCEMENT

If a beam cross section is limited because of architectural or other considerations, it may happen that the concrete cannot develop the compression force required to resist the given bending moment. In this case, reinforcement is added in the compression zone, resulting in a *doubly reinforced* beam, i.e., one with compression as well as tension reinforcement (see Fig. 3.14). The use of compression reinforcement has decreased markedly with the use of strength design methods, which account for the full-strength potential of the concrete on the compressive side of the neutral axis. However, there are situations in which compressive reinforcement is used for reasons other than strength. It has been found that the inclusion of some compression steel will reduce the long-term deflections of members (see Section 6.5). In addition, in some cases, bars will be placed in the compression zone for minimum-moment loading (see Section 12.2) or as stirrup support bars continuous throughout the beam span (see Chapter 4). It may be desirable to account for the presence of such reinforcement in flexural design, although in many cases they are neglected in flexural calculations.

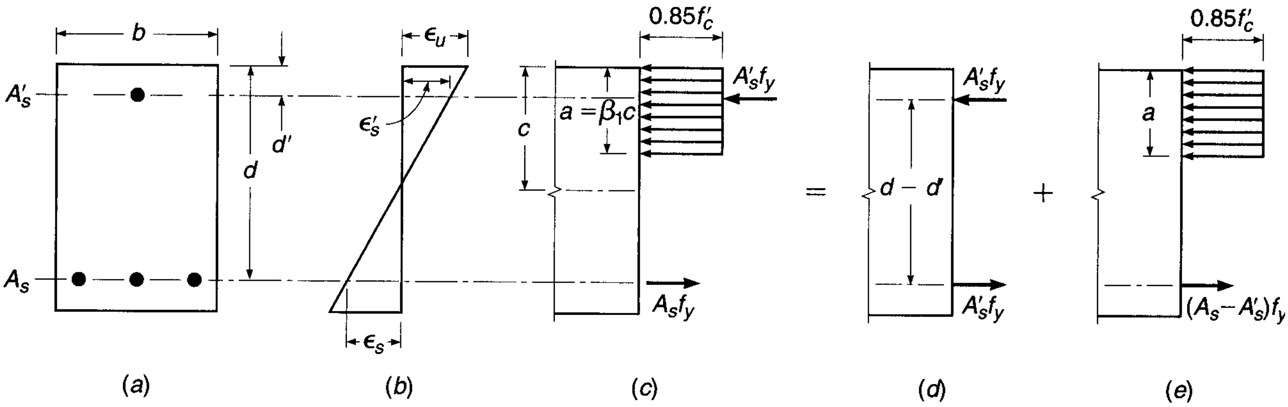


FIGURE 3.14
Doubly reinforced rectangular beam.

a. Tension and Compression Steel Both at Yield Stress

If, in a doubly reinforced beam, the tensile reinforcement ratio ρ is less than or equal to ρ_b, the strength of the beam may be approximated within acceptable limits by disregarding the compression bars. The strength of such a beam will be controlled by tensile yielding, and the lever arm of the resisting moment will ordinarily be little affected by the presence of the compression bars.

If the tensile reinforcement ratio is larger than ρ_b, a somewhat more elaborate analysis is required. In Fig. 3.14*a*, a rectangular beam cross section is shown with compression steel A'_s placed a distance d' from the compression face and with tensile steel A_s at effective depth d. It is assumed initially that both A'_s and A_s are stressed to f_y at failure. The total resisting moment can be thought of as the sum of two parts. The first part, M_{n1}, is provided by the couple consisting of the force in the compression steel A'_s and the force in an equal area of tension steel

$$M_{n1} = A'_s f_y (d - d') \tag{3.46a}$$

as shown in Fig. 3.14*d*. The second part, M_{n2}, is the contribution of the remaining tension steel $A_s - A'_s$ acting with the compression concrete:

$$M_{n2} = (A_s - A'_s) f_y \left(d - \frac{a}{2} \right) \tag{3.46b}$$

as shown in Fig. 3.14*e*, where the depth of the stress block is

$$a = \frac{(A_s - A'_s) f_y}{0.85 f'_c b} \tag{3.47a}$$

With the definitions $\rho = A_s/bd$ and $\rho' = A'_s/bd$, this can be written

$$a = \frac{(\rho - \rho') f_y d}{0.85 f'_c} \tag{3.47b}$$

The total nominal resisting moment is then

$$M_n = M_{n1} + M_{n2} = A'_s f_y (d - d') + (A_s - A'_s) f_y \left(d - \frac{a}{2} \right) \tag{3.48}$$

In accordance with the safety provisions of the ACI Code, the net tensile strain is checked; and if $\epsilon_t \geq 0.005$, this nominal capacity is reduced by the factor $\phi = 0.90$ to obtain the design strength. For ϵ_t between 0.005 and 0.004, ϕ must be adjusted, as discussed earlier.

It is highly desirable, for reasons given earlier, that failure, should it occur, be precipitated by tensile yielding rather than crushing of the concrete. This can be ensured by setting an *upper limit* on the tensile reinforcement ratio. By setting the tensile steel strain in Fig. 3.14*b* equal to ϵ_y to establish the location of the neutral axis for the failure condition and then summing horizontal forces shown in Fig. 3.14*c* (still assuming the compressive steel to be at the yield stress at failure), it is easily shown that the balanced reinforcement ratio $\bar{\rho}_b$ for a doubly reinforced beam is

$$\bar{\rho}_b = \rho_b + \rho' \tag{3.49}$$

where ρ_b is the balanced reinforcement ratio for the corresponding singly reinforced beam and is calculated from Eq. (3.28). The ACI Code limits the net tensile strain, not the reinforcement ratio. To provide the same margin against brittle failure as for singly reinforced beams, the maximum reinforcement ratio should be limited to

$$\bar{\rho}_{\max} = \rho_{\max} + \rho' \tag{3.50a}$$

Because $\rho_{\max}$ establishes the location of the neutral axis, the limitation in Eq. (3.50*a*) will provide acceptable net tensile strains. A check of ϵ_t is required to determine the strength reduction factor ϕ and to verify net tensile strain requirements are satisfied. Substituting $\rho_{0.005}$ for $\rho_{\max}$ in Eq. (3.50*a*) will give the maximum reinforcement ratio for $\phi = 0.90$.

$$\bar{\rho}_{0.005} = \rho_{0.005} + \rho' \tag{3.50b}$$

b. Compression Steel below Yield Stress

The preceding equations, through which the fundamental analysis of doubly reinforced beams is developed clearly and concisely, are valid *only* if the compression steel has yielded when the beam reached its nominal capacity. In many cases, such as for wide, shallow beams, beams with more than the usual concrete cover over the compression bars, beams with high yield strength steel, or beams with relatively small amounts of tensile reinforcement, the compression bars will be below the yield stress at failure. It is necessary, therefore, to develop more generally applicable equations to account for the possibility that the compression reinforcement has not yielded when the doubly reinforced beam fails in flexure.

Whether or not the compression steel will have yielded at failure can be determined as follows. Referring to Fig. 3.14*b*, and taking as the limiting case $\epsilon_s' = \epsilon_y$, one obtains, from geometry,

$$\frac{c}{d'} = \frac{\epsilon_u}{\epsilon_u - \epsilon_y} \qquad \text{or} \qquad c = \frac{\epsilon_u}{\epsilon_u - \epsilon_y} d'$$

Summing forces in the horizontal direction (Fig. 3.14*c*) gives the *minimum* tensile reinforcement ratio $\bar{\rho}_{cy}$ that will ensure yielding of the compression steel at failure:

$$\bar{\rho}_{cy} = 0.85\beta_1 \frac{f_c'}{f_y} \frac{d'}{d} \frac{\epsilon_u}{\epsilon_u - \epsilon_y} + \rho' \tag{3.51}$$

If the *tensile reinforcement ratio* is *less than* this limiting value, the neutral axis is sufficiently high that the compression steel stress at failure is less than the yield stress. In this case, it can easily be shown on the basis of Fig. 3.14*b* and *c* that the balanced reinforcement ratio is

$$\bar{\rho}_b = \rho_b + \rho' \frac{f'_s}{f_y} \tag{3.52}$$

where

$$f'_s = E_s \epsilon'_s = E_s \left[\epsilon_u - \frac{d'}{d}(\epsilon_u + \epsilon_y) \right] \le f_y \tag{3.53a}$$

To determine $\rho_{\max}$, $\epsilon_t = 0.004$ is substituted for ϵ_y in Eq. (3.53*a*), giving

$$f'_s = E_s \left[\epsilon_u - \frac{d'}{d}(\epsilon_u + 0.004) \right] \le f_y \tag{3.53b}$$

Likewise, for $\epsilon_t = 0.005$,

$$f'_s = E_s \left[\epsilon_u - \frac{d'}{d}(\epsilon_u + 0.005) \right] \le f_y \tag{3.53c}$$

Hence, the maximum reinforcement ratio permitted by the ACI Code is

$$\bar{\rho}_{\max} = \rho_{\max} + \rho' \frac{f'_s}{f_y} \tag{3.54a}$$

and the maximum reinforcement ratio for $\phi = 0.90$ is

$$\bar{\rho}_{0.005} = \rho_{0.005} + \rho' \frac{f'_s}{f_y} \tag{3.54b}$$

where f'_s is given in Eq. (3.53*b*). A simple comparison shows that Eqs. (3.52), (3.54*a*), and (3.54*b*), with f'_s given by Eqs. (3.53*a*), (3.53*b*), and (3.53*c*), respectively, are the generalized forms of Eqs. (3.49), (3.50*a*), and (3.50*b*).

It should be emphasized that Eqs. (3.53*a*), (3.53*b*), and (3.53*c*) for compression steel stress apply *only for beams with exact strain values in the extreme tensile steel of* ϵ_y, $\epsilon_t = 0.004$, *or* $\epsilon_t = 0.005$.

If the tensile reinforcement ratio is less than $\bar{\rho}_b$, as given by Eq. (3.52), and less than $\bar{\rho}_{cy}$, as given by Eq. (3.51), then the tensile steel is at the yield stress at failure but the compression steel is not, and new equations must be developed for compression steel stress and flexural strength. The compression steel stress can be expressed in terms of the still-unknown neutral axis depth as

$$f'_s = \epsilon_u E_s \frac{c - d'}{c} \tag{3.55}$$

Consideration of horizontal force equilibrium (Fig. 3.14*c* with compression steel stress equal to f'_s) then gives

$$A_s f_y = 0.85 \beta_1 f'_c bc + A'_s \epsilon_u E_s \frac{c - d'}{c} \tag{3.56}$$

This is a quadratic equation in c, the only unknown, and is easily solved for c. The nominal flexural strength is found using the value of f'_s from Eq. (3.55), and $a = \beta_1 c$ in the expression

$$M_n = 0.85 f'_c ab \left(d - \frac{a}{2} \right) + A'_s f'_s (d - d') \tag{3.57}$$

TABLE 3.2
Minimum beam depths for compression reinforcement to yield

	$\epsilon_t = 0.004$		$\epsilon_t = 0.005$	
f_y, psi	Maximum d'/d	Minimum d for $d' = 2.5$ in., in.	Maximum d'/d	Minimum d for $d' = 2.5$ in., in.
40,000	0.23	10.8	0.20	12.3
60,000	0.13	18.8	0.12	21.5
75,000	0.06	42.7	0.05	48.8

This nominal capacity is reduced by the strength reduction factor ϕ to obtain the design strength.

If compression bars are used in a flexural member, precautions must be taken to ensure that these bars will not buckle outward under load, spalling off the outer concrete. ACI Code 7.11.1 imposes the requirement that such bars be anchored in the same way that compression bars in columns are anchored by lateral ties (Section 8.2). Such ties must be used throughout the distance where the compression reinforcement is required.

For the compression steel to yield, the reinforcement ratio must lie below $\bar{\rho}_{max}$ and above $\bar{\rho}_{cy}$. The ratio between d' and the steel centroidal depth d to allow yielding of the compression reinforcement can be found by equating $\bar{\rho}_{cy}$ to $\bar{\rho}_{max}$ (or $\bar{\rho}_{0.005}$) and solving for d'/d. Furthermore, if d' is assumed to be 2.5 in., as is often the case, the minimum depth of beam necessary for the compression steel to yield may be found for each grade of steel. The ratios and minimum beam depths are summarized in Table 3.2. Values are included for $\epsilon_t = 0.004$, the minimum tensile yield strain permitted for flexural members, and $\epsilon_t = 0.005$, the net tensile strain needed to ensure that $\phi = 0.90$. For beams with less than the minimum depth, the compression reinforcement cannot yield unless the tensile reinforcement exceeds ρ_{max}. The compression reinforcement may yield in beams that exceed the minimum depth in Table 3.2, depending on the relative distribution of the tensile and compressive reinforcement.

c. Examples of Analysis and Design of Beams with Tension and Compression Steel

As was the case for beams with only tension reinforcement, doubly reinforced beam problems can be placed in one of two categories: analysis problems or design problems. For *analysis*, in which the concrete dimensions, reinforcement, and material strengths are given, one can find the flexural strength directly from the equations in Section 3.7a or 3.7b. First, it must be confirmed that the tensile reinforcement ratio is less than $\bar{\rho}_b$ given by Eq. (3.52), with compression steel stress from Eq. (3.53*a*). Once it is established that the tensile steel has yielded, the tensile reinforcement ratio defining compression steel yielding is calculated from Eq. (3.51), and the actual tensile reinforcement ratio is compared. If it is greater than $\bar{\rho}_{cy}$, then $f'_s = f_y$, and M_n is found from Eq. (3.48). If it is less than $\bar{\rho}_{cy}$, then $f'_s < f_y$. In this case, c is calculated by solving Eq. (3.56), f'_s comes from Eq. (3.55), and M_n is found from Eq. (3.57).

For the *design* case, in which the factored load moment M_u to be resisted is known and the section dimensions and reinforcement are to be found, a direct solution is impossible. The steel areas to be provided depend on the steel stresses, which are not known before the section is proportioned. It can be assumed that the compression steel stress is equal to the yield stress, but this must be confirmed; if it has not yielded, the design must be adjusted. The design procedure can be outlined as follows:

1. Calculate the maximum moment that can be resisted by the underreinforced section with $\rho = \rho_{max}$, or $\rho_{0.005}$ to ensure that $\phi = 0.90$. The corresponding tensile steel area is $A_s = \rho bd$, and, as usual,

$$M_n = A_s f_y\left(d - \frac{a}{2}\right)$$

with

$$a = \frac{A_s f_y}{0.85 f'_c b}$$

2. Find the excess moment, if any, that must be resisted, and set $M_2 = M_n$, as calculated in step 1.

$$M_1 = \frac{M_u}{\phi} - M_2$$

Now A_s from step 1 is defined as A_{s2}, i.e., that part of the tension steel area in the doubly reinforced beam that works with the compression force in the concrete. In Fig. 3.14*e*, $A_s - A'_s = A_{s2}$.
3. Tentatively assume that $f'_s = f_y$. Then

$$A'_s = \frac{M_1}{f_y(d - d')}$$

Alternatively, if from Table 3.2 the compression reinforcement is known not to yield, go to step 6.
4. Add an additional amount of tensile steel $A_{s1} = A'_s$. Thus, the total tensile steel area A_s is A_{s2} from step 2 plus A_{s1}.
5. Analyze the doubly reinforced beam to see if $f'_s = f_y$; that is, check the tensile reinforcement ratio against $\bar{\rho}_{cy}$.
6. If $\rho < \bar{\rho}_{cy}$, then the compression steel stress is less than f_y and the compression steel area must be increased to provide the needed force. This can be done as follows. The stress block depth is found from the requirement of horizontal equilibrium (Fig. 3.14*e*),

$$a = \frac{(A_s - A'_s)f_y}{0.85 f'_c b} \quad \text{or} \quad a = \frac{\left[A_s - A'_s\,(f'_s/f_y)\right]f_y}{0.85 f'_c b}$$

and the neutral axis depth is $c = a/\beta_1$. From Eq. (3.55),

$$f'_s = \epsilon_u E_s \frac{c - d'}{c}$$

The revised compression steel area, acting at f'_s, must provide the same force as the trial steel area that was assumed to act at f_y. Therefore,

$$A'_{s,\text{revised}} = A'_{s,\text{trial}} \frac{f_y}{f'_s}$$

The tensile steel area need not be revised, because it acts at f_y as assumed.

EXAMPLE 3.12 **Flexural strength of a given member.** A rectangular beam, shown in Fig. 3.15, has a width of 12 in. and an effective depth to the centroid of the tension reinforcement of 24 in. The tension reinforcement consists of six No. 10 (No. 32) bars in two rows. Compression reinforcement consisting of two No. 8 (No. 25) bars is placed 2.5 in. from the compression face of the beam. If $f_y = 60{,}000$ psi and $f'_c = 5000$ psi, what is the design moment capacity of the beam?

Solution. The steel areas and ratios are

$$A_s = 7.62 \text{ in}^2 \qquad \rho = \frac{7.62}{12 \times 24} = 0.0265$$

$$A'_s = 1.58 \text{ in}^2 \qquad \rho' = \frac{1.58}{12 \times 24} = 0.0055$$

Check the beam first as a singly reinforced beam to see if the compression bars can be disregarded,

$$\rho_{\max} = 0.0243 \qquad \text{from Table A.4 or Eq. (3.30}c\text{)}$$

The actual $\rho = 0.0265$ is larger than $\rho_{\max}$, so the beam must be analyzed as doubly reinforced. From Eq. (3.51), with $\beta_1 = 0.80$,

$$\bar{\rho}_{cy} = 0.85 \times 0.80 \times \frac{5}{60} \times \frac{2.5}{24} \times \frac{0.003}{0.003 - 0.00207} + 0.0055 = 0.0245$$

The tensile reinforcement ratio is greater than this, so the compression bars will yield when the beam fails. The maximum reinforcement ratio thus can be found from Eq. (3.50),

$$\bar{\rho}_{\max} = 0.0243 + 0.0055 = 0.0298$$

The actual tensile reinforcement ratio is below the maximum value, as required. Then, from Eq. (3.47a),

$$a = \frac{(7.62 - 1.58)60}{0.85 \times 5 \times 12} = 7.11 \text{ in.}$$

$$c = a/\beta_1 = \frac{7.11}{0.80} = 8.89 \text{ in.}$$

$$\epsilon_t = 0.003\left(\frac{24 - 8.89}{8.89}\right) = 0.0051$$

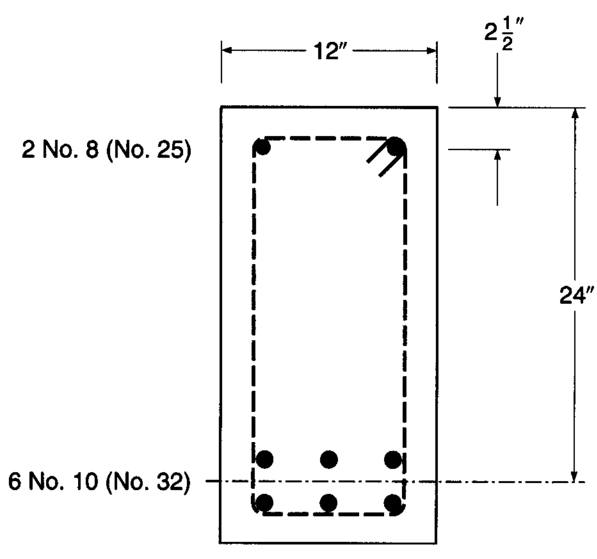


FIGURE 3.15
Doubly reinforced beam of Example 3.12.

and

$$\phi = 0.90$$

and from Eq. (3.48),

$$M_n = 1.58 \times 60(24 - 2.5) + 6.04 \times 60\left(24 - \frac{7.11}{2}\right) = 9450 \text{ in-kips}$$

The design strength is

$$\phi M_n = 0.90 \times 9450 = 8500 \text{ in-kips}$$

EXAMPLE 3.13 **Design of a doubly reinforced beam.** A rectangular beam that must carry a service live load of 2.47 kips/ft and a calculated dead load of 1.05 kips/ft on an 18 ft simple span is limited in cross section for architectural reasons to 10 in. width and 20 in. total depth. If $f_y = 60{,}000$ psi and $f'_c = 4000$ psi, what steel area(s) must be provided?

SOLUTION. The service loads are first increased by load factors to obtain the factored load of $1.2 \times 1.05 + 1.6 \times 2.47 = 5.21$ kips/ft. Then $M_u = 5.21 \times 18^2/8 = 211$ ft-kips $= 2530$ in-kips. To satisfy spacing and cover requirements (see Section 3.6), assume that the tension steel centroid will be 4 in. above the bottom face of the beam and that compression steel, if required, will be placed 2.5 in. below the beam's top surface. Then $d = 16$ in. and $d' = 2.5$ in.

First, check the capacity of the section if singly reinforced. Table A.4 shows that $\rho_{0.005}$, the maximum value of ρ for $\phi = 0.90$, to be 0.0181. While the maximum reinforcement ratio is slightly higher, Example 3.8 demonstrated there was no economic efficiency of using $\epsilon_t \leq 0.005$. So $A_s = 10 \times 16 \times 0.0181 = 2.90$ in^2. Then with

$$a = \frac{2.90 \times 60}{0.85 \times 4 \times 10} = 5.12 \text{ in.}$$

$c = a/\beta_1 = 5.12/0.85 = 6.02$ in., and the maximum nominal moment that can be developed is

$$M_n = 2.90 \times 60(16 - 5.12/2) = 2340 \text{ in-kips}$$

Alternatively, using $R = 913$ from Table A.5b, the nominal moment is $M_n = 913 \times 10 \times 16^2/1000 = 2340$ in-kips. Because the corresponding design moment $\phi M_n = 2100$ in-kips is less than the required capacity 2530 in-kips, compression steel is needed as well as additional tension steel.

The remaining moment to be carried by the compression steel couple is

$$M_1 = \frac{2530}{0.90} - 2340 = 470 \text{ in-kips}$$

As d is less than the value required to develop the compression reinforcement yield stress (Table 3.2), a reduced stress in the compression reinforcement will be used. Using the strain distribution in Fig. 3.14*b*, ϵ'_s and f'_c can be computed as

$$\epsilon'_s = 0.003\,\frac{6.02 - 2.5}{6.02} = 0.00175 \qquad \text{and} \qquad f'_s = 0.00175 \times 29{,}000 = 50.9 \text{ ksi}$$

Try $f'_s = 50$ ksi for the compression reinforcement to obtain the required area of compression steel.

$$A'_s = \frac{470}{50(16 - 2.5)} = 0.70 \text{ in}^2$$

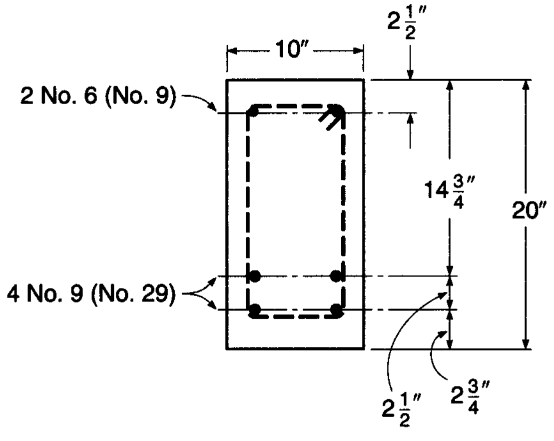


FIGURE 3.16
Doubly reinforced beam of Example 3.13.

The total area of tensile reinforcement at 60 ksi is

$$A_s = 2.90 + 0.70\left(\frac{50}{60}\right) = 3.48 \text{ in}^2$$

Two No. 6 (No. 19) bars will be used for the compression reinforcement and four No. 9 (No. 29) bars will provide the tensile steel area, as shown in Fig. 3.16. To place the tension bars in a 10 in. beam width, two rows of two bars each are used.

A final check is made to ensure that the selection of reinforcement does not create a lower compressive stress than the assumed 50 ksi.

$$A_s - A'_s\left(\frac{f'_s}{f_y}\right) = 4.0 - 0.88\left(\frac{50}{60}\right) = 3.27 \text{ in}^2$$

which is greater than 2.90 in^2 for $\epsilon_t = 0.005$, so $\phi < 0.90$.

$$a = \frac{3.27 \times 60}{0.85 \times 4 \times 10} = 5.77 \text{ in.}$$

$$c = \frac{5.77}{0.85} = 6.79 \text{ in.}$$

$$\epsilon'_s = 0.003\,\frac{6.79 - 2.5}{6.79} = 0.0019$$

$$f'_s = 29{,}000 \times 0.0019 = 55.0 \text{ ksi}$$

which is greater than assumed. Check ϕ, using $d_t = 17.25$ from the strain distribution in Fig. 3.14*b*, and compute the revised M_u. For simplicity, the area of tensile reinforcement is not modified.

$$\epsilon_t = 0.003\,\frac{17.25 - 6.79}{6.79} = 0.0046$$

for which $\phi = 0.87$. Then

$$M_u = 0.87\left[3.27 \times 60\left(16.0 - \frac{5.77}{2}\right) + 0.88 \times 55.0(16 - 2.5)\right] = 2810 \text{ in-kips}$$

This is greater than M_u, so no further refinement is necessary.

d. Tensile Steel below the Yield Stress

All doubly reinforced beams designed according to the ACI Code must be underreinforced, in the sense that the tensile reinforcement ratio is limited to ensure yielding at beam failure. Two cases were considered in Sections 3.7a and 3.7b, respectively: (a) both tension steel and compression steel yield and (b) tension steel yields but compression steel does not. Two other combinations may be encountered in analyzing the capacity of existing beams: (c) tension steel does not yield, but compression steel does, and (d) neither tension steel nor compression steel yields. The last two cases are unusual, and in fact, it would be difficult to place sufficient tension reinforcement to create such conditions, but it is possible. The solution in such cases is obtained as a simple extension of the treatment of Section 3.7b. An equation for horizontal equilibrium is written, in which both tension and compression steel stress are expressed in terms of the unknown neutral axis depth c. The resulting quadratic equation is solved for c, after which steel stresses can be calculated and the nominal flexural strength determined.

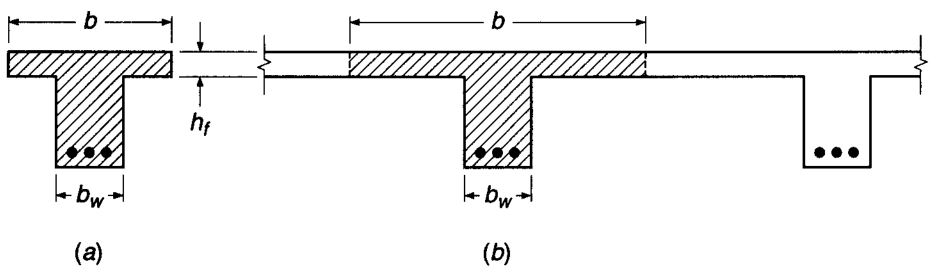
3.8 T BEAMS

With the exception of precast systems, reinforced concrete floors, roofs, decks, etc., are almost always monolithic. Forms are built for beam soffits and sides and for the underside of slabs, and the entire construction is cast at once, from the bottom of the deepest beam to the top of the slab. Beam stirrups and bent bars extend up into the slab. It is evident, therefore, that a part of the slab will act with the upper part of the beam to resist longitudinal compression. The resulting beam cross section is T-shaped rather than rectangular. The slab forms the beam flange, while the part of the beam projecting below the slab forms what is called the *web* or *stem*. The upper part of such a T beam is stressed laterally due to slab action in that direction. Although transverse compression at the level of the bottom of the slab may increase the longitudinal compressive strength by as much as 25 percent, transverse tension at the top surface reduces the longitudinal compressive strength (see Section 2.10). Neither effect is usually taken into account in design.

a. Effective Flange Width

The next issue to be resolved is that of the effective width of flange. In Fig. 3.17*a*, it is evident that if the flange is but little wider than the stem width, the entire flange can be considered effective in resisting compression. For the floor system shown in Fig. 3.17*b*, however, it may be equally obvious that elements of the flange midway between the beam stems are less highly stressed in longitudinal compression than those elements directly over the stem. This is so because of shearing deformation of the flange, which relieves the more remote elements of some compressive stress.

FIGURE 3.17
Effective flange width of T beams.



Although the actual longitudinal compression varies because of this effect, it is convenient in design to make use of an *effective flange width*, which may be smaller than the actual flange width but is considered to be uniformly stressed at the maximum value. This effective width has been found to depend primarily on the beam span and on the relative thickness of the slab.

The criteria for effective width given in ACI Code 8.12 are as follows:

1. For symmetric T beams, the effective width b shall not exceed one-fourth the span length of the beam. The overhanging slab width on either side of the beam web shall not exceed 8 times the thickness of the slab or go beyond one-half the clear distance to the next beam.
2. For beams having a slab on one side only, the effective overhanging slab width shall not exceed one-twelfth the span length of the beam, 6 times the slab thickness, or one-half the clear distance to the next beam.
3. For isolated beams in which the flange is used only for the purpose of providing additional compressive area, the flange thickness shall not be less than one-half the width of the web, and the total flange width shall not be more than 4 times the web width.

b. Strength Analysis

The neutral axis of a T beam may be either in the flange or in the web, depending upon the proportions of the cross section, the amount of tensile steel, and the strengths of the materials. If the calculated depth to the neutral axis is less than or equal to the flange thickness h_f, the beam can be analyzed as if it were a rectangular beam of width equal to b, the effective flange width. The reason is illustrated in Fig. 3.18*a*, which shows a T beam with the neutral axis in the flange. The compressive area is indicated by the shaded portion of the figure. If the additional concrete indicated by areas 1 and 2 had been added when the beam was cast, the physical cross section would have been rectangular with a width b. No bending strength would have been added because areas 1 and 2 are entirely in the tension zone, and tension concrete is disregarded in flexural calculations. The original T beam and the rectangular beam are equal in flexural strength, and rectangular beam analysis for flexure applies.

When the neutral axis is in the web, as in Fig. 3.18*b*, the preceding argument is no longer valid. In this case, methods must be developed to account for the actual T-shaped compressive zone.

In treating T beams, it is convenient to adopt the same equivalent stress distribution that is used for beams of rectangular cross section. The rectangular stress block, having a uniform compressive-stress intensity $0.85f'_c$, was devised originally on the

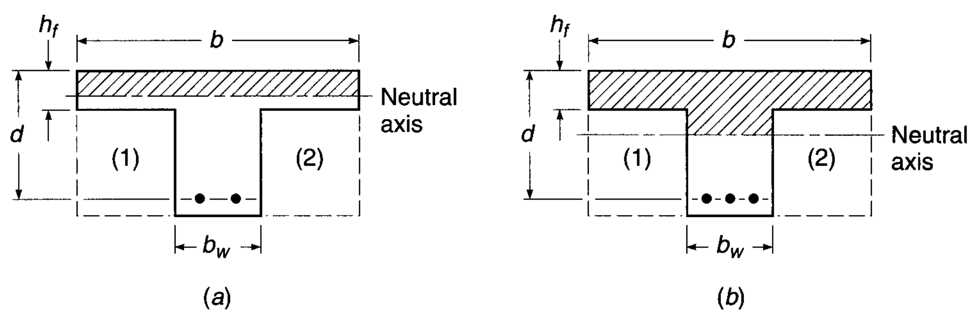


FIGURE 3.18
Effective cross sections of T beams.

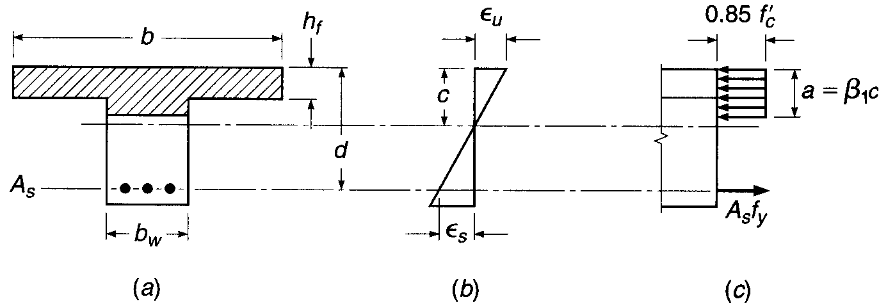


FIGURE 3.19
Strain and equivalent stress distributions for T beams.

basis of tests of rectangular beams (see Section 3.4a), and its suitability for T beams may be questioned. However, extensive calculations based on actual stress-strain curves (reported in Ref. 3.12) indicate that its use for T beams, as well as for beams of circular or triangular cross section, introduces only minor error and is fully justified.

Accordingly, a T beam may be treated as a rectangular beam if the depth of the equivalent stress block is less than or equal to the flange thickness. Figure 3.19 shows a tensile-reinforced T beam with effective flange width b, web width b_w, effective depth to the steel centroid d, and flange thickness h_f. If for trial purposes the stress block is assumed to be completely within the flange,

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{\rho f_y d}{0.85 f_c'} \tag{3.58}$$

where $\rho = A_s/bd$. If a is less than or equal to the flange thickness h_f, the member may be treated as a rectangular beam of width b and depth d. If a is greater than h_f, a T beam analysis is required as follows.

It will be assumed that the strength of the T beam is controlled by yielding of the tensile steel. This will nearly always be the case because of the large compressive concrete area provided by the flange. In addition, an upper limit can be established for the reinforcement ratio to ensure that this is so, as will be shown.

As a computational device, it is convenient to divide the total tensile steel into two parts, as shown in Fig. 3.20. The first part, A_{sf}, represents the steel area that, when stressed to f_y, is required to balance the longitudinal compressive force in the overhanging portions of the flange that are stressed uniformly at $0.85f_c'$ (Fig. 3.20b). Thus,

$$A_{sf} = \frac{0.85 f_c' (b - b_w) h_f}{f_y} \tag{3.59}$$

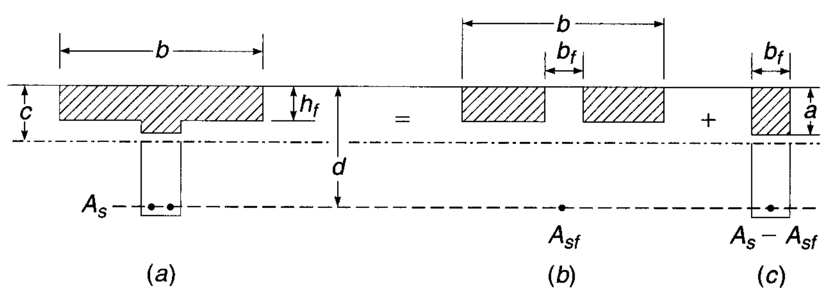


FIGURE 3.20
Computational model for design and analysis of T beams.

The force $A_{sf}f_y$ and the equal and opposite force $0.85\,f_c'(b - b_w)h_f$ act with a lever arm $d - h_f/2$ to provide the nominal resisting moment

$$M_{n1} = A_{sf}f_y\left(d - \frac{h_f}{2}\right) \tag{3.60}$$

The remaining steel area $A_s - A_{sf}$, at a stress f_y, is balanced by the compression in the rectangular portion of the beam (Fig. 3.20*c*). The depth of the equivalent rectangular stress block in this zone is found from horizontal equilibrium.

$$a = \frac{(A_s - A_{sf})f_y}{0.85 f_c'\, b_w} \tag{3.61}$$

An additional moment M_{n2} is thus provided by the forces $(A_s - A_{sf})f_y$ and $0.85f_c'ab_w$ acting at the lever arm $d - a/2$.

$$M_{n2} = (A_s - A_{sf})f_y\left(d - \frac{a}{2}\right) \tag{3.62}$$

and the total nominal resisting moment is the sum of the parts:

$$M_n = M_{n1} + M_{n2} = A_{sf}f_y\left(d - \frac{h_f}{2}\right) + (A_s - A_{sf})f_y\left(d - \frac{a}{2}\right) \tag{3.63}$$

This moment is reduced by the strength reduction factor ϕ in accordance with the safety provisions of the ACI Code to obtain the design strength.

As for rectangular beams, the tensile steel should yield prior to sudden crushing of the compression concrete, as assumed in the preceding development. Yielding of the tensile reinforcement and Code compliance are ensured if the net tensile strain is greater than 0.004. From the geometry of the section,

$$\frac{c}{d_t} \leq \frac{\epsilon_u}{\epsilon_u + \epsilon_t} \tag{3.64}$$

Setting $\epsilon_u = 0.003$ and $\epsilon_t = 0.004$ provides a maximum c/d_t ratio of 0.429, as seen in Fig. 3.10. Thus, as long as the depth to the neutral axis is less than $0.429d_t$, the net tensile strain requirements are satisfied, as they are for rectangular beam sections. This will occur if $\rho_w = A_s/b_wd$ is less than

$$\rho_{w,\max} = \rho_{\max} + \rho_f \tag{3.65}$$

where $\rho_f = A_{sf}/b_wd$ and $\rho_{\max}$ is as previously defined for a rectangular cross section [Eq. (3.30*c*)]. For c/d_t ratios between 0.429 and 0.375, equivalent to ρ_w between the $\rho_{w,\max}$ from Eq. (3.65) and $\rho_{w,0.005}$, calculated by substituting $\rho_{0.005}$ from Eq. (3.30*d*) for $\rho_{\max}$ in Eq. (3.65), the strength reduction factor ϕ must be adjusted for ϵ_t, as shown in Fig. 3.9. For $\rho_w \leq \rho_{w,0.005}$ or $c/d_t \leq 0.375$, $\phi = 0.90$.

The practical result of applying Eq. (3.65) is that the stress block of T beams will almost always be within the flange, except for unusual geometry or combinations of material strength. Consequently, rectangular beam equations may be applied in most cases.

The ACI Code restriction that the tensile reinforcement ratio for beams not be less than $\rho_{\min} = 3\sqrt{f_c'}/f_y$ and $\geq 200/f_y$ (see Section 3.4d) applies to T beams as well as rectangular beams. For T beams, the ratio ρ should be computed for this purpose based on the web width b_w.

c. Proportions of Cross Section

When designing T beams, in contrast to analyzing the capacity of a given section, normally the slab dimensions and beam spacing will have been established by transverse flexural requirements. Consequently, the only additional section dimensions that must be determined from flexural considerations are the width and depth of the web and the area of the tensile steel.

If the stem dimensions were selected on the basis of concrete stress capacity in compression, they would be very small because of the large compression flange width furnished by the presence of the slab. Such a design would not represent the optimum solution because of the large tensile steel requirement resulting from the small effective depth, because of the excessive web reinforcement that would be required for shear, and because of large deflections associated with such a shallow member. It is better practice to select the proportions of the web (1) so as to keep an arbitrarily low web reinforcement ratio ρ_w or (2) so as to keep web-shear stress at desirably low limits (Chapter 4) or (3) for continuous T beams, on the basis of the flexural requirements at the supports, where the effective cross section is rectangular and of width b_w.

In addition to the main reinforcement calculated according to the preceding requirements, it is necessary to ensure the integrity of the compressive flange of T beams by providing steel in the flange in the direction transverse to the main span. In typical construction, the slab steel serves this purpose. In other cases, additional bars must be added to permit the overhanging flanges to carry, as cantilever beams, the loads directly applied. According to ACI Code 8.12.5, the spacing of such bars must not exceed 5 times the thickness of the flange or in any case exceed 18 in.

d. Examples of Analysis and Design of T Beams

For *analyzing* the capacity of a T beam with known concrete dimensions and tensile steel area, it is reasonable to start with the assumption that the stress block depth a does not exceed the flange thickness h_f. In that case, all ordinary rectangular beam equations (see Section 3.4) apply, with beam width taken equal to the effective width of the flange. If, upon checking that assumption, a proves to exceed h_f, then T beam analysis must be applied. Equations (3.59) through (3.63) can be used, in sequence, to obtain the nominal flexural strength, after which the design strength is easily calculated.

For *design*, the following sequence of calculations may be followed:

1. Establish flange thickness h_f based on flexural requirements of the slab, which normally spans transversely between parallel T beams.
2. Determine the effective flange width b according to ACI limits.
3. Choose web dimensions b_w and d based on either of the following:
   (a) Negative bending requirements at the supports, if a continuous T beam
   (b) Shear requirements, setting a reasonable upper limit on the nominal unit shear stress v_u in the beam web (see Chapter 4)
4. With all concrete dimensions thus established, calculate a trial value of A_s, assuming that a does not exceed h_f, with beam width equal to flange width b. Use ordinary rectangular beam design methods.
5. For the trial A_s, check the depth of stress block a to confirm that it does not exceed h_f. If it should exceed that value, revise A_s, using the T beam equations.
6. Check to ensure that $\epsilon_t \geq 0.005$ or $c/d \leq 0.375$ to ensure that $\phi = 0.90$. (This will almost invariably be the case.)
7. Check to ensure that $\rho_w \geq \rho_{w,\min}$.

EXAMPLE 3.14 **Moment capacity of a given section.** The isolated T beam shown in Fig. 3.21 is composed of a flange 28 in. wide and 6 in. deep cast monolithically with a web of 10 in. width that extends 24 in. below the bottom surface of the flange to produce a beam of 30 in. total depth. Tensile reinforcement consists of six No. 10 (No. 32) bars placed in two horizontal rows separated by 1 in. clear spacing. The centroid of the bar group is 26 in. from the top of the beam. The concrete has a strength of 3000 psi, and the yield strength of the steel is 60,000 psi. What is the design moment capacity of the beam?

Solution. It is easily confirmed that the flange dimensions are satisfactory according to the ACI Code for an isolated beam. The entire flange can be considered effective. For six No. 10 (No. 32) bars, $A_s = 7.62 \text{ in}^2$. First check the location of the neutral axis, on the assumption that rectangular beam equations may be applied. Using Eq. (3.32)

$$a = \frac{7.62 \times 60}{0.85 \times 3 \times 28} = 6.40 \text{ in.}$$

This exceeds the flange thickness, and so a T beam analysis is required. From Eq. (3.59) and Fig. 3.19*b*,

$$A_{sf} = 0.85 \times \frac{3}{60}(28 - 10) \times 6 = 4.59 \text{ in}^2$$

Then, from Eq. (3.60),

$$M_{n1} = 4.59 \times 60(26 - 3) = 6330 \text{ in-kips}$$

Then, from Fig. 3.19*c*,

$$A_s - A_{sf} = 7.62 - 4.59 = 3.03 \text{ in}^2$$

and from Eqs. (3.58) and (3.59)

$$a = \frac{3.03 \times 60}{0.85 \times 3 \times 10} = 7.13 \text{ in.}$$

$$M_{n2} = 3.03 \times 60(26 - 3.56) = 4080 \text{ in-kips}$$

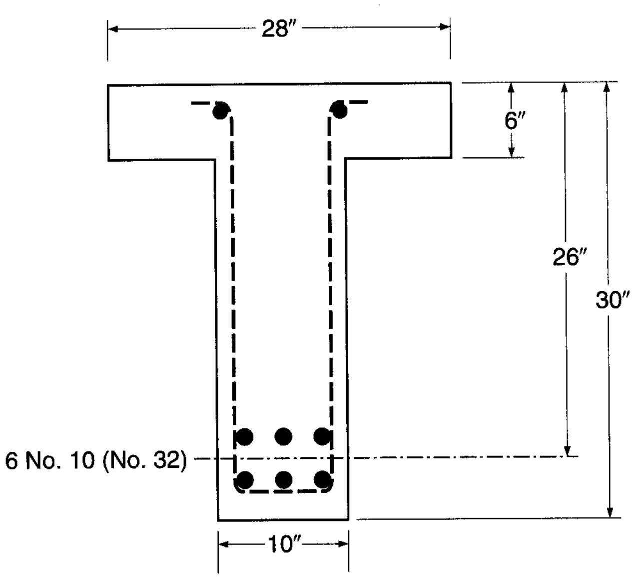


FIGURE 3.21
T beam of Example 3.14.

The depth to the neutral axis is $c = a/\beta_1 = 7.13/0.85 = 8.39$ and $d_t = 27.5$ in. to the lowest bar. The c/d_t ratio is $8.39/27.5 = 0.305 < 0.375$, so the $\epsilon_t > 0.005$ requirement is met and $\phi = 0.90$. When the ACI strength reduction factor is incorporated, the design strength is

$$\phi M_n = 0.90(6330 + 4080) = 9370 \text{ in-kips}$$

EXAMPLE 3.15 **Determination of steel area for a given moment.** A floor system, shown in Fig. 3.22, consists of a 3 in. concrete slab supported by continuous T beams with a 24 ft span, 47 in. on centers. Web dimensions, as determined by negative-moment requirements at the supports, are $b_w = 11$ in. and $d = 20$ in. What tensile steel area is required at midspan to resist a factored moment of 6400 in-kips if $f_y = 60{,}000$ psi and $f'_c = 3000$ psi?

Solution. First determining the effective flange width,

$$16h_f + b_w = 16 \times 3 + 11 = 59 \text{ in.}$$

$$\frac{\text{Span}}{4} = 24 \times \frac{12}{4} = 72 \text{ in.}$$

$$\text{Centerline beam spacing} = 47 \text{ in.}$$

The centerline T beam spacing controls in this case, and $b = 47$ in. The concrete dimensions b_w and d are known to be adequate in this case, since they have been selected for the larger negative support moment applied to the effective rectangular section $b_w d$. The tensile steel at midspan is most conveniently found by trial. Assuming the stress-block depth a is equal to the flange thickness of $h_f = 3$ in., one gets

$$d - \frac{a}{2} = 20 - 1.50 = 18.50 \text{ in.}$$

Trial:

$$A_s = \frac{M_u}{\phi f_y(d - a/2)} = \frac{6400}{0.90 \times 60 \times 18.50} = 6.41 \text{ in}^2$$

Checking the assumed value for a,

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{6.41 \times 60}{0.85 \times 3 \times 47} = 3.21 \text{ in.}$$

Since a is greater than h_f, a T beam design is required and $\phi = 0.90$ is assumed.

$$A_{sf} = \frac{0.85 f'_c (b - b_w) h_f}{f_y} = \frac{0.85 \times 3 \times 36 \times 3}{60} = 4.59 \text{ in}^2$$

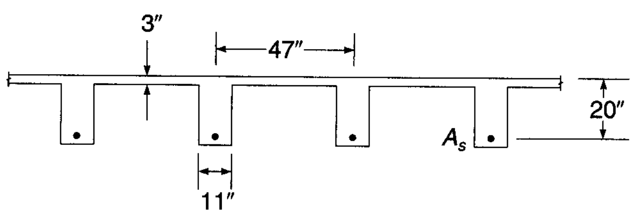


FIGURE 3.22
T beam of Example 3.15.

$$\phi M_{n1} = \phi A_{sf} f_y \left(d - \frac{h_f}{2} \right) = 0.90 \times 4.59 \times 60 \times 18.50 = 4590 \text{ in-kips}$$

$$\phi M_{n2} = M_u - \phi M_{n1} = 6400 - 4590 = 1810 \text{ in-kips}$$

Assume $a = 4.00$ in.:

$$A_s - A_{sf} = \frac{\phi M_{n2}}{\phi f_y (d - a/2)} = \frac{1810}{0.90 \times 60 \times (20 - 4.0/2)} = 1.86 \text{ in}^2$$

Check:

$$a = \frac{(A_s - A_{sf}) f_y}{0.85 f'_c b_w} = \frac{1.86 \times 60}{0.85 \times 3 \times 11} = 3.98 \text{ in.}$$

This is satisfactorily close to the assumed value of 4 in. Then

$$A_s = A_{sf} + A_s - A_{sf} = 4.59 + 1.86 = 6.45 \text{ in}^2$$

Checking to ensure that the net tensile strain of 0.005 is met to allow $\phi = 0.90$,

$$c = \frac{a}{\beta_1} = \frac{3.98}{0.85} = 4.68$$

$$\frac{c}{d_t} = \frac{4.68}{20} = 0.23 < 0.325$$

indicating that the design is satisfactory.

The close agreement should be noted between the approximate tensile steel area of 6.41 in^2 found by assuming the stress-block depth equal to the flange thickness and the more exact value of 6.45 in^2 found by T beam analysis. The approximate solution would be satisfactory in most cases.

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PROBLEMS

3.1. A rectangular beam made using concrete with $f'_c = 6000$ psi and steel with $f_y = 60{,}000$ psi has a width $b = 20$ in., an effective depth of $d = 17.5$ in., and a total depth of $h = 20$ in. The concrete modulus of rupture $f_r = 530$ psi. The elastic moduli of the concrete and steel are, respectively, $E_c = 4{,}030{,}000$ psi and $E_s = 29{,}000{,}000$ psi. The tensile steel consists of four No. 11 (No. 36) bars.

(*a*) Find the maximum service load moment that can be resisted without stressing the concrete above $0.45f'_c$ or the steel above $0.40f_y$.

(*b*) Determine whether the beam will crack before reaching the service load.

(*c*) Compute the nominal flexural strength of the beam.

(*d*) Compute the ratio of the nominal flexural strength of the beam to the maximum service load moment, and compare your findings to the ACI load factors and strength reduction factor.

3.2. A rectangular, tension-reinforced beam is to be designed for dead load of 500 lb/ft plus self-weight and service live load of 1200 lb/ft, with a 22 ft simple span. Material strengths will be $f_y = 60$ ksi and $f'_c = 3$ ksi for steel and concrete, respectively. The total beam depth must not exceed 16 in. Calculate the required beam width and tensile steel requirement, using a reinforcement ratio of $0.60\rho_{max}$. Use ACI load factors and strength reduction factors. The effective depth may be assumed to be 2.5 in. less than the total depth.

3.3. A beam with a 20 ft simple span has cross-sectional dimensions $b = 12$ in., $d = 23$ in., and $h = 25$ in. (see Fig. 3.2*b* for notation). It carries a uniform service load of 2450 lb/ft in addition to its own weight. Material strengths are $f'_c = 4000$ psi and $f_y = 60{,}000$ psi. Assume a weight of 150 pcf for reinforced concrete.

(*a*) Check whether this beam, if reinforced with three No. 9 (No. 29) bars, is adequate to carry this load with a minimum factor of safety against flexural failure of 1.85. If this requirement is not met, select a three-bar reinforcement of diameter or diameters adequate to provide this safety.

(*b*) Determine the maximum stress in the steel and in the concrete under service load, i.e., when the beam carries its own weight and the specified uniform load.

(*c*) Will the beam show hairline cracks on the tension side under service load?

3.4. A rectangular reinforced concrete beam with dimensions $b = 14$ in., $d = 25$ in., and $h = 28$ in. is reinforced with three No. 10 (No. 32) bars. Material strengths are $f_y = 60{,}000$ psi and $f'_c = 5000$ psi.

(*a*) Find the moment that will produce the first cracking at the bottom surface of the beam, basing your calculation on I_g, the moment of inertia of the gross concrete section.

(*b*) Repeat the calculation, using I_{ut}, the moment of inertia of the uncracked transformed section.

(*c*) Determine the maximum moment that can be carried without stressing the concrete beyond $0.45f'_c$ or the steel beyond $0.60f_y$.

(*d*) Find the nominal flexural strength of this beam.

(*e*) Compute the ratio of the flexural strength from part (*d*) to the service capacity from part (*c*).

(*f*) Comment on your results, paying particular attention to comparing parts (*a*) and (*b*) and comparing the result in part (*e*) with the load factors in the ACI Code.

3.5. A tensile-reinforced beam has $b = 12$ in. and $d = 20$ in. to the center of the bars, which are placed all in one row. If $f_y = 60{,}000$ psi and $f'_c = 5000$ psi, find the nominal flexural strength M_n for *(a)* A_s = two No. 8 (No. 25) bars, *(b)* A_s = two No. 10 (No. 32) bars, *(c)* A_s = three No. 10 (No. 32) bars.

3.6. A singly reinforced rectangular beam is to be designed, with effective depth approximately 1.5 times the width, to carry a service live load of 2000 lb/ft in addition to its own weight, on a 24 ft simple span. The ACI Code load factors are to be applied as usual. With $f_y = 60{,}000$ psi and $f'_c = 4000$ psi, determine the required concrete dimensions b, d, and h, and steel reinforcing bars *(a)* for $\rho = 0.60\rho_{max}$ and *(b)* for $\rho = \rho_{0.005}$. Include a sketch of each cross section drawn to scale. Allow for No. 4 (No. 13) stirrups. Comment on your results.

3.7. A four-span continuous beam of constant rectangular cross section is supported at A, B, C, D, and E. The factored moments resulting from analysis are as follows:

At Supports, ft-kips	At Midspan, ft-kips
$M_a = 138$	$M_{ab} = 158$
$M_b = 220$	$M_{bc} = 138$
$M_c = 200$	$M_{cd} = 138$
$M_d = 220$	$M_{de} = 158$
$M_e = 138$	

Determine the required final concrete dimensions for this beam, using $d = 1.75b$, and determine the reinforcement requirements at each critical moment section. Your final reinforcement ratio should not exceed $= 0.6\rho_{0.005}$. Use $f_y = 60{,}000$ psi and $f_c{}' = 6000$ psi.

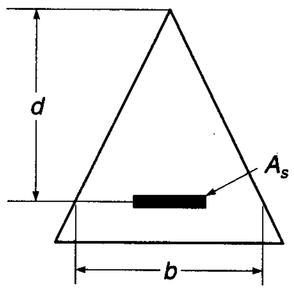
3.8. A two-span continuous concrete beam is to be supported by three concrete walls spaced 30 ft on centers. A service live load of 1.5 kips/ft is to be carried in addition to the self-weight of the beam. Use pattern loading; i.e., consider two loading conditions: (1) live load on both spans and (2) live load on a single span. A constant rectangular cross section is to be used with $d = 2b$, but reinforcement is to be varied according to requirements. Find the required concrete dimensions and reinforcement at all critical sections. Allow for No. 3 (No. 10) stirrups. Use a span-to-depth ratio of 15 as the first estimate of the depth. Adjust the depth if the reinforcement ratio is too high. Include sketches, drawn to scale, of the critical cross sections. Use $f_y = 60{,}000$ psi and $f'_c = 6000$ psi.

3.9. A rectangular concrete beam measures 12 in. wide and has an effective depth of 18 in. Compression steel consisting of two No. 8 (No. 25) bars is located 2.5 in. from the compression face of the beam. If $f'_c = 4000$ psi and $f_y = 60{,}000$ psi, what is the design moment capacity of the beam, according to the ACI Code, for the following alternative tensile steel areas? *(a)* A_s = three No. 10 (No. 32) bars in one layer, *(b)* A_s = four No. 10 (No. 32) bars in two layers, *(c)* A_s = six No. 9 (No. 29) bars in two layers. (*Note*: Check for yielding of compression steel in each case.) Plot M_n versus ρ and comment on your findings.

3.10. A rectangular concrete beam of width $b = 24$ in. is limited by architectural considerations to a maximum total depth $h = 16$ in. It must carry a total factored load moment $M_u = 400$ ft-kips. Design the flexural reinforcement for this member, using compression steel if necessary. Allow 3 in. to the center of the bars from the compression or tension face of the beam. Material strengths are $f_y = 60{,}000$ psi and $f'_c = 4000$ psi. Select reinforcement to provide the needed areas, and show a sketch of your final design, including provision for No. 4 (No. 13) stirrups.

3.11. For the beam with the triangular cross section shown in Fig. P3.11, determine (*a*) the balanced reinforcement ratio and (*b*) the maximum reinforcement ratio if $\epsilon_t = 0.005$. The dimensions of the triangle are such that the width of the triangle equals the distance from the apex. Thus, the width at the effective depth b equals the effective depth d. Express the reinforcement ratio ρ in terms of b and d. Draw the strain distribution, and stress distribution, and define your notation.

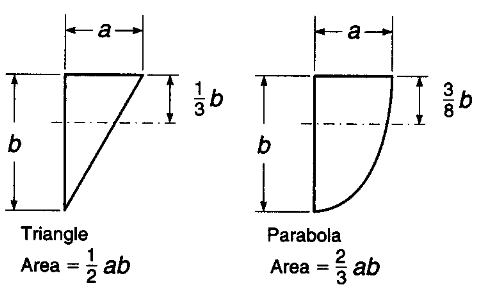
FIGURE P3.11



3.12. Develop a design table and graph for the moment capacity of rectangular concrete beams based on the use of the flexural resistance factor R. (See Table A.5a and Graph A.1a for examples.) Material strengths are $f_y = 60{,}000$ psi and $f'_c = 8000$ psi. The table and graph should begin with ρ_{min} and end at ρ_{max}. Your work must show how the maximum and minimum values of ρ were computed. You may use Excel or MathCAD to perform your calculations. Your submittal must include a table, a graph, and commentary on how you checked the work.

3.13. A rectangular beam made using concrete with $f'_c = 5000$ psi and steel with $f_y = 60{,}000$ psi has a width $b = 18$ in., an effective depth $d = 21$ in., and a total depth $h = 24$ in. The beam is reinforced with four No. 9 (No. 29) bars. Compute the nominal moment capacity, assuming (*a*) an equivalent rectangular stress block, (*b*) a triangular stress block with a peak value of f'_c, and (*c*) a parabolic stress block with a peak value of f'_c (see Fig. P3.13). Compare and comment on your results, knowing that the rectangular stress block correlates within 4 percent with test results.

FIGURE P3.13



3.14. A precast T beam is to be used as a bridge over a small roadway. Concrete dimensions are $b = 48$ in., $b_w = 16$ in., $h_f = 5$ in., and $h = 25$ in. The effective depth $d = 20$ in. Concrete and steel strengths are 6000 psi and 60,000 psi, respectively. Using approximately one-half the maximum tensile reinforcement permitted by the ACI Code (select the actual size of bar and number to be used), determine the design moment capacity of the girder. If the beam is used on a 30 ft simple span, and if in addition to its own weight it must support railings, curbs, and suspended loads totaling 0.475 kip/ft, what uniform service live load limit should be posted?

3.15. A rectangular beam with a width of 8 in., an effective depth of 10 in., and a total depth of 12 in. is reinforced with a single fiberglass reinforcing bar that has a cross-sectional area of 0.45 in^2. The bar has a nominal tensile strength of 140,000 psi, a linear stress-strain curve to failure, and a strain at failure of 1.8 percent. The concrete strength $f_c' = 6000$ psi. Determine the nominal flexural strength of the section.

3.16. Compute the maximum and minimum reinforcement ratios for reinforcement with an 80 ksi yield point and $f_c' = 4000$ to 8000 psi in 1000 psi increments, similar to those shown in Table A.4. Using the maximum and minimum reinforcement ratios, develop resistance factors and design graphs similar to Table A.5b and Graph A.1a.

4 Shear and Diagonal Tension in Beams

4.1 INTRODUCTION

Chapter 3 dealt with the flexural behavior and flexural strength of beams. Beams must also have an adequate safety margin against other types of failure, some of which may be more dangerous than flexural failure. This may be so because of greater uncertainty in predicting certain other modes of collapse, or because of the catastrophic nature of some other types of failure, should they occur.

Shear failure of reinforced concrete, more properly called *diagonal tension failure*, is one example. Shear failure is difficult to predict accurately. In spite of many decades of experimental research (Refs. 4.1 to 4.6) and the use of highly sophisticated analytical tools (Refs. 4.7 and 4.8), it is not yet fully understood. Furthermore, if a beam without properly designed shear reinforcement is overloaded to failure, shear collapse is likely to occur suddenly, with no advance warning of distress. This is in strong contrast with the nature of flexural failure. For typically underreinforced beams, flexural failure is initiated by gradual yielding of the tension steel, accompanied by obvious cracking of the concrete and large deflections, giving ample warning and providing the opportunity to take corrective measures. Because of these differences in behavior, reinforced concrete beams are generally provided with special *shear reinforcement* to ensure that flexural failure would occur before shear failure if the member were severely overloaded.

Figure 4.1 shows a shear-critical beam tested under third point loading. With no shear reinforcement provided, the member failed immediately upon formation of the critical crack in the high-shear region near the right support.

It is important to realize that shear analysis and design are not really concerned with shear as such. The shear stresses in most beams are far below the direct shear strength of the concrete. The real concern is with *diagonal tension stress*, resulting from the combination of shear stress and longitudinal flexural stress. Most of this chapter deals with analysis and design for diagonal tension, and it provides background for understanding and using the shear provisions of the 2008 ACI Code. Members without web reinforcement are studied first to establish the location and orientation of cracks and the diagonal cracking load. Methods are then developed for the design of shear reinforcement according to the present ACI Code, both in ordinary beams and in special types of members, such as deep beams.

Over the years, alternative methods of shear design have been proposed, based on variable angle truss models and diagonal compression field theory (Refs. 4.9 and 4.10). These approaches will be reviewed briefly later in this chapter, with one such approach, the modified compression field theory, presented in detail.

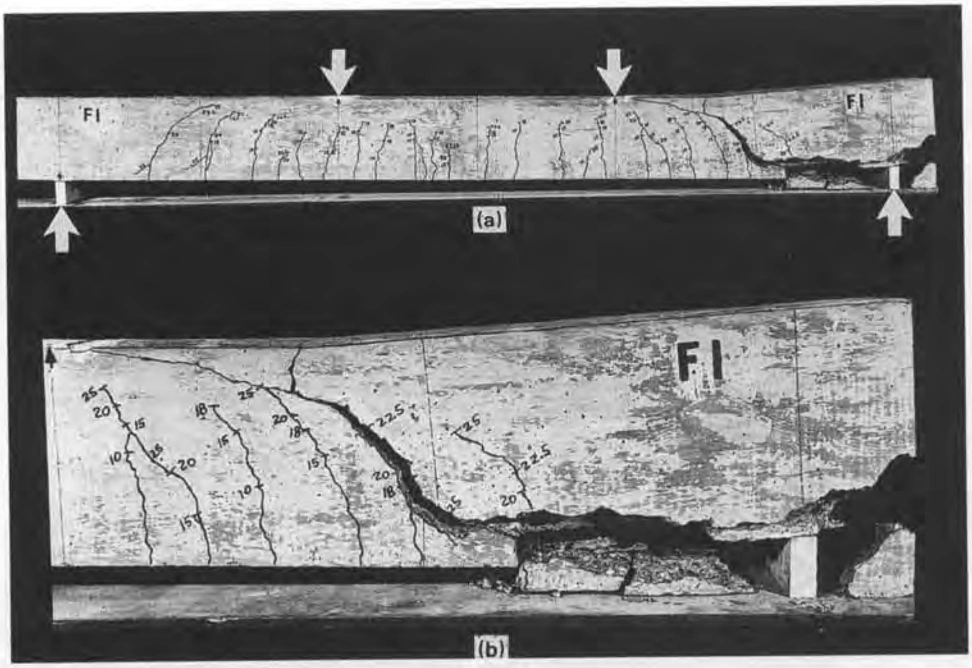


FIGURE 4.1
Shear failure of reinforced concrete beam: (*a*) overall view, (*b*) detail near right support.

Finally, there are some circumstances in which consideration of direct shear is appropriate. One example is in the design of composite members combining precast beams with a cast-in-place top slab. Horizontal shear stresses on the interface between components are important. The shear-friction theory, useful in this and other cases, will be presented following development of methods for the analysis and design of beams for diagonal tension.

4.2 DIAGONAL TENSION IN HOMOGENEOUS ELASTIC BEAMS

The stresses acting in homogeneous beams were briefly reviewed in Section 3.2. It was pointed out that when the material is elastic (stresses proportional to strains), shear stresses

$$v = \frac{VQ}{Ib} \tag{3.4}$$

act at any section in addition to the bending stresses

$$f = \frac{My}{I} \tag{3.2}$$

except for those locations at which the shear force V happens to be zero.

The role of shear stresses is easily visualized by the performance under load of the laminated beam of Fig. 4.2; it consists of two rectangular pieces bonded together along the contact surface. If the adhesive is strong enough, the member will deform as one single beam, as shown in Fig. 4.2*a*. On the other hand, if the adhesive is weak, the two pieces will separate and slide relative to each other, as shown in Fig. 4.2*b*.

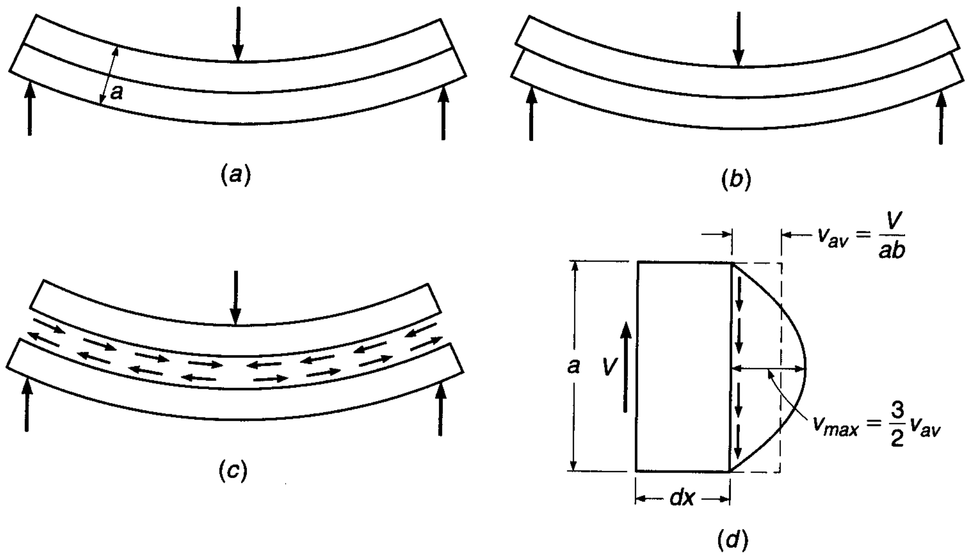


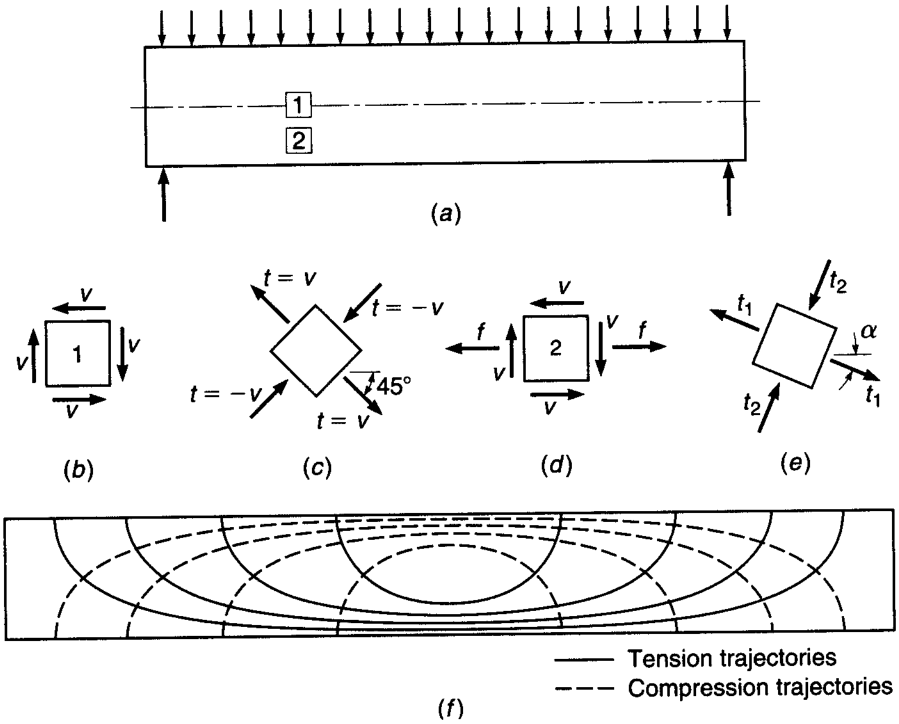
FIGURE 4.2
Shear in homogeneous rectangular beams.

Evidently, then, when the adhesive is effective, there are forces or stresses acting in it that prevent this sliding or shearing. These horizontal shear stresses are shown in Fig. 4.2*c* as they act, separately, on the top and bottom pieces. The same stresses occur in horizontal planes in single-piece beams; they are different in intensity at different distances from the neutral axis.

Figure 4.2*d* shows a differential length of a single-piece rectangular beam acted upon by a shear force of magnitude V. Upward translation is prevented; i.e., vertical equilibrium is provided by the vertical shear stresses v. Their average value is equal to the shear force divided by the cross-sectional area $v_{av} = V/ab$, but their intensity varies over the depth of the section. As is easily computed from Eq. (3.4), the shear stress is zero at the outer fibers and has a maximum of $1.5v_{av}$ at the neutral axis, the variation being parabolic as shown. Other values and distributions are found for other shapes of the cross section, the shear stress always being zero at the outer fibers and of maximum value at the neutral axis. If a small square element located at the neutral axis of such a beam is isolated, as shown in Fig. 4.3*b*, the vertical shear stresses on it, equal and opposite on the two faces for reasons of equilibrium, act as shown. However, if these were the only stresses present, the element would not be in equilibrium; it would spin. Therefore, on the two horizontal faces there exist equilibrating horizontal shear stresses of the same magnitude. That is, at any point within the beam, the horizontal shear stresses of Fig. 4.3*b* are equal in magnitude to the vertical shear stresses of Fig. 4.2*d*.

It is proved in any strength-of-materials text that on an element cut at 45° these shear stresses combine in such a manner that their effect is as shown in Fig. 4.3*c*. That is, the action of the two pairs of shear stresses on the vertical and horizontal faces is the same as that of two pairs of normal stresses, one tensile and one compressive, acting on the 45° faces and of numerical value equal to that of the shear stresses. If an element of the beam is considered that is located neither at the neutral axis nor at the outer edges, its vertical faces are subject not only to the shear stresses but also to the familiar bending stresses whose magnitude is given by Eq. (3.2) (Fig. 4.3*d*). The six stresses that now act on the element can again be combined into a pair of inclined

FIGURE 4.3
Stress trajectories in homogeneous rectangular beam.



compressive stresses and a pair of inclined tensile stresses that act at right angles to each other. They are known as *principal stresses* (Fig. 4.3*e*). Their value, as mentioned in Section 3.2, is given by

$$t = \frac{f}{2} \pm \sqrt{\frac{f^2}{4} + v^2} \tag{3.1}$$

and their inclination α by $\tan 2\alpha = 2v/f$.

Since the magnitudes of the shear stresses v and the bending stresses f change both along the beam and vertically with distance from the neutral axis, the inclinations as well as the magnitudes of the resulting principal stresses t also vary from one place to another. Figure 4.3*f* shows the inclinations of these principal stresses for a rectangular beam uniformly loaded. That is, these stress trajectories are lines which, at any point, are drawn in that direction in which the particular principal stress, tension or compression, acts at that point. It is seen that at the neutral axis the principal stresses in a beam are always inclined at 45° to the axis. In the vicinity of the outer fibers they are horizontal near midspan.

An important point follows from this discussion. Tensile stresses, which are of particular concern in view of the low tensile strength of the concrete, are not confined to the horizontal bending stresses f that are caused by bending alone. Tensile stresses of various inclinations and magnitudes, resulting from shear alone (at the neutral axis) or from the combined action of shear and bending, exist in all parts of a beam and can impair its integrity if not adequately provided for. It is for this reason that the inclined tensile stresses, known as *diagonal tension*, must be carefully considered in reinforced concrete design.

4.3 REINFORCED CONCRETE BEAMS WITHOUT SHEAR REINFORCEMENT

The discussion of shear in a homogeneous elastic beam applies very closely to a plain concrete beam *without* reinforcement. As the load is increased in such a beam, a tension crack will form where the tensile stresses are largest and will immediately cause the beam to fail. Except for beams of very unusual proportions, the largest tensile stresses are those caused at the outer fiber by bending alone, at the section of maximum bending moment. In this case, shear has little, if any, influence on the strength of a beam.

However, when tension reinforcement is provided, the situation is quite different. Even though tension cracks form in the concrete, the required flexural tension strength is furnished by the steel, and much higher loads can be carried. Shear stresses increase proportionally to the loads. In consequence, diagonal tension stresses of significant intensity are created in regions of high shear forces, chiefly close to the supports. The longitudinal tension reinforcement has been so calculated and placed that it is chiefly effective in resisting longitudinal tension near the tension face. It does not reinforce the tensionally weak concrete against the diagonal tension stresses that occur elsewhere, caused by shear alone or by the combined effect of shear and flexure. Eventually, these stresses attain magnitudes sufficient to open additional tension cracks in a direction perpendicular to the local tension stress. These are known as *diagonal* cracks, in distinction to the vertical flexural cracks. The latter occur in regions of large moments, the former in regions in which the shear forces are high. In beams in which no reinforcement is provided to counteract the formation of large diagonal tension cracks, their appearance has far-reaching and detrimental effects. For this reason, methods of predicting the loads at which these cracks will form are desired.

a. Criteria for Formation of Diagonal Cracks

It is seen from Eq. (3.1) that the diagonal tension stresses t represent the combined effect of the shear stresses v and the bending stresses f. These in turn are, respectively, proportional to the shear force V and the bending moment M at the particular location in the beam [Eqs. (3.2) and (3.4)]. Depending on configuration, support conditions, and load distribution, a given location in a beam may have a large moment combined with a small shear force, or the reverse, or large or small values for both shear and moment. Evidently, the relative values of M and V will affect the magnitude as well as the direction of the diagonal tension stresses. Figure 4.4 shows a few typical beams and their moment and shear diagrams and draws attention to locations at which various combinations of high or low V and M occur.

At a location of large shear force V and small bending moment M, there will be little flexural cracking, if any, prior to the development of a diagonal tension crack. Consequently, the average shear stress prior to crack formation is

$$v = \frac{V}{bd} \tag{4.1}$$

The exact distribution of these shear stresses over the depth of the cross section is not known. It cannot be computed from Eq. (3.4) because this equation does not account for the influence of the reinforcement and because concrete is not an elastic homogeneous material. The value computed from Eq. (4.1) must therefore be regarded merely as a measure of the average intensity of shear stresses in the section. The maximum

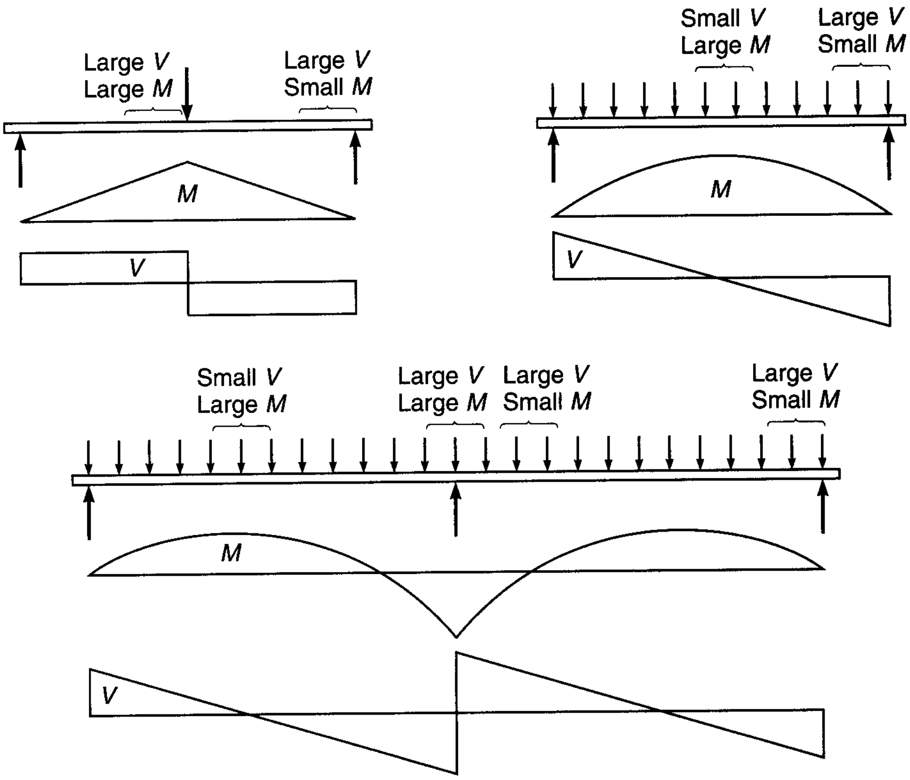


FIGURE 4.4
Typical locations of critical combinations of shear and moment.

value, which occurs at the neutral axis, will exceed this average by an unknown but moderate amount.

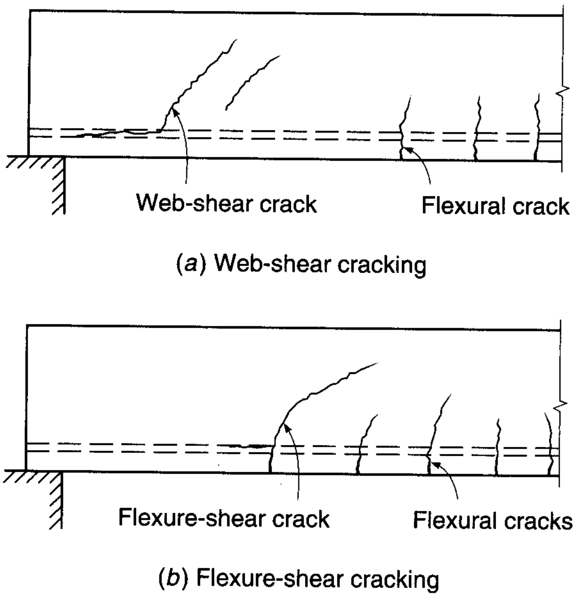
If flexural stresses are negligibly small at the particular location, the diagonal tensile stresses, as in Fig. 4.3*b* and *c*, are inclined at about 45° and are numerically equal to the shear stresses, with a maximum at the neutral axis. Consequently, diagonal cracks form mostly at or near the neutral axis and propagate from that location, as shown in Fig. 4.5*a*. These *web-shear* cracks can be expected to form when the diagonal tension stress in the vicinity of the neutral axis becomes equal to the tensile strength of the concrete. The former, as was indicated, is of the order of, and somewhat larger than, $v = V/bd$; the latter, as discussed in Section 2.9, varies from about $3\sqrt{f_c'}$ to about $5\sqrt{f_c'}$. An evaluation of a very large number of beam tests is in fair agreement with this reasoning (Ref. 4.1). It was found that in regions with large shear and small moment, diagonal tension cracks form at an average or nominal shear stress v_{cr} of about $3.5\sqrt{f_c'}$, that is,

$$v_{cr} = \frac{V_{cr}}{bd} = 3.5\sqrt{f_c'} \tag{4.2a}$$

where V_{cr} is that shear force at which the formation of the crack was observed.† Web-shear cracking is relatively rare and occurs chiefly near supports of deep, thin-webbed beams or at inflection points of continuous beams.

† Actually, diagonal tension cracks form at places where a compressive stress acts in addition to and perpendicular to the diagonal tension stress, as shown in Fig. 4.3*d* and *e*. The crack, therefore, occurs at a location of biaxial stress rather than uniaxial tension. However, the effect of this simultaneous compressive stress on the cracking strength appears to be small, in agreement with the information in Fig. 2.8.

FIGURE 4.5
Diagonal tension cracking in reinforced concrete beams.



The situation is different when both the shear force and the bending moment have large values. At such locations, in a well-proportioned and reinforced beam, flexural tension cracks form first. Their width and length are well controlled and kept small by the presence of longitudinal reinforcement. However, when the diagonal tension stress at the upper end of one or more of these cracks exceeds the tensile strength of the concrete, the crack bends in a diagonal direction and continues to grow in length and width (see Fig. 4.5*b*). These cracks are known as *flexure-shear* cracks and are more common than web-shear cracks.

It is evident that at the instant at which a diagonal tension crack of this type develops, the average shear stress is larger than that given by Eq. (4.1). This is so because the preexisting tension crack has reduced the area of uncracked concrete that is available to resist shear to a value smaller than that of the uncracked area bd used in Eq. (4.1). The amount of this reduction will vary, depending on the unpredictable length of the preexisting flexural tension crack. Furthermore, the simultaneous bending stress f combines with the shear stress v to increase the diagonal tension stress t further [see Eq. (3.1)]. No way has been found to calculate reliable values of the diagonal tension stress under these conditions, and recourse must be made to test results.

A large number of beam tests have been evaluated for this purpose (Ref. 4.1). They show that in the presence of large moments (for which adequate longitudinal reinforcement has been provided) the nominal shear stress at which diagonal tension cracks form and propagate is, in most cases, conservatively given by

$$v_{cr} = \frac{V_{cr}}{bd} = 1.9\sqrt{f'_c} \tag{4.2b}$$

Comparison with Eq. (4.2*a*) shows that large bending moments can reduce the shear force at which diagonal cracks form to roughly one-half the value at which they would form if the moment were zero or nearly so. This is in qualitative agreement with the discussion just given.

It is evident, then, that the shear at which diagonal cracks develop depends on the ratio of shear force to bending moment, or, more precisely, on the ratio of shear stress v to bending stress f near the top of the flexural crack. Neither of these can be accurately calculated. It is clear, though, that $v = K_1(V/bd)$, where, by comparison with Eq. (4.1), constant K_1 depends chiefly on the depth of penetration of the flexural crack. On the other hand [see Eq. (3.10)], $f = K_2(V/bd^2)$, where K_2 also depends on crack configuration. Hence, the ratio

$$\frac{v}{f} = \frac{K_1}{K_2}\frac{Vd}{M}$$

must be expected to affect that load at which flexural cracks develop into flexure-shear cracks, the unknown quantity K_1/K_2 to be explored by tests. Equation (4.2*a*) gives the cracking shear for very large values of Vd/M, and Eq. (4.2*b*) for very small values. Moderate values of Vd/M result in magnitudes of v_{cr} intermediate between these extremes. Again, from evaluations of large numbers of tests (Ref. 4.1), it has been found that the nominal shear stress at which diagonal flexure-shear cracking develops can be predicted from

$$v_{cr} = \frac{V_{cr}}{bd} = 1.9\sqrt{f'_c} + 2500\,\frac{\rho Vd}{M} \leq 3.5\sqrt{f'_c} \qquad (4.3a)$$

where

$$V_{cr} = v_{cr}\,bd$$

and $\rho = A_s/bd$, as before, and 2500 is an empirical constant in psi units. A graph of this relation and comparison with test data are given in Fig. 4.6.

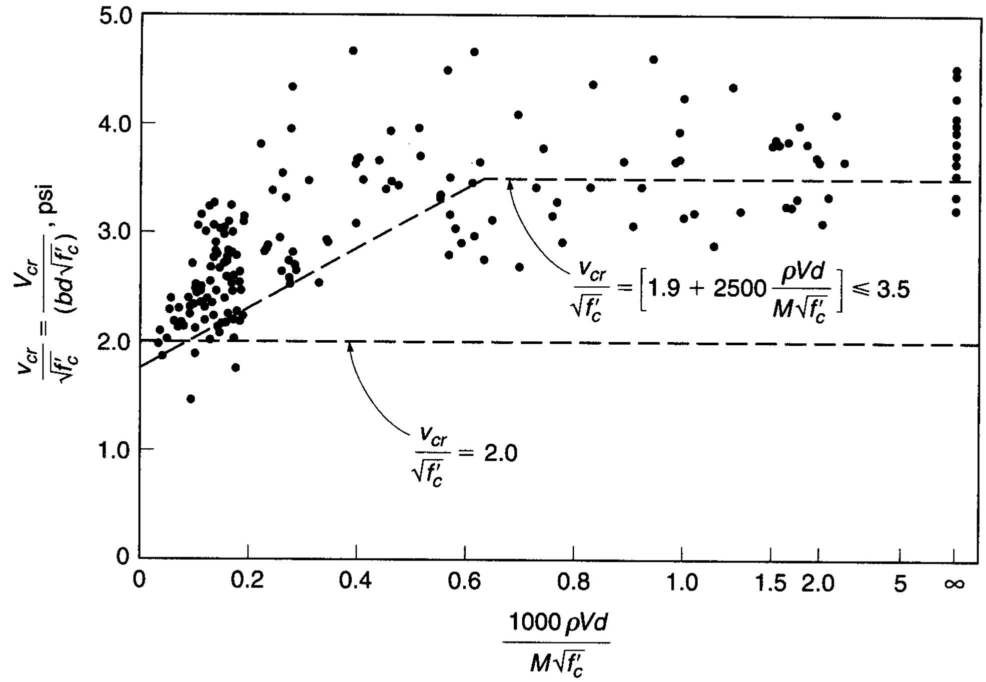


FIGURE 4.6
Correlation of Eq. (4.3*a*) with test results.

Apart from the influence of Vd/M, it is seen from Eq. (4.3*a*) that increasing amounts of tension reinforcement, i.e., increasing values of the reinforcement ratio ρ, have a beneficial effect in that they increase the shear at which diagonal cracks develop. This is so because larger amounts of longitudinal steel result in smaller and narrower flexural tension cracks prior to the formation of diagonal cracking, leaving a larger area of uncracked concrete available to resist shear. [For more details on the development of Eq. (4.3*a*), see Ref. 4.1.]

A brief study of Fig. 4.6 will show that although Eq. (4.3*a*) captures the overall effects of the controlling variables on v_{cr}, the match with actual data is far from perfect. Of particular concern is the tendency of Eq. (4.3*a*) to overestimate the shear strength of beams with reinforcement ratios $\rho < 1.0$ percent, values that are commonly used in practice. The cracking stress predicted in Eq. (4.3*a*) becomes progressively less conservative as f_c' increases above 5000 psi and as beam depth d increases above 18 in. On the other hand, Eq. (4.3*a*) underestimates the effect of Vd/M on v_{cr} and ignores the positive effect of flanges (present on most reinforced concrete beams) on shear strength. The conservatism of Eq. (4.3*a*) increases as both flange thickness and web width increase (Ref. 4.3), although these factors have less of an effect than f_c', ρ, or Vd/M on v_{cr}.

Considering the three main variables, an improved match with test results is obtained with the empirical relationship (Ref. 4.11)

$$v_{cr} = \frac{V_{cr}}{bd} = 59\left(f_c'\rho\frac{Vd}{M}\right)^{1/3} \tag{4.3b}$$

Equation (4.3*b*) was calibrated based on beams with $d \approx 12$ in. It can be modified to account for the lower average shear cracking stress exhibited by deeper beams with the addition of one term.

$$v_{cr} = \frac{V_{cr}}{bd} = 59\left(\frac{12}{d}\right)^{1/4}\left(f_c'\rho\frac{Vd}{M}\right)^{1/3} \tag{4.3c}$$

b. Behavior of Diagonally Cracked Beams

In regard to flexural cracks, as distinct from diagonal tension cracks, it was explained in Section 3.3 that cracks on the tension side of a beam are permitted to occur and are in no way detrimental to the strength of the member. One might expect a similar situation in regard to diagonal cracking caused chiefly by shear. The analogy, however, is not that simple. Flexural tension cracks are harmless only because adequate longitudinal reinforcement has been provided to resist the flexural tension stresses that the cracked concrete is no longer able to transmit. In contrast, the beams now being discussed, although furnished with the usual longitudinal reinforcement, are not equipped with any other reinforcement to offset the effects of diagonal cracking. This makes the diagonal cracks much more decisive in subsequent performance and strength of the beam than the flexural cracks.

Two types of behavior have been observed in the many tests on which present knowledge is based:

1. The diagonal crack, once formed, spreads either immediately or at only slightly higher load, traversing the entire beam from the tension reinforcement to the compression face, splitting it in two and failing the beam. This process is sudden and without warning and occurs chiefly in the shallower beams, i.e., beams with

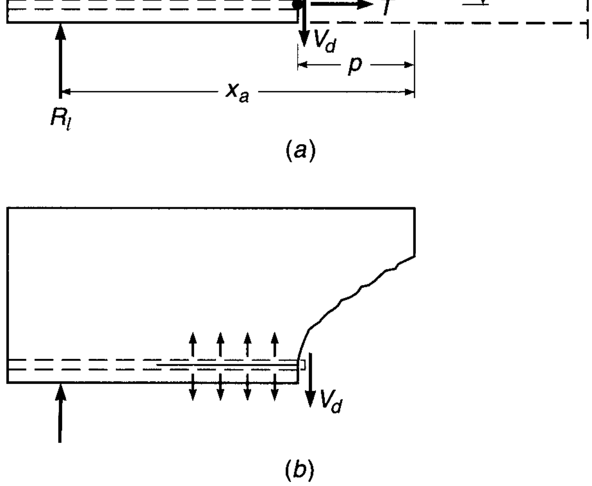
span-depth ratios of about 8 or more. Beams in this range of dimensions are very common. Complete absence of shear reinforcement would make them very vulnerable to accidental large overloads, which would result in catastrophic failures without warning. For this reason it is good practice to provide a minimum amount of shear reinforcement even if calculation does not require it, because such reinforcement restrains growth of diagonal cracks, thereby increasing ductility and providing warning in advance of actual failure. Only in situations where an unusually large safety factor against inclined cracking is provided, i.e., where actual shear stresses are very small compared with v_{cr}, as in some slabs and most footings, is it permissible to omit shear reinforcement.

2. Alternatively, the diagonal crack, once formed, spreads toward and partially into the compression zone but stops short of penetrating to the compression face. In this case no sudden collapse occurs, and the failure load may be significantly higher than that at which the diagonal crack first formed. This behavior is chiefly observed in the deeper beams with smaller span-depth ratios and will be analyzed now.

Figure 4.7*a* shows a portion of a beam, arbitrarily loaded, in which a diagonal tension crack has formed. Consider the part of the beam to the left of the crack, shown in solid lines. There is an external upward shear force $V_{\text{ext}} = R_l - P_1$ acting on this portion.

Once a crack is formed, no tension force perpendicular to the crack can be transmitted across it. However, as long as the crack is narrow, it can still transmit forces in its own plane through interlocking of the surface roughnesses. Sizable interlock forces V_i of this kind have in fact been measured, amounting to one-third and more of the total shear force. The components V_{ix} and V_{iy} of V_i are shown in Fig. 4.7*a*. The other

FIGURE 4.7
Forces at a diagonal crack in a beam without web reinforcement.



internal vertical forces are those in the uncracked portion of the concrete V_{cz} and across the longitudinal steel, acting as a dowel, V_d. Thus, the internal shear force is

$$V_{\text{int}} = V_{cz} + V_d + V_{iy}$$

Equilibrium requires that $V_{\text{int}} = V_{\text{ext}}$ so that the part of the shear resisted by the uncracked concrete is

$$V_{cz} = V_{\text{ext}} - V_d - V_{iy} \tag{4.4}$$

In a beam provided with longitudinal reinforcement only, the portion of the shear force resisted by the steel in dowel action is usually quite small. In fact, the reinforcing bars on which the dowel force V_d acts are supported against vertical displacement chiefly by the thin concrete layer below. The bearing pressure caused by V_d creates, in this concrete, vertical tension stresses as shown in Fig. 4.7*b*. Because of these stresses, diagonal cracks often result in splitting of the concrete along the tension reinforcement, as shown. (See also Fig. 4.1.) This reduces the dowel force V_d and also permits the diagonal crack to widen. This, in turn, reduces the interface force V_i and frequently leads to immediate failure.

Next consider moments about point a at the intersection of V_{cz} and C; the external moment $M_{\text{ext},a}$ acts at a and happens to be $R_l x_a - P_1(x_a - x_1)$ for the loading shown. The internal moment is

$$M_{\text{int},a} = T_b z + V_d p - V_i m$$

Here p is the horizontal projection of the diagonal crack and m is the moment arm of the force V_i with respect to point a. The designation T_b for T is meant to emphasize that this force in the steel acts at point b rather than vertically below point a. Equilibrium requires that $M_{\text{int},a} = M_{\text{ext},a}$ so that the longitudinal tension in the steel at b is

$$T_b = \frac{M_{\text{ext},a} - V_d p + V_i m}{z} \tag{4.5}$$

Neglecting the forces V_d and V_i, which decrease with increasing crack opening, one has, with very little error,

$$T_b = \frac{M_{\text{ext},a}}{z} \tag{4.6}$$

The formation of the diagonal crack, then, is seen to produce the following redistribution of internal forces and stresses:

1. At the vertical section through point a, the average shear stress before crack formation was V_{ext}/bd. After crack formation, the shear force is resisted by a combination of the dowel shear, the interface shear, and the shear force on the much smaller area by of the remaining uncracked concrete. As tension splitting develops along the longitudinal bars, V_d and V_i decrease; this, in turn, increases the shear force and the resulting shear stress on the remaining uncracked concrete area.
2. The diagonal crack, as described previously, usually rises above the neutral axis and traverses some part of the compression zone before it is arrested by the compression stresses. Consequently, the compression force C also acts on an area by smaller than that on which it acted before the crack was formed. Correspondingly, formation of the crack has increased the compression stresses in the remaining uncracked concrete.
3. Prior to diagonal cracking, the tension force in the steel at point b was caused by, and was proportional to, the bending moment in a vertical section through the

same point *b*. As a consequence of the diagonal crack, however, Eq. (4.6) shows that the tension in the steel at *b* is now caused by, and is proportional to, the bending moment at *a*. Since the moment at *a* is evidently larger than that at *b*, formation of the crack has caused a sudden increase in the steel stress at *b*.

If the two materials are capable of resisting these increased stresses, equilibrium will establish itself after internal redistribution and further load can be applied before failure occurs. Such failure can then develop in various ways. For one, if only enough steel has been provided at *b* to resist the moment at that section, the increase of the steel force, described in item 3, will cause the steel to yield because of the larger moment at *a*, thus failing the beam. If the beam is properly designed to prevent this occurrence, it is usually the concrete at the head of the crack that will eventually crush. This concrete is subject simultaneously to large compression and shear stresses, and this biaxial stress combination is conducive to earlier failure than would take place if either of these stresses were acting alone. Finally, if there is splitting along the reinforcement, it will cause the bond between steel and concrete to weaken to such a degree that the reinforcement may pull loose. This either may be the cause of failure of the beam or may occur simultaneously with crushing of the remaining uncracked concrete.

It was noted earlier that relatively deep beams will usually show continued and increasing resistance after formation of a critical diagonal tension crack, but relatively shallow beams will fail almost immediately upon formation of the crack. The amount of reserve strength, if any, was found to be erratic. In fact, in several test series in which two specimens as identical as one can make them were tested, one failed immediately upon formation of a diagonal crack, while the other reached equilibrium under the described redistribution and failed at a higher load.

For this reason, this reserve strength is discounted in modern design procedures. As previously mentioned, most beams are furnished with at least a minimum of web reinforcement. For those flexural members that are not, such as slabs, footings, and others, design is based on that shear force V_{cr} or shear stress v_{cr} at which formation of inclined cracks must be expected. Thus, Eq. (4.3*a*), or some equivalent of it, has become the design criterion for such members.

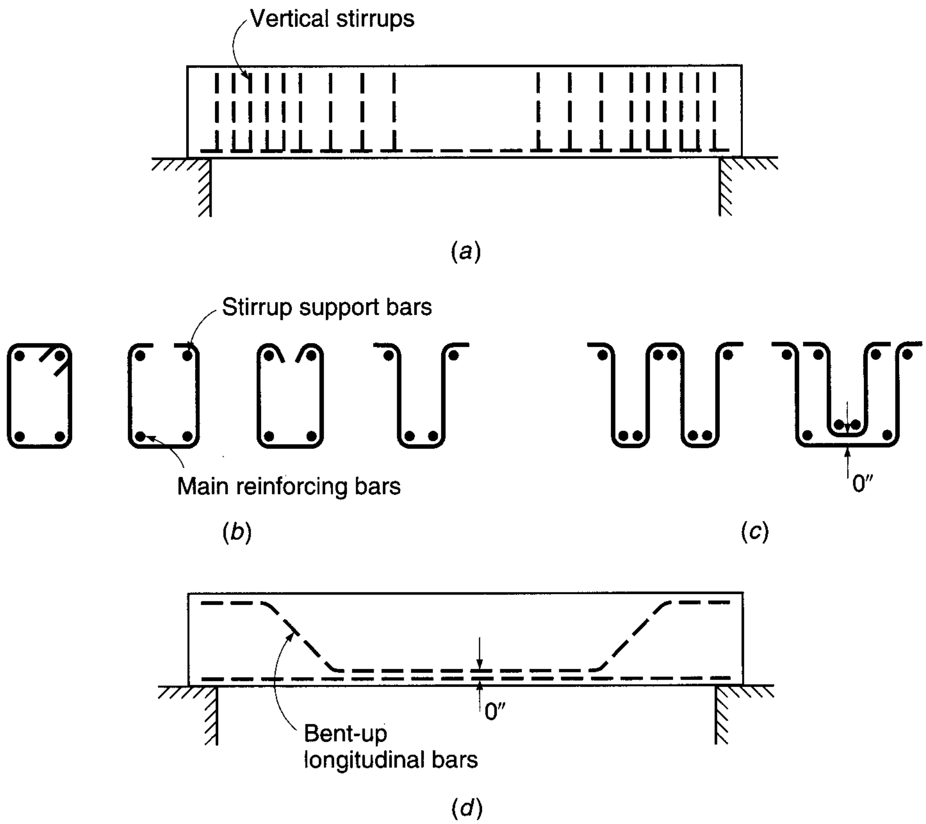
4.4 REINFORCED CONCRETE BEAMS WITH WEB REINFORCEMENT

Economy of design demands, in most cases, that a flexural member be capable of developing its full moment capacity rather than having its strength limited by premature shear failure. This is also desirable because structures, if overloaded, should not fail in the sudden and explosive manner characteristic of many shear failures, but should show adequate ductility and warning of impending distress. The latter, as pointed out earlier, is typical of flexural failure caused by yielding of the longitudinal bars, which is preceded by gradual excessively large deflections and noticeable widening of cracks. Therefore, if a fairly large safety margin relative to the available shear strength as given by Eq. (4.3*a*) or its equivalent does not exist, special shear reinforcement, known as *web reinforcement*, is used to increase this strength.

a. Types of Web Reinforcement

Typically, web reinforcement is provided in the form of vertical *stirrups*, spaced at varying intervals along the axis of the beam depending on requirements, as shown in Fig. 4.8*a*. Relatively small bars are used, generally Nos. 3 to 5 (Nos. 10 to 16). Simple

FIGURE 4.8
Types of web reinforcement.



U-shaped bars similar to Fig. 4.8*b* are most common, although multiple-leg stirrups such as shown in Fig. 4.8*c* are sometimes necessary. Stirrups are formed to fit around the main longitudinal bars at the bottom and hooked or bent around longitudinal bars at the top of the member to improve anchorage and provide support during construction. Detailed requirements for anchorage of stirrups will be discussed in Chapter 5.

Alternatively, shear reinforcement may be provided by bending up a part of the longitudinal steel where it is no longer needed to resist flexural tension, as suggested by Fig. 4.8*d*. In continuous beams, these bent-up bars may also provide all or part of the necessary reinforcement for negative moments. The requirements for longitudinal flexural reinforcement often conflict with those for diagonal tension, and because the savings in steel resulting from use of the capacity of bent bars as shear resistance is small, most designers prefer to include vertical stirrups to provide for all the shear requirement, counting on the bent part of the longitudinal bars, if bent bars are used, only to increase the overall safety against diagonal tension failure.

Welded wire reinforcement is also used for shear reinforcement, particularly for small, lightly loaded members with thin webs, and for certain types of precast, prestressed beams.

b. Behavior of Web-Reinforced Concrete Beams

Web reinforcement has no noticeable effect prior to the formation of diagonal cracks. In fact, measurements show that the web steel is practically free of stress prior to crack formation. After diagonal cracks have developed, web reinforcement augments the shear resistance of a beam in four separate ways:

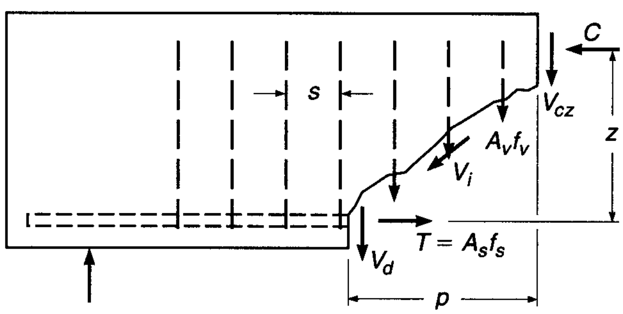
1. Part of the shear force is resisted by the bars that traverse a particular crack. The mechanism of this added resistance is discussed below.
2. The presence of these same bars restricts the growth of diagonal cracks and reduces their penetration into the compression zone. This leaves more uncracked concrete available at the head of the crack for resisting the combined action of shear and compression, already discussed.
3. The stirrups also counteract the widening of the cracks, so that the two crack faces stay in close contact. This makes for a significant and reliable interface force V_i (see Fig. 4.7).
4. As shown in Fig. 4.8, the stirrups are arranged so that they tie the longitudinal reinforcement into the main bulk of the concrete. This provides some measure of restraint against the splitting of concrete along the longitudinal reinforcement, shown in Figs. 4.1 and 4.7*b*, and increases the share of the shear force resisted by dowel action.

From this it is clear that failure will be imminent when the stirrups start yielding. This not only exhausts their own resistance but also permits a wider crack opening with consequent reduction of the beneficial restraining effects, points 2 to 4, above.

It becomes clear from this description that member behavior, once a crack is formed, is quite complex and dependent in its details on the particulars of crack configuration (length, inclination, and location of the main or critical crack). The latter, in turn, is quite erratic and has so far defied purely analytical prediction. For this reason, the concepts that underlie present design practice are not wholly rational. They are based partly on rational analysis, partly on test evidence, and partly on successful long-time experience with structures in which certain procedures for designing web reinforcement have resulted in satisfactory performance.

Beams with Vertical Stirrups. Since web reinforcement is ineffective in the uncracked beam, the magnitude of the shear force or stress that causes cracking to occur is the same as in a beam without web reinforcement and is approximated by Eq. (4.3*a*). Most frequently, web reinforcement consists of *vertical stirrups*; the forces acting on the portion of such a beam between the crack and the nearby support are shown in Fig. 4.9. They are the same as those of Fig. 4.7, except that each stirrup traversing the crack exerts a force $A_v f_v$ on the given portion of the beam. Here A_v is the cross-sectional area of the stirrup (in the case of the U-shaped stirrup of Fig. 4.8*b* it is

FIGURE 4.9
Forces at a diagonal crack in a beam with vertical stirrups.



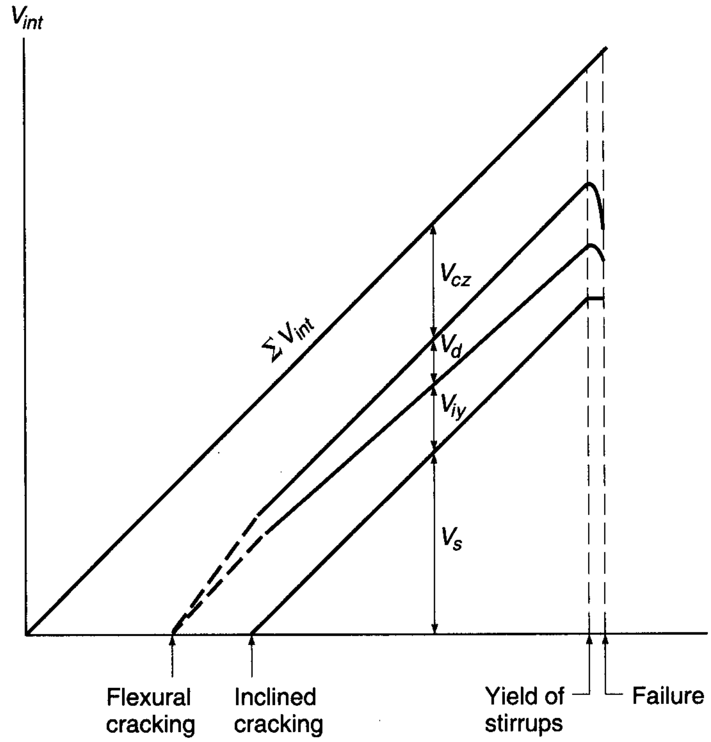


FIGURE 4.10
Redistribution of internal shear forces in a beam with stirrups. (*Adapted from Ref. 4.3.*)

twice the area of one bar), and f_v is the tensile stress in the stirrup. Equilibrium in the vertical direction requires

$$V_{ext} = V_{cz} + V_d + V_{iy} + V_s \tag{a}$$

where $V_s = nA_v f_v$ is the vertical force in the stirrups, n being the number of stirrups traversing the crack. If s is the stirrup spacing and p the horizontal projection of the crack, as shown, then $n = p/s$.

The approximate distribution of the four components of the internal shear force with increasing external shear V_{ext} is shown schematically in Fig. 4.10. It is seen that after inclined cracking, the portion of the shear $V_s = nA_v f_v$ carried by the stirrups increases linearly, while the sum of the three other components, $V_{cz} + V_d + V_{iy}$, stays nearly constant. When the stirrups yield, their contribution remains constant at the yield value $V_s = nA_v f_{yt}$, where f_{yt} represents the yield strength of the stirrup (or transverse) reinforcement. However, because of widening of the inclined cracks and longitudinal splitting, V_{iy} and V_d fall off rapidly. This overloads the remaining uncracked concrete and very soon precipitates failure.

While total shear carried by the stirrups at yielding is known, the individual magnitudes of the three other components are not. Limited amounts of test evidence have led to the conservative assumption in present-day methods that just prior to failure of a web-reinforced beam, the sum of these three internal shear components is equal to the cracking shear V_{cr}, as given by Eq. (4.3*a*). This sum is generally (somewhat loosely) referred to as the *contribution of the concrete* to the total shear resistance and is denoted V_c. Thus $V_c = V_{cr}$ and

$$V_c = V_{cz} + V_d + V_{iy} \tag{b}$$

The number of stirrups n spaced a distance s apart was seen to depend on the length p of the horizontal projection of the diagonal crack. This length is conservatively assumed to be equal to the effective depth of the beam; thus $n = d/s$, implying a crack somewhat flatter than 45°. Then, at failure, when $V_{ext} = V_n$, Eqs. (*a*) and (*b*) yield for the nominal shear strength

$$V_n = V_c + \frac{A_v f_{yt} d}{s} \tag{4.7a}$$

where V_c is taken equal to the cracking shear V_{cr} given by Eq. (4.3*a*); that is,

$$V_c = \left(1.9\sqrt{f_c'} + 2500\,\frac{\rho V d}{M}\right) bd \le 3.5\sqrt{f_c'}\,bd \tag{4.3a}$$

Dividing both sides of Eq. (4.7*a*) by *bd*, the same relation is expressed in terms of the nominal shear stress:

$$v_n = \frac{V_n}{bd} = v_c + \frac{A_v f_{yt}}{bs} \tag{4.7b}$$

In Ref. 4.1, the results of 166 beam tests are compared with Eq. (4.7*b*). It is shown that the equation predicts the actual shear strength quite conservatively, the observed strength being on average 45 percent larger than predicted; a very few of the individual test beams developed strength just slightly below that of Eq. (4.7*b*).

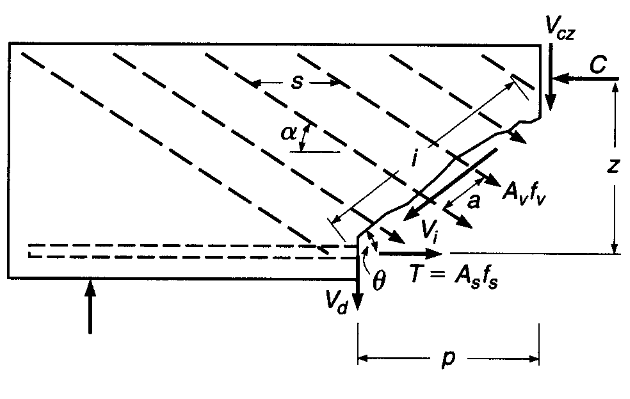
Beams with Inclined Bars. The function of *inclined web reinforcement* (Fig. 4.8*d*) can be discussed in very similar terms. Figure 4.11 again indicates the forces that act on the portion of the beam to one side of the diagonal crack that results in eventual failure. The crack with horizontal projection p and inclined length $i = p/\cos\theta$ is crossed by inclined bars horizontally spaced a distance s apart. The inclination of the bars is α and that of the crack θ, as shown. The distance between bars measured parallel to the direction of the crack is seen from the irregular triangle to be

$$a = \frac{s}{\sin\theta(\cot\theta + \cot\alpha)} \tag{a}$$

The number of bars crossing the crack $n = i/a$, after some transformation, is

$$n = \frac{p}{s}(1 + \cot\alpha\tan\theta) \tag{b}$$

FIGURE 4.11
Forces at a diagonal crack in a beam with inclined web reinforcement.



The vertical component of the force in one bar or stirrup is $A_v f_v \sin \alpha$, so that the total vertical component of the forces in all bars that cross the crack is

$$V_s = nA_v f_v \sin \alpha = A_v f_v \frac{p}{s}(\sin \alpha + \cos \alpha \tan \theta) \tag{4.8}$$

As in the case of vertical stirrups, shear failure occurs when the stress in the web reinforcement reaches the yield point. Also, the same assumptions are made as in the case of stirrups, namely, that the horizontal projection of the diagonal crack is equal to the effective depth d, and that $V_{cz} + V_d + V_{iy}$ is equal to V_c. Lastly, the inclination θ of the diagonal crack, which varies somewhat depending on various influences, is generally assumed to be 45°. On this basis, when failure is caused by shear, the nominal strength is

$$V_n = V_c + \frac{A_v f_{yt} d(\sin \alpha + \cos \alpha)}{s} \tag{4.9}$$

It is seen that Eq. (4.7*a*), developed for vertical stirrups, is only a special case, for $\alpha = 90°$, of the more general expression (4.9).

Note that Eqs. (4.7) and (4.9) apply only if web reinforcement is so spaced that any conceivable diagonal crack is traversed by at least one stirrup or inclined bar. Otherwise web reinforcement would not contribute to the shear strength of the beam, because diagonal cracks that could form between widely spaced web reinforcement would fail the beam at the load at which it would fail if no web reinforcement were present. This imposes upper limits on the permissible spacing s to ensure that the web reinforcement is actually effective as calculated.

To summarize, at this time the nature and mechanism of diagonal tension failure are clearly understood qualitatively, but some of the quantitative assumptions that have been made in the preceding development cannot be proved by rational analysis. However, the calculated results are in acceptable and generally conservative agreement with a very large body of empirical data, and structures designed on this basis have proved satisfactory. Newer methods, introduced in Section 4.8, provide alternatives that are slowly being incorporated into the ACI Code and the AASHTO Bridge Specifications (Ref. 4.12). Chapter 10 presents a detailed description of one such alternative, the so-called strut-and-tie model, which appears in Appendix A of the 2008 ACI Code.

4.5 ACI CODE PROVISIONS FOR SHEAR DESIGN

According to ACI Code 11.1.1, the design of beams for shear is to be based on the relation

$$V_u \leq \phi V_n \tag{4.10}$$

where V_u is the total shear force applied at a given section of the beam due to factored loads and $V_n = V_c + V_s$ is the nominal shear strength, equal to the sum of the contributions of the concrete and the web steel if present. Thus for vertical stirrups

$$V_u \leq \phi V_c + \frac{\phi A_v f_{yt} d}{s} \tag{4.11a}$$

and for inclined bars

$$V_u \leq \phi V_c + \frac{\phi A_v f_{yt} d(\sin \alpha + \cos \alpha)}{s} \tag{4.11b}$$

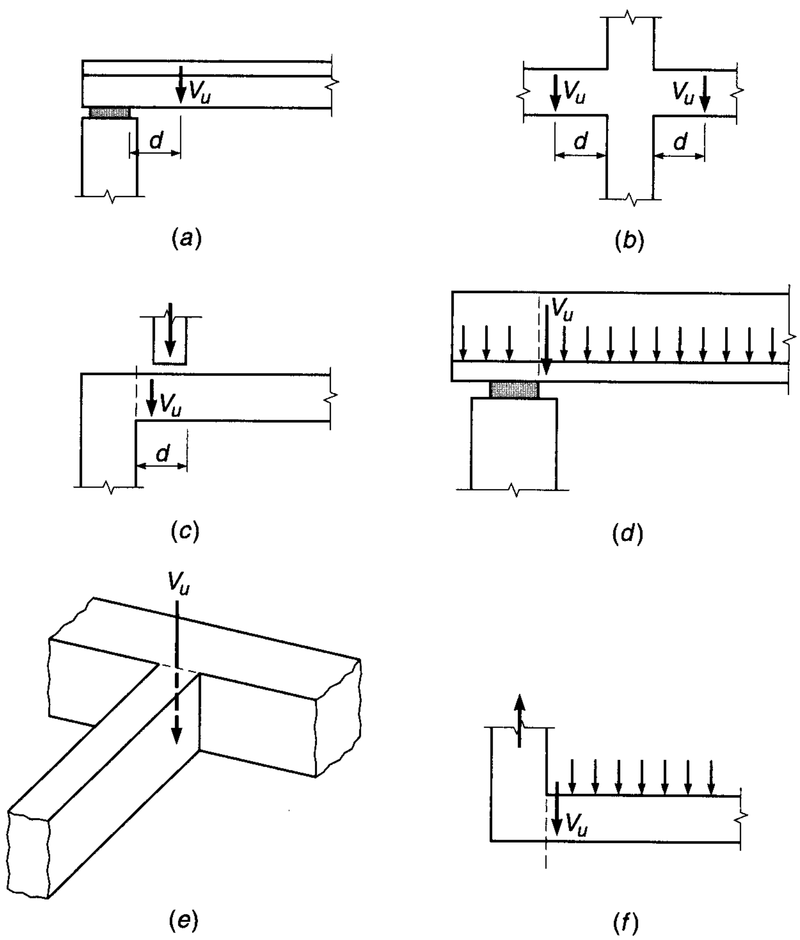


FIGURE 4.12
Location of critical section for shear design: (*a*) end-supported beam; (*b*) beam supported by columns; (*c*) concentrated load within *d* of the face of the support; (*d*) member loaded near the bottom; (*e*) beam supported by girder of similar depth; (*f*) beam supported by monolithic vertical element.

where all terms are as previously defined. The strength reduction factor ϕ is to be taken equal to 0.75 for shear. The additional conservatism, compared with the value of $\phi = 0.90$ for bending for typical beam designs, reflects both the sudden nature of diagonal tension failure and the large scatter of test results.

For typical support conditions, where the reaction from the support surface or from a monolithic column introduces vertical compression at the end of the beam, sections located less than a distance d from the face of the support may be designed for the same shear V_u as that computed at a distance d, as shown in Fig. 4.12*a* and *b*. However, the critical design section should be taken at the face of the support if concentrated loads act within that distance (Fig. 4.12*c*), if the beam is loaded near its bottom edge (as may occur for an inverted T beam, as shown in Fig. 4.12*d*), or if the reaction causes vertical tension rather than compression [e.g., if the beam is supported by a girder of similar depth (Fig. 4.12*e*) or at the end of a monolithic vertical element (Fig. 4.12*f*)].

a. Shear Strength Provided by the Concrete

The nominal shear strength contribution of the concrete (including the contributions from aggregate interlock, dowel action of the main reinforcing bars, and that of the uncracked concrete) is basically the same as Eq. (4.3*a*) with slight notational changes.

To permit application of Eq. (4.3*a*) to T beams having web width b_w, the rectangular beam width b is replaced by b_w with the understanding that for rectangular beams b is used for b_w. For T beams with a tapered web width, such as typical concrete joists, the average web width is used, unless the narrowest part of the web is in compression, in which case b_w is taken as the minimum width. Further, in Eq. (4.3*a*), the shear V and moment M are designated V_u and M_u to emphasize that they are the values computed at factored loads. Thus, for members subject to shear and flexure, according to ACI Code 11.2.2, the concrete contribution to shear strength is

$$V_c = \left(1.9\lambda\sqrt{f_c'} + 2500\,\frac{\rho_w V_u d}{M_u}\right)b_w d \le 3.5\lambda\sqrt{f_c'}\,b_w d \qquad (4.12a)$$

where ρ_w = longitudinal reinforcement ratio $A_s/b_w d$ or A_s/bd. With the section dimension b_w and d in inches and $V_u d$ and M_u in consistent units, V_c is expressed in pounds. In Eq. (4.12*a*), the quantity $V_u d/M_u$ is not to be taken greater than 1.0.

The term λ in Eq. (4.12*a*) is a modification factor reflecting the lower tensile strength of lightweight concrete compared with normalweight concrete of the same compressive strength (see Table 2.2 and Ref. 4.13). Lightweight aggregate concretes having densities from 90 to 120 pcf are used widely, particularly for precast elements. In accordance with ACI Code 8.6.1, λ = 0.85 for "sand-lightweight" concrete and 0.75 for "all-lightweight" concrete. Linear interpolation between 0.75 and 0.85, based on volumetric fractions, is permitted when a portion of the lightweight fine aggregate is replaced by normalweight fine aggregate. Linear interpolation between 0.85 and 1.0 is also permitted for concretes containing normalweight fine aggregate and a blend of lightweight and normalweight coarse aggregate. If the average split-cylinder strength of lightweight concrete (a good measure of its direct tensile strength) is specified, $\lambda = f_{ct}/(6.7\sqrt{f_c'}) \le 1.0$. For normalweight concert, $\lambda = 1.0$.

While Eq. (4.12*a*) is perfectly well suited to computerized design or for research, for manual calculations its use is tedious because ρ_w, V_u, and M_u generally change along the span, requiring that V_c be calculated at frequent intervals. For this reason, an alternative equation for V_c is permitted by ACI Code 11.2.1:

$$V_c = 2\lambda\sqrt{f_c'}\,b_w d \qquad (4.12b)$$

Referring to Fig. 4.6, it is clear that Eq. (4.12*b*) is very conservative in regions where the shear-moment ratio is high, such as near the ends of simple spans or near the inflection points of continuous spans; however, because of its simplicity, it is often used in practice.

For members with a circular cross section, ACI Code 11.2.3 provides that the area used to calculate V_c in Eqs. (4.12*a*) and (4.12*b*) be the product of the diameter and the effective depth. The latter may be taken as 0.8 times the diameter of the member.

The tests on which Eqs. (4.12*a*) and (4.12*b*) are based used beams with concrete compressive strength mostly in the range of 3000 to 5000 psi. More recent experimental results (Refs. 4.14 to 4.17) have shown that in beams constructed using high-strength concrete (see Section 2.12) with f_c' above 6000 psi, the concrete contribution to shear strength V_c is less than predicted by those equations. Differences become increasingly significant, the higher the concrete strength. For this reason, ACI Code 11.1.2 places an upper limit of 100 psi on the value of $\sqrt{f_c'}$ to be used in Eqs. (4.12*a*) and (4.12*b*), *as well as in all other ACI Code shear provisions*. However, values of $\sqrt{f_c'}$ greater than 100 psi may be used in computing V_c if a minimum amount of web reinforcement is used (see Section 4.5b).

b. Minimum Web Reinforcement

If V_u, the shear force at factored loads, is no larger than ϕV_c, calculated by Eq. (4.12*a*) or alternatively by Eq. (4.12*b*), then theoretically no web reinforcement is required. Even in such a case, however, ACI Code 11.4.6 requires provision of at least a minimum area of web reinforcement equal to

$$A_{v,\min} = 0.75\sqrt{f'_c}\,\frac{b_w s}{f_{yt}} \geq 50\,\frac{b_w s}{f_{yt}} \tag{4.13}$$

where s = longitudinal spacing of web reinforcement, in.
f_{yt} = yield strength of web steel, psi
$A_{v,\min}$ = total cross-sectional area of web steel within distance s, in^2

This provision holds unless V_u is one-half or less of the design shear strength provided by the concrete ϕV_c. Specific exceptions to this requirement for minimum web steel are made for slabs and footings; for concrete joist floor construction; for beams with total depth h not greater than 10 in.; and for beams integral with slabs with h not greater than 24 in. and not greater than the larger of 2.5 times the thickness of the flange and 0.5 times the thickness of the web. These members are excluded because of their capacity to redistribute internal forces before diagonal tension failure, as confirmed by both tests and successful design experience. In addition, beams constructed of steel fiber reinforced, normalweight concrete with f'_c not exceeding 6000 psi, total depth h not greater than 24 in., and V_u not greater than $\phi 2\sqrt{f'_c}\,b_w d$ are not required to meet the requirements for minimum web reinforcement because beams meeting these requirements have been shown to have shear strength in excess of $3.5\sqrt{f'_c}\,b_w d$ (Ref. 4.18).†

For high-strength concrete beams, the limitation of 100 psi imposed on the value of $\sqrt{f'_c}$ used in calculating V_c by Eq. (4.12*a*) or (4.12*b*) is waived by ACI Code 11.1.2.1 if such beams are designed with minimum web reinforcement equal to the amount required by Eq. (4.13). In this case, the concrete contribution to shear strength may be calculated based on the full concrete compressive strength. Tests described in Refs. 4.14 and 4.17 indicate that for beams with concrete strength above about 6000 psi, the concrete contribution V_c was significantly less than predicted by the ACI Code equations, although the steel contribution V_s was higher. The total nominal shear strength V_n was greater than predicted by ACI Code methods in all cases. The use of minimum web steel for high-strength concrete beams is intended to enhance the post-cracking capacity, thus resulting in safe designs even though the concrete contribution to shear strength is overestimated.‡

EXAMPLE 4.1 **Beam without web reinforcement.** A rectangular beam is to be designed to carry a shear force V_u of 27 kips. No web reinforcement is to be used, and f'_c is 4000 psi. What is the minimum cross section if controlled by shear?

† To qualify, the fiber-reinforced concrete must conform to requirements in ACI Code 5.6.6.2 that specify a minimum deformed steel fiber content of 100 lb/yd^3 and minimum residual flexural strength values when the concrete is tested in accordance with ASTM C1609, "*Standard Test Method for Flexural Performance of Fiber-Reinforced Concrete (Using Beam with Third-Point Loading)*."

‡ The shortcomings of the ACI Code "$V_c + V_s$" approach to shear design, particularly the provisions relating to the concrete contribution V_c, have provided motivation for the development of more rational procedures, as will be discussed in Section 4.8.

Solution. If no web reinforcement is to be used, the cross-sectional dimensions must be selected so that the applied shear V_u is no larger than one-half the design shear strength ϕV_c. The calculations will be based on Eq. (4.12*b*). Thus,

$$V_u = \frac{1}{2}\phi(2\lambda\sqrt{f'_c}b_w d)$$

$$b_w d = \frac{27{,}000}{0.75 \times 1.0\sqrt{4000}} = 569 \text{ in}^2$$

A beam with $b_w = 18$ in. and $d = 32$ in. is required. Alternately, if the minimum amount of web reinforcement given by Eq. (4.13) is used, the concrete shear resistance may be taken at its full value ϕV_c, and it is easily confirmed that a beam with $b_w = 12$ in. and $d = 24$ in. will be sufficient.

c. Region in Which Web Reinforcement Is Required

If the required shear strength V_u is greater than the design shear strength ϕV_c provided by the concrete in any portion of a beam, there is a theoretical requirement for web reinforcement. Elsewhere in the span, web steel at least equal to the amount given by Eq. (4.13) must be provided, unless the factored shear force is less than $\frac{1}{2}\phi V_c$.

The portion of any span through which web reinforcement is theoretically necessary can be found from the shear diagram for the span, superimposing a plot of the shear strength of the concrete. Where the shear force V_u exceeds ϕV_c, shear reinforcement must provide for the excess. The additional length through which at least the minimum web steel is needed can be found by superimposing a plot of $\phi V_c/2$.

EXAMPLE 4.2 **Limits of web reinforcement.** A simply supported rectangular beam 16 in. wide having an effective depth of 22 in. carries a total factored load of 9.4 kips/ft on a 20 ft clear span. It is reinforced with 7.62 in^2 of tensile steel, which continues uninterrupted into the supports. If $f'_c = 4000$ psi, throughout what part of the beam is web reinforcement required?

Solution. The maximum external shear force occurs at the ends of the span, where $V_u = 9.4 \times 20/2 = 94$ kips. At the critical section for shear, a distance d from the support, $V_u = 94 - 9.4 \times 1.83 = 76.8$ kips. The shear force varies linearly to zero at midspan. The variation of V_u is shown in Fig. 4.13*a*. Adopting Eq. (4.12*b*) gives

$$V_c = 2\lambda\sqrt{f'_c}b_w d = 2 \times 1.0\sqrt{4000} \times 16 \times 22 = 44{,}500 \text{ lb}$$

Hence $\phi V_c = 0.75 \times 44.5 = 33.4$ kips. This value is superimposed on the shear diagram, and, from geometry, the point at which web reinforcement theoretically is no longer required is

$$10\left(\frac{94.0 - 33.4}{94.0}\right) = 6.45 \text{ ft}$$

from the support face. However, according to the ACI Code, at least a minimum amount of web reinforcement is required wherever the shear force exceeds $\phi V_c/2$, or 16.7 kips in this case. As seen from Fig. 4.13*a*, this applies to a distance

$$10\left(\frac{94.0 - 16.7}{94.0}\right) = 8.22 \text{ ft}$$

from the support face. To summarize, at least the minimum web steel must be provided within a distance of 8.22 ft from the supports, and within 6.45 ft the web steel must provide for the shear force corresponding to the shaded area.

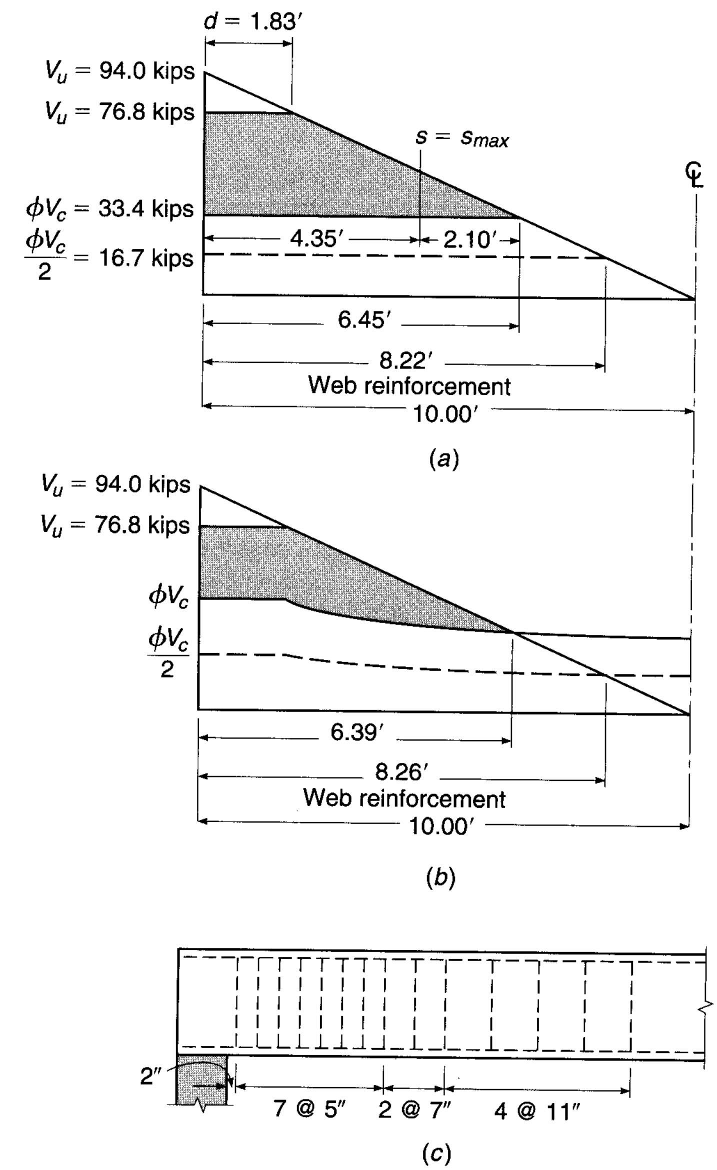


FIGURE 4.13
Shear design example.

If the alternative Eq. (4.12*a*) is used, the variation along the span of ρ_w, V_u, and M_u must be known so that V_c can be calculated. This is shown in tabular form in Table 4.1.

The factored shear V_u and the design shear capacity ϕV_c are plotted in Fig. 4.13*b*. From the graph it is found that stirrups are theoretically no longer required 6.39 ft from the support face. However, from the plot of $\phi V_c/2$ it is found that at least the minimum web steel is to be provided within a distance of 8.26 ft.

When Figs. 4.13*a* and *b* are compared, it is evident that the length over which web reinforcement is needed is nearly the same for this example whether Eq. (4.12*a*) or (4.12*b*) is used. However, the smaller shaded area of Fig. 4.13*b* indicates that substantially less web-steel area would be needed within that required distance if the more accurate Eq. (4.12*a*) were adopted.

TABLE 4.1
Shear design example

Distance from Support, ft	M_u, ft-kips	V_u, kips	V_c[a]	ϕV_c
0	0	94.0	61.3	46.0
1	89	84.6	61.3	46.0
2	169	75.2	57.8	43.4
3	240	65.8	51.9	38.9
4	301	56.4	48.8	36.6
5	353	47.0	47.0	35.2
6	395	37.6	45.6	34.2
7	428	28.2	44.6	33.5
8	451	18.8	43.8	32.8
9	465	9.4	43.0	32.3
10	470	0	42.3	31.7

[a] $V_c = (1.9\lambda\sqrt{f'_c} + 2500\rho_w V_u d/M_u)b_w d \le 3.5\lambda\sqrt{f'_c}\, b_w d$ and $V_u d/M_u \le 1.0$

d. Design of Web Reinforcement

The design of web reinforcement, under the provisions of the ACI Code, is based on Eq. (4.11*a*) for vertical stirrups and Eq. (4.11*b*) for inclined stirrups or bent bars. In design, it is usually convenient to select a trial web-steel area A_v based on standard stirrup sizes [usually in the range from No. 3 to 5 (No. 10 to 16) for stirrups, and according to the longitudinal bar size for bent-up bars], for which the required spacing s can be found. Equating the design strength ϕV_n to the required strength V_u and transposing Eqs. (4.11*a*) and (4.11*b*) accordingly, one finds that the required spacing of web reinforcement is, for vertical stirrups,

$$s = \frac{\phi A_v f_{yt} d}{V_u - \phi V_c} \tag{4.14a}$$

and for bent bars

$$s = \frac{\phi A_v f_{yt} d(\sin\alpha + \cos\alpha)}{V_u - \phi V_c} \tag{4.14b}$$

It should be emphasized that when conventional U stirrups such as in Fig. 4.8*b* are used, the web area A_v provided by each stirrup is *twice* the cross-sectional area of the bar; for stirrups such as those of Fig. 4.8*c*, A_v is 4 times the area of the bar used. Equation (4.14*a*) is applicable to members with circular, as well as rectangular, cross sections. For circular members, d is taken as the effective depth, as defined earlier in Section 4.5a, and A_v is taken as 2 times the area of the bar, hoop, or spiral.

While the ACI Code requires only that the inclined part of a bent bar make an angle of at least 30° with the longitudinal part, bars are usually bent at a 45° angle. Only the center three-fourths of the inclined part of any bar is to be considered effective as web reinforcement.

It is undesirable to space vertical stirrups closer than about 4 in.; the size of the stirrups should be chosen to avoid a closer spacing. When vertical stirrups are required over a comparatively short distance, it is good practice to space them uniformly over the entire distance, the spacing being calculated for the point of greatest shear

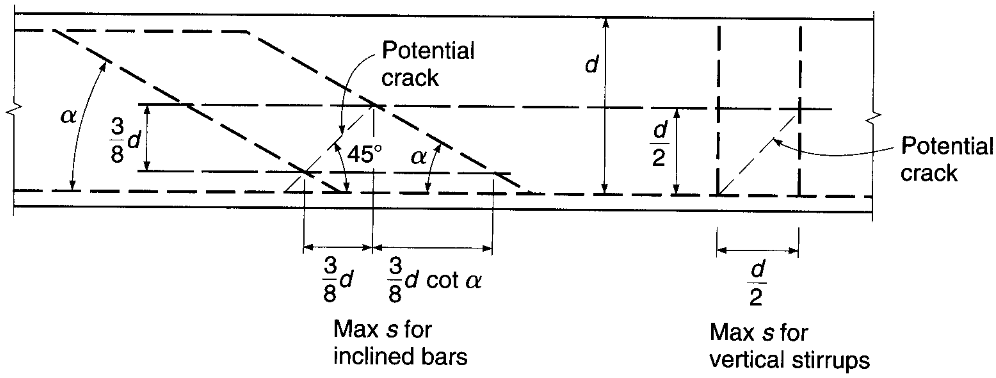


FIGURE 4.14
Maximum spacing of web reinforcement as governed by diagonal crack interception.

(minimum spacing). If the web reinforcement is required over a long distance, and if the shear varies materially throughout this distance, it is more economical to compute the spacings required at several sections and to place the stirrups accordingly, in groups of varying spacing.

Where web reinforcement is needed, the Code requires it to be spaced so that every 45° line, representing a potential diagonal crack and extending from the middepth $d/2$ of the member to the longitudinal tension bars, is crossed by at least one line of web reinforcement; in addition, the Code specifies a maximum spacing of 24 in. When V_s exceeds $4\sqrt{f_c'}b_wd$, these maximum spacings are halved. These limitations are shown in Fig. 4.14 for both vertical stirrups and inclined bars, for situations in which the excess shear does not exceed the stated limit.

For design purposes, Eq. (4.13) giving the minimum web-steel area A_v is more conveniently inverted to permit calculation of maximum spacing s for the selected A_v. Thus, for the usual case of vertical stirrups, with $V_s \leq 4\sqrt{f_c'}b_wd$, the maximum spacing of stirrups is the smallest of

$$s_{\max} = \frac{A_v f_{yt}}{0.75\sqrt{f_c'}\,b_w} \leq \frac{A_v f_{yt}}{50 b_w} \qquad (4.15a)$$

$$s_{\max} = \frac{d}{2} \qquad (4.15b)$$

$$s_{\max} = 24 \text{ in.} \qquad (4.15c)$$

For longitudinal bars bent at 45°, Eq. (4.15*b*) is replaced by $s_{\max} = 3d/4$, as confirmed by Fig. 4.14.

To avoid excessive crack width in beam webs, the ACI Code limits the yield strength of the reinforcement to f_{yt} = 60,000 psi or less for reinforcing bars and 80,000 psi or less for welded wire reinforcement. In no case, according to the ACI Code, is V_s to exceed $8\sqrt{f_c'}\,b_wd$, regardless of the amount of web steel used.

EXAMPLE 4.3

Design of web reinforcement. Using vertical U stirrups with f_{yt} = 60,000 psi, design the web reinforcement for the beam in Example 4.2.

Solution. The solution will be based on the shear diagram in Fig. 4.13*a*. The stirrups must be designed to resist that part of the shear shown shaded. With No. 3 (No. 10) stirrups used for trial, the three maximum spacing criteria are first applied. For $\phi V_s = V_u - \phi V_c$ = 43,400 lb, which is less than $4\phi\sqrt{f'_c}\,b_w d = 66{,}800$ lb, the maximum spacing must exceed neither $d/2 =$ 11 in. nor 24 in. Also, from Eq. (4.15*a*),

$$s_{max} = \frac{A_v f_{yt}}{0.75\sqrt{f'_c}\,b_w} = \frac{0.22 \times 60{,}000}{0.75\sqrt{4000} \times 16} = 17.4 \text{ in.}$$

$$\leq \frac{A_v f_{yt}}{50 b_w} = \frac{0.22 \times 60{,}000}{50 \times 16} = 16.5 \text{ in.}$$

The first criterion controls in this case, and a maximum spacing of 11 in. is imposed. From the support to a distance d from the support, the excess shear $V_u - \phi V_c$ is 43,400 lb. In this region, the required spacing is

$$s = \frac{\phi A_v f_{yt} d}{V_u - \phi V_c} = \frac{0.75 \times 0.22 \times 60{,}000 \times 22}{43{,}400} = 5.0 \text{ in.}$$

This is neither so small that placement problems would result nor so large that maximum spacing criteria would control, and the choice of No. 3 (No. 10) stirrups is confirmed. Solving Eq. (4.14*a*) for the excess shear at which the maximum spacing can be used gives

$$V_u - \phi V_c = \frac{\phi A_v f_{yt} d}{s} = \frac{0.75 \times 0.22 \times 60{,}000 \times 22}{11} = 19{,}800 \text{ lb}$$

With reference to Fig. 4.13*a*, this is attained at a distance x_1 from the point of zero excess shear, where $x_1 = 6.45 \times 19{,}800/60{,}600 = 2.10$ ft. This is 4.35 ft from the support face. With this information, a satisfactory spacing pattern can be selected. The first stirrup is usually placed at a distance $s/2$ from the support. The following spacing pattern is satisfactory:

1 space at 2 in. = 2 in.

7 spaces at 5 in. = 35 in.

2 spaces at 7 in. = 14 in.

4 spaces at 11 in. = 44 in.

Total = 95 in. = 7 ft 11 in.

The resulting stirrup pattern is shown in Fig. 4.13*c*. As an alternative solution, it is possible to plot a curve showing required spacing as a function of distance from the support. Once the required spacing at some reference section, say at the support, is determined,

$$s_0 = \frac{0.75 \times 0.22 \times 60{,}000 \times 22}{94{,}000 - 33{,}400} = 3.59 \text{ in.}$$

it is easy to obtain the required spacings elsewhere. In Eq. (4.14*a*), only $V_u - \phi V_c$ changes with distance from the support. For uniform load, this quantity is a linear function of distance from the point of zero excess shear, 6.45 ft from the support face. Hence, at 1 ft intervals,

$$s_1 = 3.59 \times 6.45/5.45 = 4.25 \text{ in.}$$

$$s_2 = 3.59 \times 6.45/4.45 = 5.20 \text{ in.}$$

$$s_3 = 3.59 \times 6.45/3.45 = 6.70 \text{ in.}$$

$$s_4 = 3.59 \times 6.45/2.45 = 9.45 \text{ in.}$$

$$s_5 = 3.59 \times 6.45/1.45 = 15.97 \text{ in.}$$

This is plotted in Fig. 4.15 together with the maximum spacing of 11 in., and a practical spacing pattern is selected. The spacing at a distance d from the support face is selected as the minimum

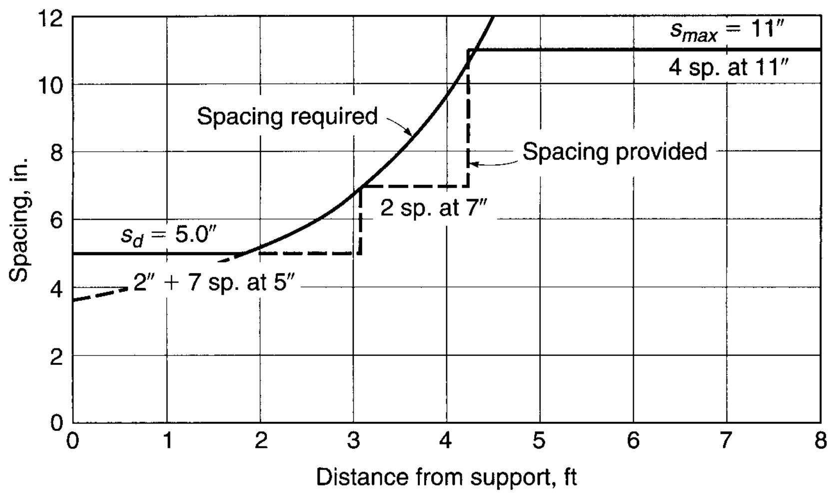


FIGURE 4.15
Required stirrup spacings for Example 4.3.

requirement, in accordance with the ACI Code. The pattern of No. 3 (No. 10) U-shaped stirrups selected (shown on the graph) is identical with the previous solution. In most cases, the experienced designer would find it unnecessary actually to plot the spacing diagram of Fig. 4.15 and would select a spacing pattern directly after calculating the required spacing at intervals along the beam.

If the web steel were to be designed on the basis of the excess-shear diagram in Fig. 4.13*b*, the second approach illustrated above would necessarily be selected, and spacings would be calculated at intervals along the span. In this particular case, a spacing of 7.07 in. is calculated up to 20 in. from the face of the support. The calculated spacing drops to 6.76 in. at *d* from the face of the support, and then increases to 11 in., the maximum permissible spacing, 4 ft from the support. The following practical spacing could be used:

1 space at 3 in. =	3 in.
6 spaces at 7 in. =	42 in.
4 spaces at 11 in. =	44 in.
Total =	89 in. = 7 ft 5 in.

Thus, 11 No. 3 (No. 10) stirrups would be used, rather than the 14 previously calculated, in each half of the span.

The number of stirrups just calculated represents the minimum for each of the two expressions for V_c. Although not required by the ACI Code, it is good design practice to continue the stirrups (at maximum spacing) through the middle region of the beam, even though the calculated shear is low. Doing so satisfies the dual purposes of providing continuing support for the top longitudinal reinforcement that is required wherever stirrups are used and providing additional shear capacity in the region to handle load cases not considered in developing the shear diagram. If this were done, the number of stirrups would increase from 14 and 11 to $16\frac{1}{2}$ and $13\frac{1}{2}$ per half-span (i.e., one stirrup at midspan), respectively.

4.6 EFFECT OF AXIAL FORCES

The beams considered in the preceding sections were subjected to shear and flexure only. Reinforced concrete beams may also be subjected to axial forces, acting simultaneously with shear and flexure, due to a variety of causes. These include external axial loads, longitudinal prestressing, and restraint forces introduced as a result of shrinkage of the concrete or temperature changes. Beams may have their strength in shear significantly modified in the presence of axial tension or compression, as is evident from a review of Sections 4.1 through 4.4.

Prestressed concrete members are treated by somewhat specialized methods, according to present practice, based largely on results of testing prestressed concrete beams. They will be considered separately in Chapter 19, and only nonprestressed reinforced concrete beams will be treated here.

The main effect of axial load is to modify the diagonal cracking load of the member. It was shown in Section 4.3 that diagonal tension cracking will occur when the principal tensile stress in the web of a beam, resulting from combined action of shear and bending, reaches the tensile strength of the concrete. It is clear that the introduction of longitudinal force, which modifies the magnitude and direction of the principal tensile stresses, may significantly alter the diagonal cracking load. Axial compression will increase the cracking load, while axial tension will decrease it.

For members carrying only flexural and shear loading, the shear force at which diagonal cracking occurs V_{cr} is predicted by Eq. (4.3*a*), based on a combination of theory and experimental evidence. Furthermore, for reasons that were explained in Section 4.4b, in beams with web reinforcement, the contribution of the concrete to shear strength V_c is taken equal to the diagonal cracking load V_{cr}. Thus, according to the ACI Code, the concrete contribution is calculated by Eq. (4.12*a*) or (4.12*b*). For members carrying flexural and shear loading plus axial loads, V_c can be calculated by suitable modifications of these equations as follows.

a. Axial Compression

In developing Eq. (4.3*a*) for V_{cr}, it was pointed out that the diagonal cracking load depends on the ratio of shear stress v to bending stress f at the top of the flexural crack. While these stresses were never actually determined, they were conveniently expressed as

$$v = K_1\left(\frac{V}{bd}\right) \tag{a}$$

and

$$f = K_2\left(\frac{M}{bd^2}\right) \tag{b}$$

Equation (*a*) relates the concrete shear stress at the top of the flexural crack to the average shear stress; Eq. (*b*) can be used to relate the flexural tension in the concrete at the top of the crack to the tension in the flexural steel, through the modular ratio $n = E_s/E_c$, as follows:

$$f = K_0\frac{f_s}{n} = K_0\frac{M}{nA_sjd}$$

or

$$f = K_0\frac{M}{n\rho jbd^2} \tag{c}$$

where jd is the internal lever arm between C and T, and K_0 is an unknown constant. Thus, the previous constant K_2 is equal to $K_0/n\rho j$.

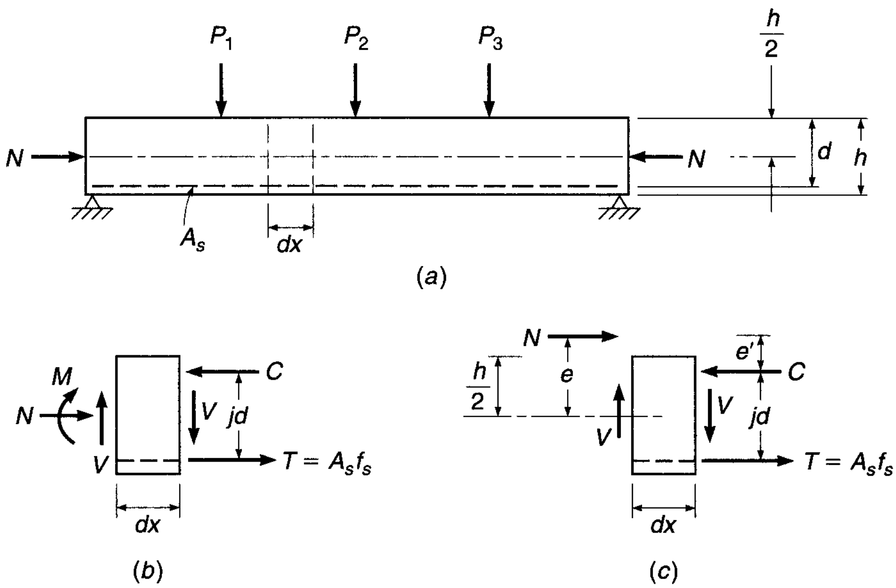


FIGURE 4.16
Beams subject to axial compression plus bending and shear loads.

Now consider a beam subject to axial compression N as well as M and V, as shown in Fig. 4.16*a*. In Fig. 4.16*b*, the external moment, shear, and thrust acting on the left side of a small element of the beam, having length dx, are equilibrated by the internal stress resultants T, C, and V acting on the right. It is convenient to replace the external loads M and N with the statically equivalent load N acting at eccentricity $e = M/N$ from the middepth, as shown in Fig. 4.16*c*. The lever arm of the eccentric force N with respect to the compressive resultant C is

$$e' = e + d - \frac{h}{2} - jd \qquad (d)$$

The steel stress f_s can now be found taking moments about the point of application of C.

$$f_s = \frac{Ne'}{A_s jd}$$

from which

$$f_s = \frac{M + N(d - h/2 - jd)}{A_s jd}$$

Noting that j is very close to $\frac{7}{8}$ for loads up to that producing diagonal cracking, the term in parentheses in the last equation above can be written as $(d - 4h)/8$. Then with $f = K_0 f_s/n$ as before, the concrete tensile stress at the head of the flexural crack is

$$f = K_0 \frac{M - N(4h - d)/8}{n\rho jbd^2} = K_2 \frac{M - N(4h - d)/8}{bd^2} \qquad (e)$$

Comparing Eq. (*e*) with Eqs. (*c*) and (*b*) makes it clear that the previous derivation for flexural tension f holds for the present case including axial loads if a modified moment $M - N(4h - d)/8$ is substituted for M. It follows that Eq. (4.3*a*) can be used to calculate V_{cr} with the same substitution of modified for actual moment.

The ACI Code provisions are based on this development. The concrete contribution to shear strength V_c is taken equal to V_{cr} and is given by Eq. (4.12a) as before:

$$V_c = \left(1.9\lambda\sqrt{f'_c} + 2500\,\frac{\rho_w V_u d}{M_u}\right) b_w d \qquad (4.12a)$$

except that the modified moment

$$M_m = M_u - N_u\,\frac{4h - d}{8} \qquad (4.16)$$

is to be substituted for M_u and $V_u d/M_u$ need not be limited to 1.0 as before. The thrust N_u is to be taken positive for compression. For beams with axial compression, the upper limit of $3.5\lambda\sqrt{f'_c}\,b_w d$ is replaced by

$$V_c = 3.5\lambda\sqrt{f'_c}\,b_w d\sqrt{1 + \frac{N_u}{500A_g}} \qquad (4.17)$$

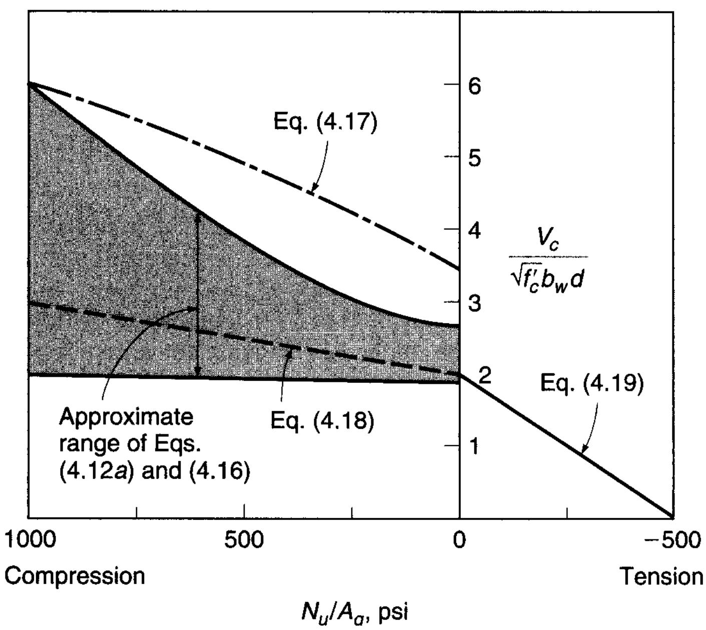
where A_g is the gross area of the concrete and N_u/A_g is expressed in psi units.

As an alternative to the rather complicated determination of V_c using Eqs. (4.12a), (4.16), and (4.17), ACI Code 11.2.1.2 permits the use of an alternative simplified expression:

$$V_c = 2\left(1 + \frac{N_u}{2000A_g}\right)\lambda\sqrt{f'_c}\,b_w d \qquad (4.18)$$

Figure 4.17 shows a comparison of V_c calculated by the more complex and simplified expressions for beams with compression load. Equation (4.18) is seen to be generally quite conservative, particularly for higher values of N_u/A_g. However, because of its simplicity, it is widely used in practice.

FIGURE 4.17
Comparison of equations for V_c for members subject to axial loads.



b. Axial Tension

The approach developed above for beams with axial compression does not correlate well with experimental evidence for beams subject to axial tension, and often predicts strengths V_c higher than actually measured. For this reason, the ACI Code provides that, for members carrying significant axial tension as well as bending and shear, the contribution of the concrete be taken as

$$V_c = 2\left(1 + \frac{N_u}{500A_g}\right)\lambda\sqrt{f'_c}\, b_w d \tag{4.19}$$

but not less than zero, where N_u is negative for tension. As a simplifying alternative, the Commentary to the Code suggests that, for beams carrying axial tension, V_c be taken equal to zero and the shear reinforcement be required to carry the total shear. The variation of V_c with N_u/A_g for beams with tension is shown in Fig. 4.17 also.

EXAMPLE 4.4 **Effect of axial forces on V_c.** A beam with dimensions b = 12 in., d = 24 in., and h = 27 in., with f'_c = 4000 psi, carries a single concentrated factored load of 100 kips at midspan. Find the maximum shear strength of the concrete V_c at the first critical section for shear at a distance d from the support (*a*) if no axial forces are present, (*b*) if axial compression of 60 kips acts, and (*c*) if axial tension of 60 kips acts. In each case, compute V_c by both the more complex and simplified expressions of the ACI Code. Neglect the self-weight of the beam. At the section considered, tensile reinforcement consists of three No. 10 (No. 32) bars with a total area of 3.81 in^2.

SOLUTION. At the critical section, V_u = 50 kips and $M_u = 50 \times 2 = 100$ ft-kips, while $\rho = 3.81/(12 \times 24) = 0.013$.

(*a*) If $N_u = 0$, Eq. (4.12*a*) predicts

$$V_c = \left(1.9 \times 1.0\sqrt{4000} + 2500\,\frac{0.013 \times 50 \times 2}{100}\right)12 \times \frac{24}{1000} = 44.0 \text{ kips}$$

not to exceed the value of

$$V_c = 3.5 \times 1.0\sqrt{4000} \times 12 \times \frac{24}{1000} = 63.8 \text{ kips}$$

If the simplified Eq. (4.12*b*) is used,

$$V_c = 2 \times 1.0\sqrt{4000} \times 12 \times \frac{24}{1000} = 36.4 \text{ kips}$$

which is about 17 percent below the more exact value of Eq. (4.12*a*).

(*b*) With a compression of 60 kips introduced, the modified moment is found from Eq. (4.16) to be

$$M_m = 100 - 60\,\frac{4 \times 27 - 24}{8 \times 12} = 47.5 \text{ ft-kips}$$

After introduction of that value into Eq. (4.12*a*) in place of M_u, the concrete shear strength is

$$V_c = \left(1.9 \times 1.0\sqrt{4000} + 2500\,\frac{0.013 \times 50 \times 2}{47.5}\right)12 \times \frac{24}{1000} = 54.3 \text{ kips}$$

and, according to Eq. (4.17), should not exceed

$$V_c = 63.8\sqrt{1 + \frac{60{,}000}{500 \times 12 \times 27}} = 74.6 \text{ kips}$$

If the simplified Eq. (4.18) is used,

$$V_c = 2\left(1 + \frac{60{,}000}{2000 \times 12 \times 27}\right) \times 1.0\sqrt{4000} \times 12 \times \frac{24}{1000} = 39.8 \text{ kips}$$

Comparing the results of the more exact calculation for (*a*) and (*b*), one sees that the introduction of an axial compressive stress of 60,000/(12 × 27) = 185 psi increases the concrete shear V_c by about 25 percent.

(*c*) With an axial tension of 60 kips acting, the reduced V_c is found from Eq. (4.19) to be

$$V_c = 2\left(1 - \frac{60{,}000}{500 \times 12 \times 27}\right) \times 1.0\sqrt{4000} \times 12 \times \frac{24}{1000} = 22.9 \text{ kips}$$

a reduction of almost 50 percent from the value for $N_u = 0$. The alternative of using Eq. (4.19) for this case, according to the ACI Commentary, would be to set $V_c = 0$.

In all cases above, the strength reduction factor $\phi = 0.75$ would be applied to V_c to obtain the design strength.

4.7 BEAMS WITH VARYING DEPTH

Reinforced concrete members having varying depth are frequently used in the form of haunched beams for bridges or portal frames, as shown in Fig. 4.18*a*, as precast roof girders such as shown in Fig. 4.18*b*, or as cantilever slabs. Generally the depth increases in the direction of increasing moments. For beams with varying depth, the inclination of the internal compressive and tensile stress resultants may significantly affect the shear for which the beam should be designed. In addition, the shear resistance of such members may differ from that of prismatic beams.

Figure 4.18*c* shows a cantilever beam, with fixed support at the left end, carrying a single concentrated load *P* at the right. The depth increases linearly in the direction

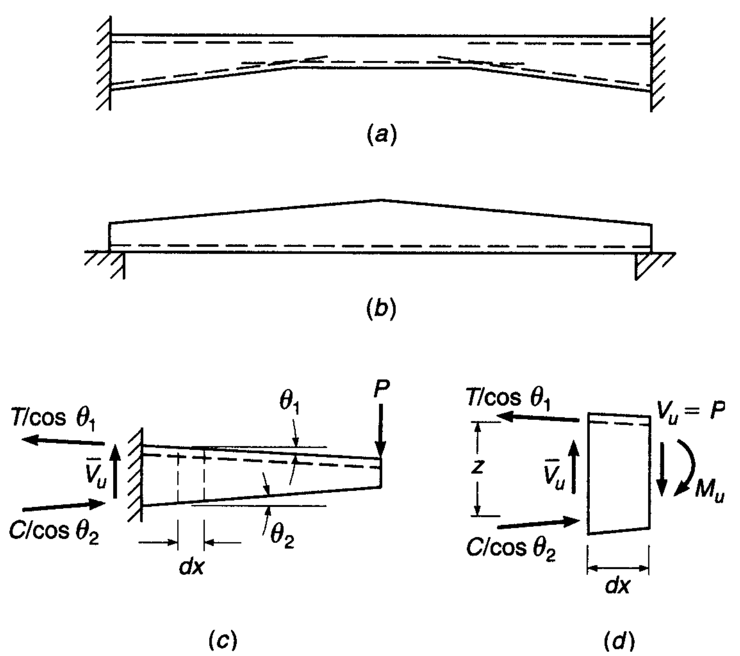


FIGURE 4.18
Effect of varying beam depth on shear.

of increasing moment. In such cases, the internal tension in the steel and the compressive stress resultant in the concrete are inclined, and introduce components transverse to the axis of the member. With reference to Fig. 4.18*d*, showing a short length *dx* of the beam, if the slope of the top surface is θ_1 and that of the bottom is θ_2, the net shear force $\overline{V}_u$ for which the beam should be designed is very nearly equal to

$$\overline{V}_u = V_u - T \tan \theta_1 - C \tan \theta_2$$

where V_u is the external shear force equal to the load P here, and $C = T = M_u/z$. The internal lever arm $z = d - a/2$ as usual. Thus, in a case for which the beam depth increases in the direction of increasing moment, the shear for which the member should be designed is approximately

$$\overline{V}_u = V_u - \frac{M_u}{z} (\tan \theta_1 + \tan \theta_2) \tag{4.20a}$$

For the infrequent case in which the member depth decreases in the direction of increasing moment, it is easily confirmed that the corresponding equation is

$$\overline{V}_u = V_u + \frac{M_u}{z} (\tan \theta_1 + \tan \theta_2) \tag{4.20b}$$

These equations are approximate because the direction of the internal forces is not exactly as assumed; however, the equations may be used without significant error provided the slope angles do not exceed about 30°.

There has been very little research studying the shear strength of beams having varying depth. Tests reported in Ref. 4.19 on simple span beams with haunches at slopes up to about 15° and with depths both increasing and decreasing in the direction of increasing moments indicate no appreciable change in the cracking load V_{cr} compared with that for prismatic members. Furthermore, the strength of the haunched beams, which contained vertical stirrups as web reinforcement, was not significantly decreased or increased, regardless of the direction of decreasing depth. Based on this information, *it appears safe to design beams with varying depth for shear using equations for V_c and V_s developed for prismatic members*, provided the actual depth *d* at the section under consideration is used in the calculations.

4.8 ALTERNATIVE MODELS FOR SHEAR ANALYSIS AND DESIGN

The ACI Code method of design for shear and diagonal tension in beams, presented in preceding sections of this chapter, is essentially empirical. While generally leading to safe designs, the ACI Code "$V_c + V_s$" approach lacks a physical model for the behavior of beams subject to shear combined with bending, and its shortcomings are now generally recognized. The "concrete contribution" V_c is generally considered to be some combination of force transfer by dowel action of the main steel, aggregate interlock along a diagonal crack, and shear in the uncracked concrete beyond the end of the crack. The values of each contribution are not identified. A rather vague rationalization is followed in adopting the diagonal cracking load of a member *without* web steel as the concrete contribution to the shear strength of an otherwise identical beam *with* web steel (see Section 4.4). Furthermore, as discussed in Section 4.3, Eqs. (4.3*a*) and (4.12*a*), used to predict the diagonal cracking load, overestimate concrete shear strength for beams with low reinforcement ratios ($\rho < 1.0$ percent), overestimate the gain in shear strength resulting from the use of high-strength concrete (Refs. 4.14 to 4.17), and underestimate the influence of $V_u d/M_u$ (Ref. 4.3). The expressions also ignore the fact that shear strength decreases as member size increases (Refs. 4.20 to 4.21).

Ad hoc procedures are built into the ACI Code to adjust for some of these deficiencies, but it follows that it is necessary to include equations, also empirically developed for the most part, for specific classes of members (e.g., deep beams vs. normal beams, beams with axial loads, prestressed vs. nonprestressed beams, high-strength concrete beams)—with restrictions on the range of applicability of such equations. And it is necessary to incorporate seemingly arbitrary provisions for the maximum nominal shear stress and for the extension of flexural reinforcement past the theoretical point of need. The end result is that the number of ACI Code equations for shear design has grown from 4 prior to 1963 to 38 in 2008.

With this as background, attention has been given to the development of design approaches based on rational behavioral models, generally applicable, rather than on empirical evidence alone (Ref. 4.6).

The *truss model* was originally introduced by Ritter (Ref. 4.22) and Morsch (Ref. 4.23) at the turn of the last century. A simplified version has long provided the basis for the ACI Code design of shear steel. The essential features of the truss model are reviewed with reference to Fig. 4.19*a*, which shows one-half the span of a simply supported, uniformly loaded beam. The combined action of flexure and shear produces the pattern of cracking shown. Reinforcement consists of the main flexural steel near the tension face and vertical stirrups distributed over the span.

The structural action can be represented by the truss of Fig. 4.19*b*, with the main steel providing the tension chord, the concrete top flange acting as the compression chord, the stirrups providing the vertical tension web members, and the concrete between inclined cracks acting as 45° compression diagonals. The truss is formed by lumping all the stirrups cut by section *a-a* into one vertical member and all the diagonal concrete struts cut by section *b-b* into one compression diagonal. Experience shows that for typical cases, the results of the model described are quite conservative, particularly for beams with small amounts of web reinforcement. As noted above, in the ACI Code the observed excess shear capacity is taken equal to the shear at the commencement of diagonal cracking and is referred to as the *concrete contribution* V_c.

Over the past 25 years, the truss concept has been greatly extended by the work of Schlaich, Marti, Collins, MacGregor, and others (Refs. 4.6, 4.24 to 4.29). It was realized that the angle of inclination of the concrete struts is generally not 45° but may range between about 25° and 65°, depending to a large extent on the arrangement of reinforcement. This led to what has become known as the *variable-angle truss model*, shown in Fig. 4.19*c*, which illustrates the five basic components of the improved model: (*a*) struts, or concrete compression members uniaxially loaded; (*b*) ties, or steel tension members; (*c*) joints at the intersection of truss members, assumed to be pin-connected; (*d*) compression fans, which form at "disturbed" regions, such as at the supports or under concentrated loads, transmitting the forces into the beam; and (*e*) diagonal compression fields, occurring where parallel compression struts transmit force from one stirrup to another. As in the ACI Code development, stirrups are typically assumed to reach yield stress at failure. With the force in all the verticals known and equal to $A_v f_{yt}$, the truss of Fig. 4.19*c* becomes statically determinate. Direct design equations can be based on the variable-angle truss model for ordinary cases. The model also permits direct numerical solution for the required reinforcement for special cases. The truss model does not include components of the shear failure mechanism such as aggregate interlock and friction, dowel action of the longitudinal steel, and shear carried across uncracked concrete. Furthermore, in the format originally proposed, the truss

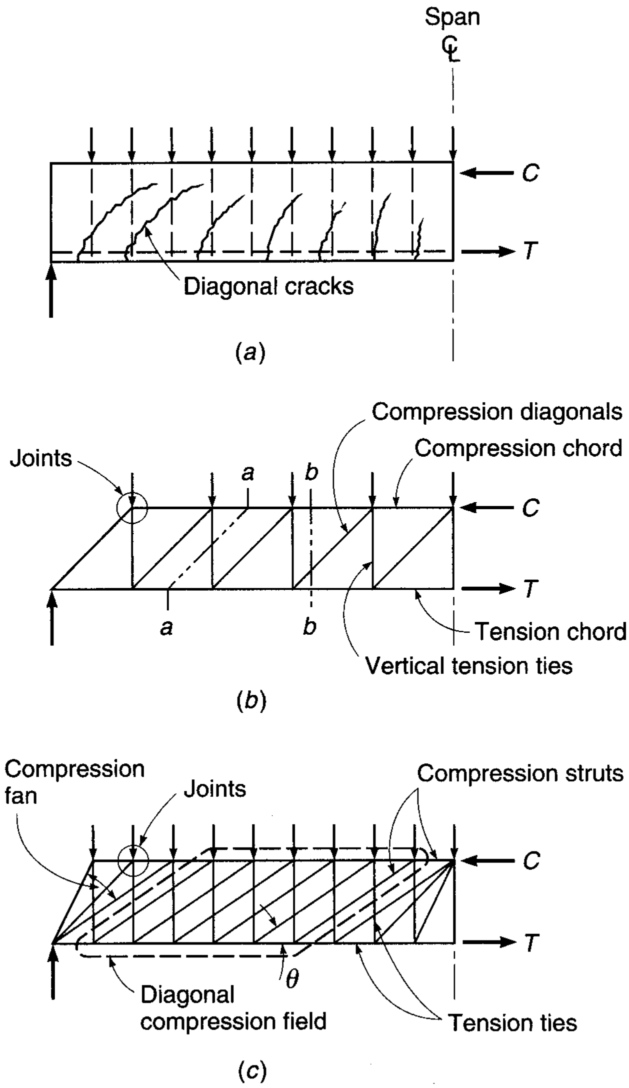


FIGURE 4.19
Truss model for beams with web reinforcement:
(*a*) uniformly loaded beam;
(*b*) simple truss model;
(*c*) more realistic model.

model does not account for compatibility requirements; i.e., it is based on *plasticity theory*. One form of the truss model is incorporated in Appendix A of the ACI Code; strut-and-tie models are discussed in detail in Chapter 10.

a. Compression Field Theory

The Canadian National Standard for reinforced concrete (Ref. 4.30) includes a method of shear design that is essentially the same as the present ACI method but also includes an alternative "general method" based on the variable-angle truss and the *compression field theory* (Refs. 4.27 and 4.31). The latter is incorporated in *AASHTO LRFD Bridge Design Specifications* (Ref. 4.12), where its use is mandatory for shear design. In its complete form, known as the *modified compression field theory*, it accounts for requirements of compatibility as well as equilibrium and incorporates stress-strain characteristics of both materials. Thus, it is capable of predicting not only the failure

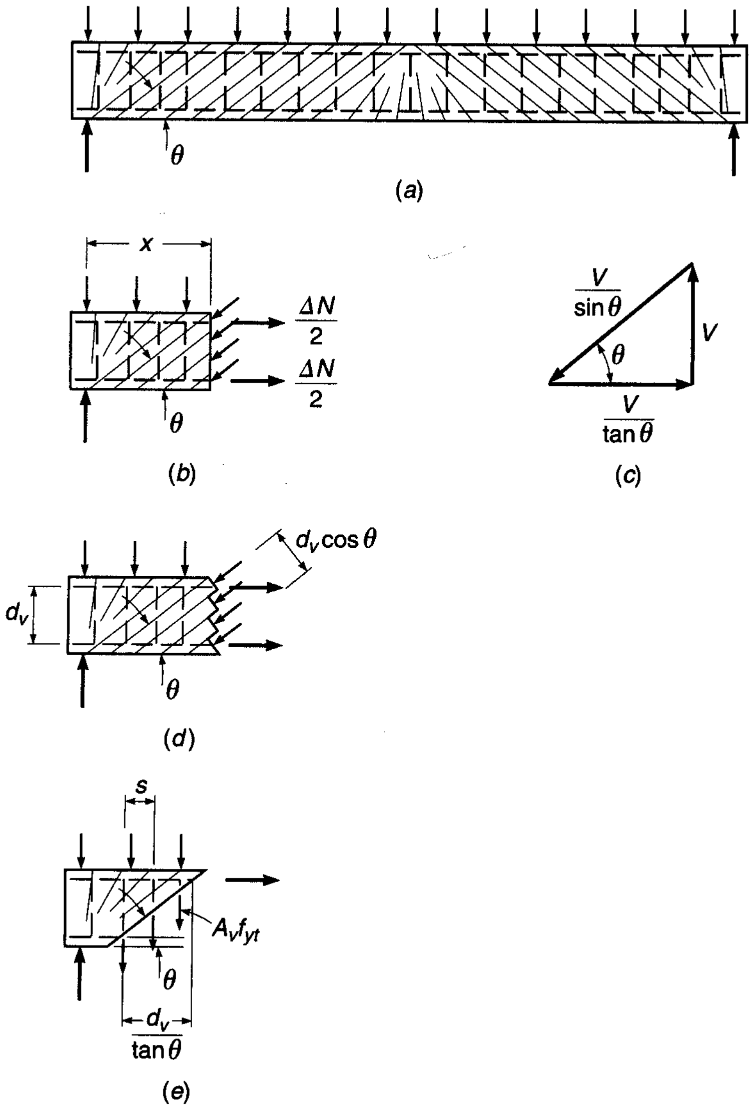


FIGURE 4.20
Basis of compression field theory for shear: (*a*) beam with shear and longitudinal steel; (*b*) tension in horizontal bars due to shear; (*c*) diagonal compression on beam web; (*d*) vertical tension in stirrups; (*e*) equilibrium diagram of forces due to shear. (*Adapted from Ref. 4.27.*)

load but also the complete load-deformation response. The most basic elements of the compression field theory, applied to members carrying combined flexure and shear, will be clear from Fig. 4.20. Figure 4.20*a* shows a simple-span concrete beam, reinforced with longitudinal bars and transverse stirrups, and carrying a uniformly distributed loading along the top face. The light diagonal lines are an idealized representation of potential tensile cracking in the concrete.

Figure 4.20*b* illustrates that the net shear V at a section a distance x from the support is resisted by the vertical component of the diagonal compression force in the concrete struts. The horizontal component of the compression in the struts must be equilibrated by the total tension force ΔN in the longitudinal steel. Thus, with reference to Fig. 4.20*b* and *c*, the magnitude of the longitudinal tension resulting from shear is

$$\Delta N = \frac{V}{\tan\theta} = V\cot\theta \tag{4.21}$$

where θ is the angle of inclination of the diagonal struts. These forces superimpose on the longitudinal forces owing to flexure, not shown in Fig. 4.20*b*.

The effective depth for shear calculations, according to this method, is taken at the distance between longitudinal force resultants d_v. Thus, from Fig. 4.20*d*, the diagonal compressive stress in a web having width b_v is

$$f_d = \frac{V}{b_v d_v \sin\theta \cos\theta} \tag{4.22}$$

The tensile force in the vertical stirrups, each having area A_v and assumed to act at the yield stress f_{yt}, can be found from the free body of Fig. 4.20*e*. With stirrups assumed to be at uniform spacing *s*,

$$A_v f_{yt} = \frac{Vs \tan\theta}{d_v} \tag{4.23}$$

Note, with reference to the free-body diagram, that the transverse reinforcement within the length $d_v/\tan\theta$ can be designed to resist the lowest shear that occurs within this length, i.e., the shear at the right end.

In the ACI Code method developed in Section 4.4, it was assumed that the angle θ was 45°. With that assumption, and if d is substituted for d_v, Eq. (4.23) is identical to that used earlier for the design of vertical stirrups. It is generally recognized, however, that the slope angle of the compression struts is not necessarily 45°, and following Refs. 4.12 and 4.30 that angle can range from 20 to 75°, provided the same value of θ is used in satisfying all requirements at a section. It is evident from Eqs. (4.21) and (4.23) that if a lower slope angle is selected, less vertical reinforcement but more horizontal reinforcement will be required. In addition, the compression in the concrete diagonals will be increased. Conversely, if a higher slope angle is used, more vertical steel but less horizontal steel will be needed, and the diagonal thrust will be less. It is generally economical to use a slope angle θ somewhat less than 45°, with the limitation that the concrete diagonal struts not be overstressed in compression.

In addition to providing an improved basis for the design of reinforcement for shear, the variable angle truss model gives important insights into detailing needs. For example, it becomes clear from the above that the increase in longitudinal steel tension resulting from the diagonal compression in the struts requires that flexural steel be extended beyond the point at which it is theoretically not needed for flexure, to account for the increased horizontal tensile force resulting from the thrust in the compression diagonals. This is not recognized explicitly in the ACI Code method for beam design. (However, the ACI Code does contain the arbitrary requirement that the flexural steel be extended a distance d or 12 bar diameters beyond the point indicated by flexural requirements.) Also, it is clear from the basic concept of the truss model that stirrups must be capable of developing their full tensile strength throughout the entire stirrup height. For wide beams, focus on truss action indicates that special attention should be given to lateral distribution of web reinforcement. It is often the practice to use conventional U stirrups for wide beams, with the vertical tension from the stirrups concentrated around the outermost bars. According to the discussion above, diagonal compression struts transmit forces only at the joints. Lack of stirrup joints at the interior of the wide-beam web would force joints to form only at the exterior longitudinal bars, which would concentrate the diagonal compression at the outer faces of the beam and possibly result in premature failure. It is best to form a truss joint at each of the longitudinal bars, and multiple leg stirrups should always be used in wide beams (see Fig. 4.8*c*).

References 4.12 and 4.30 incorporate a refined version of the approach just described, known as the modified compression field theory (MCFT), in which the cracked concrete is treated as a new material with its own stress-strain characteristics, including the ability to carry tension following crack formation. The compressive strength and the stress-strain curve of the concrete in the diagonal compression struts decrease as the diagonal tensile strain in the concrete increases. Equilibrium, compatibility, and constitutive relationships are formulated in terms of average stresses and average strains. Variability in the angle of inclination of the compression struts and stress-strain softening effects in the response of the concrete are taken into account. Consideration is also given to local stress conditions at crack locations. The method is capable of accurately predicting the response of complex elements such as shear walls, diaphragms, and membrane elements subjected to in-plane shear and axial loads through the full range of loading, from zero load to failure (Refs. 4.28 and 4.29). The version of the method adopted in Ref. 4.12 has been simplified to allow its use for routine design.

b. Design Provisions

The version of the MCFT adopted in the *AASHTO LRFD Bridge Design Specifications* (Ref. 4.12) is, like the shear provisions in the ACI Code, based on nominal shear capacity, with V_n equal to the lesser of

$$V_n = V_c + V_s \tag{4.24}$$

$$V_n = 0.25 f_c' b_v d_v \tag{4.25}$$

where b_v = web width (the same as b_w in the ACI Code) and d_v = effective depth in shear, taken as equal to the flexural lever arm (the distance between the centroids of the tensile and compressive forces), but not less than the greater of $0.9d$ or $0.72h$.

The values of V_c and V_s differ from those used by the ACI, with

$$V_c = \beta \sqrt{f_c'}\, b_v d_v \tag{4.26}$$

and

$$V_s = \frac{A_v f_{yt} d_v (\cot\theta + \cot\alpha)\sin\alpha}{s} \tag{4.27}$$

where A_v, f_{yt}, s, α, and θ are as defined before. β is the *concrete tensile stress factor* and is based on the ability of diagonally cracked concrete to resist tension, which also controls the angle of the diagonal tension crack θ. In Ref. 4.12, the values of β and θ are determined based on the strain in the longitudinal tension reinforcement, which can be approximated by†

$$\epsilon_s = \frac{|M_u|/d_v - 0.5N_u + |V_u|}{E_s A_s} \le 0.006 \tag{4.28}$$

The sign convention for N_u is the same as used in Section 4.6 and the ACI Code: compression is positive and tension is negative (the opposite sign convention is used in Ref. 4.12). M_u should not be taken less than $V_u d_v$; when calculating A_s, the area of bars terminated less than their development length (see Chapter 5) from the section

† Equation (4.28) is a simplification of $\epsilon_s = \dfrac{|M_u/d_v| - 0.5N_u + 0.5\,|V_u| \cot\theta}{E_s A_s}$, with $0.5|V_u| \cot\theta$ approximated by $|V_u|$. The simplification eliminates the need for an iterative solution between ϵ_s and θ.

under consideration should be reduced in proportion to the decreased development; ϵ_s should be taken as zero if the value calculated in Eq. (4.28) is negative; and ϵ_s should be doubled if N_u is high enough to cause cracking to the flexural compression face of the member. For sections closer than d_v to the face of the support, ϵ_s calculated at d_v from the face of the support may be used to determine β and θ.

For members with at least the minimum shear reinforcement, the concrete tensile stress factor is given by

$$\beta = \frac{4.8}{1+750\epsilon_s} \tag{4.29}$$

The angle θ, in degrees, is given by

$$\theta = 29 + 3500\epsilon_s \tag{4.30}$$

As shown in Eq. (4.21), the strength of the *longitudinal* reinforcement must be adequate to carry the additional forces induced by shear. Referring to Fig. 4.21, this leads to

$$A_s f_y \geq T = \frac{|M_u|}{\phi_f} - \frac{0.5N_u}{\phi_c} + \left(\frac{|V_u|}{\phi_v} - 0.5V_s\right)\cot\theta \tag{4.31}$$

where ϕ_f, ϕ_c, and ϕ_v are, respectively, the capacity reduction factors for flexure, axial load (tension or compression), and shear. V_s need not be taken greater than V_u/ϕ. Since the inclination of the compression struts changes, tension in the longitudinal reinforcement does not exceed that required to resist the maximum moment alone.

For members with less than the minimum transverse reinforcement, the angle θ is given by Eq. (4.30), while the value of β becomes a function of ϵ_s and a crack spacing parameter s_{xe}.

$$\beta = \frac{4.8}{1 + 750\epsilon_s} \quad \frac{51}{39 + s_{xe}} \tag{4.32}$$

The crack spacing parameter is

$$s_{xe} = s_x \frac{1.38}{a_g + 0.63} \tag{4.33}$$

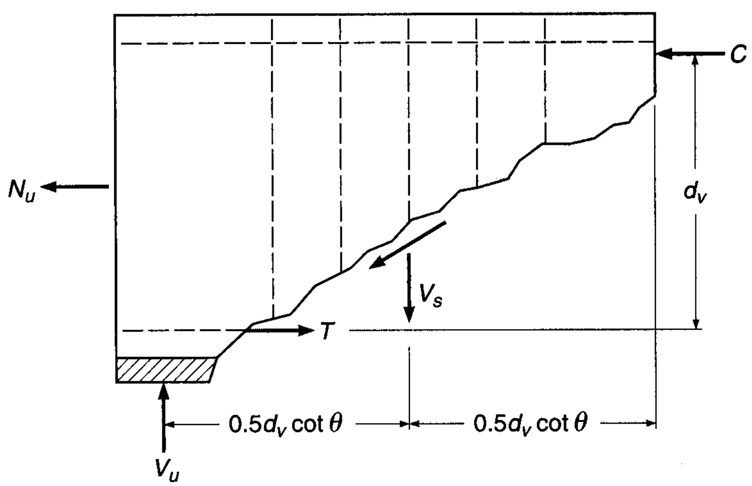


FIGURE 4.21
Equilibrium diagram for calculating tensile force in reinforcement. (*Adapted from Ref. 4.12.*)

where 12.0 in. $\leq s_{xe} \leq$ 80.0 in., s_x = lesser of the shear depth d_v or the spacing between layers of longitudinal crack control reinforcement, each layer with an area of steel of at least $0.003 b_v s_x$, and a_g = maximum size of the coarse aggregate. Note that $s_{xe} = s_x$ for $\frac{3}{4}$-in. coarse aggregate.

Since θ is not, in general, equal to 45°, the critical section might appropriately be taken as $d_v \cot \theta$ from the face of the support if all the load were applied to the upper surface of the member. For simplicity, however, the critical section is taken a distance d_v from the face of the support when the reaction introduces compression into the end region of the member, similar to the loading cases shown in Fig. 4.12*a* and *b*. For all other cases, the crucial section is taken at the face of the support, as shown in Fig. 4.12*c* to *f*.

AASHTO requires a minimum amount of transverse reinforcement $A_v = \sqrt{f'_c}\, b_v s / f_{yt}$ (compared to $0.75\sqrt{f'_c}\, b_w s/f_{yt}$ for ACI), when $V_u > 0.5\phi V_c$, and specifies maximum spacings of transverse reinforcement of $s \leq 0.8 d_v \leq 24$ in. when $v_u < 0.125 f'_c$ and $s \leq 0.4 d_v \leq 12$ in. when $v_u \geq 0.125 f'_c$. Because the predictions obtained with the MCFT are generally more accurate than those obtained with the ACI method, AASHTO allows the use of $\phi = 0.90$ for shear, the same as for flexure.

EXAMPLE 4.5

Design by modified compression field approach. Re-solve the problem given in Examples 4.2 and 4.3 based on the MCFT. Use ACI load factors and $\phi = 0.9$ for shear, as used in *AASHTO LRFD Bridge Design Specifications* (Ref. 4.12). Assume an aggregate size a_g of $\frac{3}{4}$ in.

Solution. For simplicity, the effective depth for shear d_v will be set at the minimum allowable value $= 0.9d = 0.9 \times 22 = 19.8$ in. Both M_u and V_u are as tabulated previously in Table 4.1.

The critical section for shear is located a distance d_v = 19.8 in. = 1.65 ft from the support where $V_u = 94 - 9.4 \times 1.65 = 78.5$ kips. Calculating $0.125 f'_c b_v d_v = 0.125 \times 4000 \times 16 \times 19.8 = 158{,}400$ lb leads to maximum spacing criteria for No. 3 (No. 10) stirrups equal to the smaller of $0.8 d_v = 0.8 \times 19.8 = 15.8$ in., 24 in., or

$$s_{\max} = \frac{A_v f_{yt}}{\sqrt{f'_c}\, b_v} = \frac{0.22 \times 60{,}000}{\sqrt{4000} \times 16} = 13.0 \text{ in.}$$

Using Eq. (4.28), the strain in the longitudinal tension steel is approximated as

$$\epsilon_s = \frac{|M_u|/19.8 + |V_u|}{29{,}000 \times 7.62}$$

with M_u and V_u in in-kips and kips, respectively.

The values of ϵ_s are tabulated along with M_u and V_u in Table 4.2. These values are used to calculate θ using Eq. (4.30) and β using Eqs. (4.29) and (4.32) for sections with and without minimum stirrups, respectively. Where the section meets the minimum stirrup criterion, the values of β are used to calculate the values of V_c, which are then used, along with the values of θ, to calculate V_s and the required stirrup spacing s (see Table 4.2).

For transverse reinforcement less than the minimum, the values of β are based on ϵ_s and s_x. The latter may be taken as the lesser of d_v or the spacing of longitudinal crack control reinforcement. In this case, d_v = 19.8 in. controls since crack control reinforcement is not used. The equivalent crack spacing parameter $s_{xe} = s_x$ because a_g = 0.75 in. These values of β are used to determine the point where $\phi V_c/2 \geq V_u$, the point at which stirrups may be terminated (Table 4.2). The values of V_u, ϕV_c with at least minimum stirrups, and $\phi V_c/2$ for less than minimum stirrups are plotted in Fig. 4.22*a*. The following stirrup spacings can be used for this case:

$$\begin{aligned} 1 \text{ space at 6 in.} &= 6 \text{ in.} \\ 6 \text{ spaces at 13 in.} &= \underline{78 \text{ in.}} \\ \text{Total} &= 84 \text{ in.} = 7 \text{ ft} \end{aligned}$$

TABLE 4.2
Modified compression field design example using ϕ = 0.9 for shear

Distance from Support, ft	M_u, ft-kips	V_u, kips	$\epsilon_s \times 1000$	θ	ϕV_c for at Least Minimum Stirrups: β	ϕV_c, kips	V_s, kips	s, in.	ϕV_c for Less Than Minimum Stirrups: β	ϕV_c, kips	$\phi V_c/2$, kips
0	0	94.0	0.85	32.0	2.93	52.8	45.7	9.2	2.54	45.8	22.9
1	89	84.6	0.77	31.7	3.05	55.0	32.9	12.9	2.64	47.7	23.8
1.65†	144	78.5	0.75	31.6	3.07	55.4	25.6	16.5	2.66	48.1	24.0
2	169	75.2	0.80	31.8	2.99	54.0	23.6	17.9	2.60	46.8	23.4
3	240	65.8	0.96	32.3	2.80	50.4	17.1	24.2	2.43	43.7	21.9
4	301	56.4	1.08	32.8	2.65	47.8	9.5	42.6	2.30	41.5	20.7
5	353	47.0	1.18	33.1	2.55	45.9	1.2	336	2.21	39.8	19.9
6	395	37.6	1.25	33.4	2.47	44.6	—	—	2.15	38.7	19.4
7	428	28.2	1.30	33.6	2.43	43.8	—	—	2.11	38.0	19.0
8	451	18.8	1.32	33.6	2.41	43.5	—	—	2.09	37.7	18.8
9	465	9.4	1.32	33.6	2.41	43.5	—	—	2.09	37.7	18.9
10	470	0.0	1.29	33.5	2.44	44.0	—	—	2.12	38.2	19.1

† d_v from face of support.

For this example, V_s is selected based on V_u at each point, not the minimum V_u on a crack with angle θ. This simplifies the design procedure and results in a somewhat more conservative design. Even so, only 7 No. 3 (No. 10) stirrups are needed, or 9 stirrups if the stirrups are continued at the maximum spacing through the middle region of the beam. These values compare favorably with the minimum number of stirrups per half-span, 11 and 14, previously

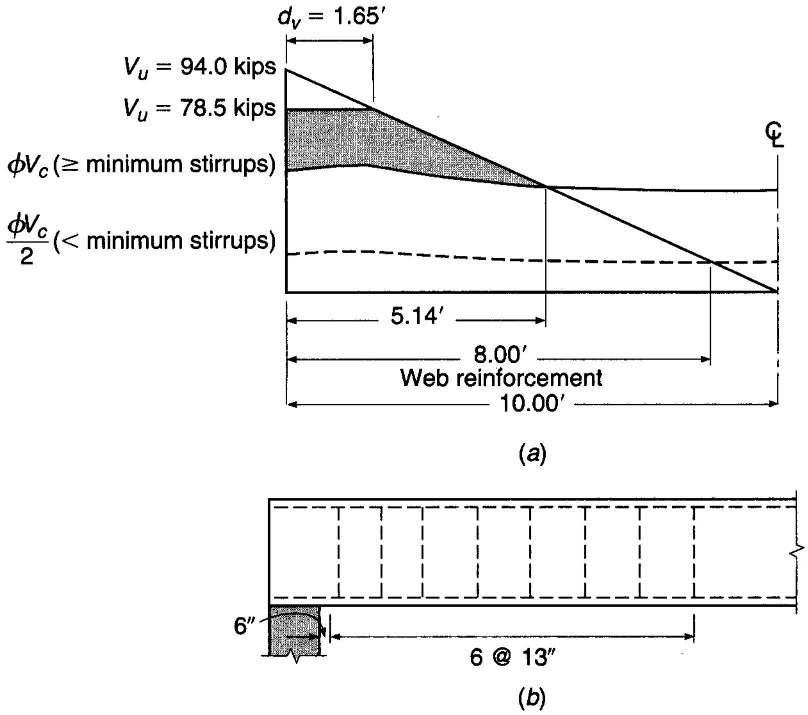


FIGURE 4.22
Modified compression field design for Example 4.5.

calculated (Example 4.3) using the two methods required by the ACI Code. The resulting stirrup pattern is shown in Fig. 4.22*b*.

By way of comparison, had $\phi_{shear} = 0.75$ been used in this example, the stirrup spacing would have been

$$\begin{aligned} 1 \text{ space at } 5 \text{ in.} &= 5 \text{ in.} \\ 3 \text{ spaces at } 10 \text{ in.} &= 30 \text{ in.} \\ 4 \text{ spaces at } 13 \text{ in.} &= \underline{52 \text{ in.}} \\ \text{Total} &= 87 \text{ in.} = 7 \text{ ft } 3 \text{ in.} \end{aligned}$$

for a total of 8 stirrups.

The MCFT recognizes that shear increases the force in the flexural steel, although, as explained earlier, the maximum tensile force in the steel is not affected. Equation (4.31) should be used to calculate the tensile force T along the span, which will then govern the locations where tensile steel may be terminated. This will be discussed further in Chapter 5.

The MCFT is not included in the 2008 ACI Code. ACI Code 1.4, however, permits the use of "any system of design or construction . . . , the adequacy of which has been shown by successful use or by analysis or test," if approved by the appropriate building official. The application of the MCFT in Canada and in U.S. bridge practice provides the evidence needed to demonstrate "successful use."

4.9 SHEAR-FRICTION DESIGN METHOD

Generally, in reinforced concrete design, shear is used merely as a convenient measure of diagonal tension, which is the real concern. In contrast, there are circumstances such that direct shear may cause failure of reinforced concrete members. Such situations occur commonly in precast concrete structures, particularly in the vicinity of connections, as well as in composite construction combining cast-in-place concrete with either precast concrete or structural steel elements. Potential failure planes can be established for such cases along which direct shear stresses are high, and failure to provide adequate reinforcement across such planes may produce disastrous results.

The necessary reinforcement may be determined on the basis of the *shear-friction method* of design (Refs. 4.32 to 4.38). The basic approach is to assume that the concrete may crack in an unfavorable manner, or that slip may occur along a predetermined plane of weakness. Reinforcement must be provided crossing the potential or actual crack or shear plane to prevent direct shear failure.

The shear-friction theory is very simple, and the behavior is easily visualized. Figure 4.23*a* shows a cracked block of concrete, with the crack crossed by reinforcement. A shear force V_n acts parallel to the crack, and the resulting tendency for the upper block to slip relative to the lower is resisted largely by friction along the concrete interface at the crack. Since the crack surface is naturally rough and irregular, the effective coefficient of friction may be quite high. In addition, the irregular surface will cause the two blocks of concrete to separate slightly, as shown in Fig. 4.23*b*.

If reinforcement is present normal to the crack, then slippage and subsequent separation of the concrete will stress the steel in tension. Tests have confirmed that well-anchored steel will be stressed to its yield strength when shear failure is obtained (Ref. 4.34). The resulting tensile force sets up an equal and opposite pressure between the concrete faces on either side of the crack. It is clear from the free body of Fig. 4.23*c* that the maximum value of this interface pressure is $A_{vf}f_y$, where A_{vf} is the total area of steel crossing the crack and f_y is its yield strength.

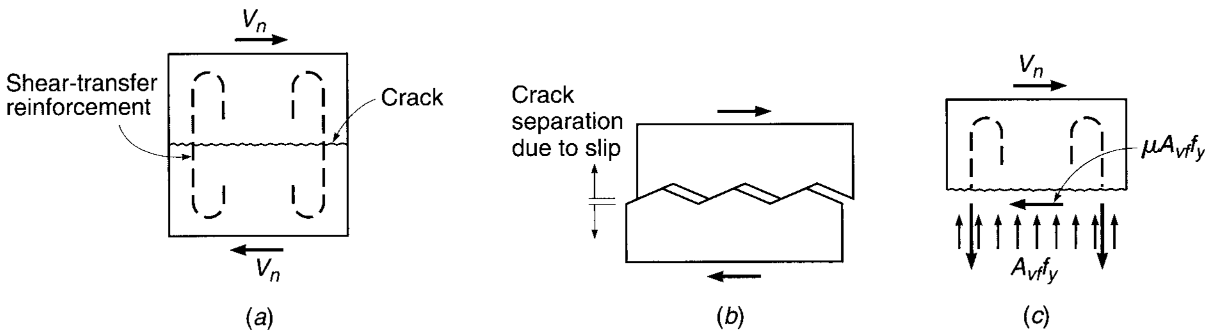


FIGURE 4.23
Basis of shear-friction design method: (*a*) applied shear; (*b*) enlarged representation of crack surface; (*c*) free-body sketch of concrete above crack.

The concrete resistance to sliding may be expressed in terms of the normal force times a coefficient of friction μ. By setting the summation of horizontal forces equal to zero

$$V_n = \mu A_{vf} f_y \tag{4.34}$$

Based on tests, μ may be taken as 1.4 for cracks in monolithic concrete, but V_n should not be assumed to be greater than $0.2f'_c A_c$, $(480 + 0.08f'_c)A_c$, or $1600A_c$ (Refs. 4.32, 4.37, and 4.38).

The relative movement of the concrete on opposite sides of the crack also subjects the individual reinforcing bars to shearing action, and the dowel resistance of the bars to this shearing action contributes to shear resistance. However, it is customary to neglect the dowel effect for simplicity in design and to compensate for this by using an artificially high value of the friction coefficient.

The provisions of ACI Code 11.6 are based on Eq. (4.34). The design strength is equal to ϕV_n, where $\phi = 0.75$ for shear-friction design, and V_n must not exceed the smallest of $0.2f'_c A_c$, $(480 + 0.08f'_c)A_c$, and $1600A_c$ for monolithic or intentionally roughened normalweight concrete or the smaller of $0.2f'_c A_c$ and $800A_c$ lb for other cases. When concretes of different strengths are cast against each another, V_n should be based on the lower value of f'_c. Recommendations for friction factor μ are as follows:

Concrete placed monolithically	1.4λ
Concrete placed against hardened concrete with surface intentionally roughened	1.0λ
Concrete placed against hardened concrete not intentionally roughened	0.6λ
Concrete anchored to as-rolled structural steel by headed studs or reinforcing bars	0.7λ

where λ is 1.0 for normalweight concrete and 0.75 for both sand-lightweight and all-lightweight concrete. In other cases, λ is determined based on volumetric proportions of lightweight and normalweight aggregates, as described in Section 4.5a and specified in ACI Code 8.6.1, but not greater than 0.85. The yield strength of the reinforcement f_y may not exceed 60,000 psi. Direct tension across the shear plane, if present, must be carried by additional reinforcement, and permanent net compression across the shear plane may be taken as additive to the force in the shear-friction reinforcement $A_{vf} f_y$ when calculating the required A_{vf}.

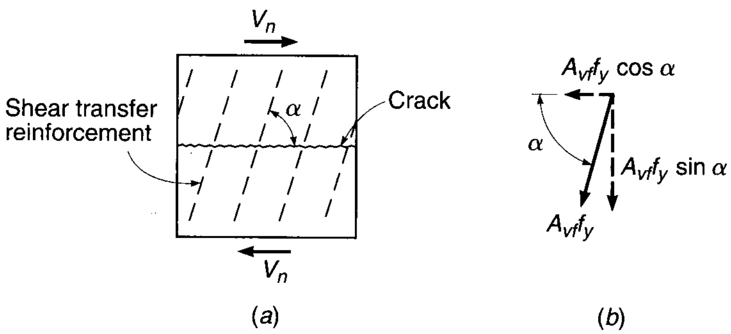


FIGURE 4.24
Shear-friction reinforcement inclined with respect to crack face.

When shear is transferred between concrete newly placed against hardened concrete, the surface roughness is an important variable; an intentionally roughened surface is defined to have a full amplitude of approximately $\frac{1}{4}$ in. In any case, the old surface must be clean and free of laitance. When shear is to be transferred between as-rolled steel and concrete, the steel must be clean and without paint, according to ACI Code 11.6.

If V_u is the shear force to be resisted at factored loads, then with $V_u = \phi V_n$, the required steel area is found by transposition of Eq. (4.34):

$$A_{vf} = \frac{V_u}{\phi \mu f_y} \tag{4.35}$$

In some cases, the shear-friction reinforcement may not cross the shear plane at 90° as described in the preceding paragraphs. If the shear-friction reinforcement is inclined to the shear plane so that the shear force is applied in the direction to increase tension in the steel, as in Fig. 4.24*a*, then the component of that tension parallel to the shear plane, shown in Fig. 4.24*b*, contributes to the resistance to slip. Then the shear strength may be computed from

$$V_n = A_{vf} f_y(\mu \sin\alpha + \cos\alpha) \tag{4.36}$$

in lieu of Eq. (4.34). Here α is the angle between the shear-friction reinforcement and the shear plane. If α is larger than 90°, i.e., if the inclination of the steel is such that the tension in the bars tends to be reduced by the shear force, then the assumption that the steel stress equals f_y is not valid, and a better arrangement of bars should be made.

Certain precautions should be observed in applying the shear-friction method of design. Reinforcement, of whatever type, should be well anchored to develop the yield strength of the steel, by the full development length or by hooks or bends, in the case of reinforcing bars, or by proper heads and welding, in the case of studs joining concrete to structural steel. The concrete should be well confined, and the liberal use of hoops has been recommended (Ref. 4.32). Care must be taken to consider all possible failure planes and to provide sufficient well-anchored steel across these planes.

EXAMPLE 4.6

Design of beam bearing detail. A precast beam must be designed to resist a support reaction, at factored loads, of $V_u = 100$ kips applied to a 3 × 3 steel angle, as shown in Fig. 4.25. In lieu of a calculated value, a horizontal force N_u, owing to restrained volume change, will be assumed to be 20 percent of the vertical reaction, or 20 kips. Determine the required auxiliary reinforcement, using steel of yield strength $f_y = 60{,}000$ psi. Concrete strength $f'_c = 5000$ psi.

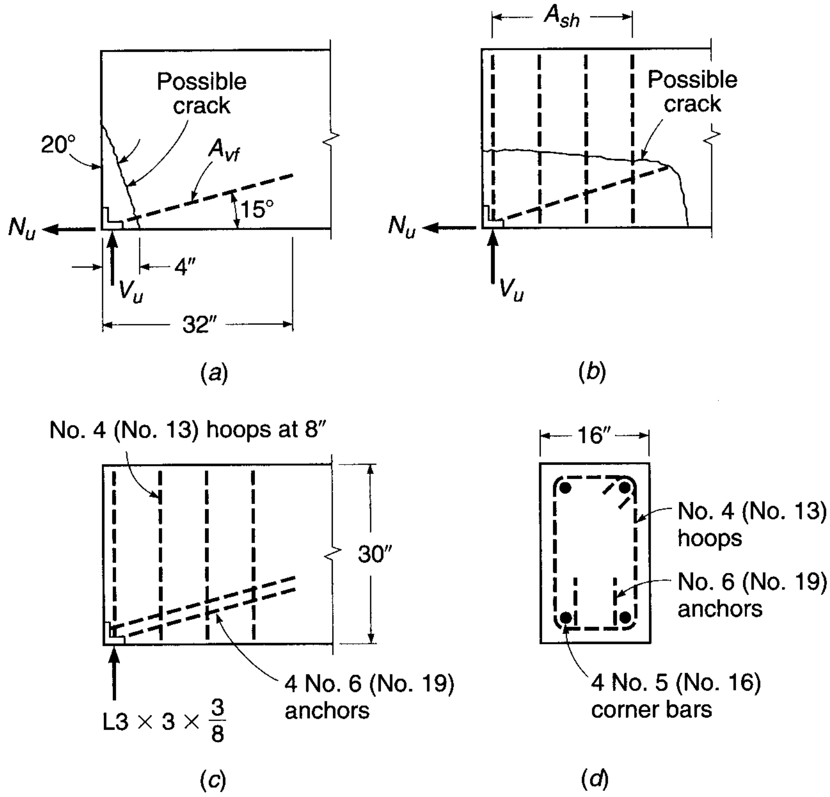


FIGURE 4.25
Design of beam bearing shoe: (*a*) diagonal crack; (*b*) horizontal crack; (*c*) reinforcement; (*d*) cross section.

Solution. A potential crack will be assumed at 20°, initiating at a point 4 in. from the end of the beam, as shown in Fig. 4.25*a*. The total required steel A_{vf} is the sum of that required to resist the resultant of V_u and N_u acting parallel to the cracks = $V_u \cos 20° + N_u \sin 20°$. Equation (4.35) is modified accordingly:

$$A_{vf} = \frac{V_u \cos 20° + N_u \sin 20°}{\phi \mu f_y}$$

$$= \frac{100 \times 0.940 + 20 \times 0.340}{0.75 \times 1.4 \times 60} = \frac{101 \text{ kips}}{63 \text{ ksi}}$$

$$= 1.60 \text{ in}^2$$

The net compression normal to the potential crack would be no less than $V_u \sin 20° - N_u \cos 20° = 15.4$ kips. This could be counted upon to reduce the required shear-friction steel, according to the ACI Code, but it will be discounted conservatively here. Four No. 6 (No. 19) bars will be used, providing an area of 1.76 in^2. They will be welded to the 3 × 3 angle and will extend into the beam a sufficient distance to develop the yield strength of the bars. According to the ACI Code, the development length for a No. 6 (No. 19) bar is 26 in., 32 in. without the ψ_s factor (see Chapter 5). Considering the uncertainty of the exact crack location, the bars will be extended 32 in. into the beam as shown in Fig. 4.25*a*. The bars will be placed at an angle of 15° with the bottom face of the member. For the crack oriented at an angle of 20°, as assumed, the area of the crack is

$$A_c = 16\left(\frac{4}{\sin 20°}\right) = 187 \text{ in}^2$$

Thus, according to the ACI Code, the maximum nominal shear strength of the surface is not to exceed $V_n = 0.2f'_cA_c = 187$ kips, $V_n = (480 + 0.08f'_c)A_c = 165$ kips, or $V_n = 1600A_c = 299$ kips. The maximum design strength to be used is $\phi V_n = 0.75 \times 165 = 124$ kips. As calculated earlier, the applied shear on the interface at factored loads is

$$V_u = 100 \cos 20° + 20 \sin 20° = 101 \text{ kips}$$

and so the design is judged satisfactory to this point.

A second possible crack must be considered, as shown in Fig. 4.25*b*, resulting from the tendency of the entire anchorage weldment to pull horizontally out of the beam.

The required steel area A_{sh} and the concrete shear stress will be calculated based on the development of the full yield tension in the bars A_{vf}. (Note that the factor ϕ need not be used here because it has already been introduced in computing A_{vf}.)

$$A_{sh} = \frac{A_{vf}f_y \cos 15°}{\mu f_y} = \frac{1.76 \times 0.966}{1.4} = 1.21 \text{ in}^2$$

Four No. 4 (No. 13) hoops will be used, providing an area of 1.60 in^2.

The maximum shear force that can be transferred, according to the ACI Code limits, will be based conservatively on a horizontal plane 32 in. long. No strength reduction factor need be included in the calculation of this maximum value because it was already introduced in determining the steel area A_{vf} by which the shear force is applied. Accordingly,

$$V_n \le (480 + 0.08f'_c) \times 16 \times 32 = 451 \text{ kips}$$

The maximum shear force that could be applied in the given instance is the value used to calculate A_{sh},

$$V_u = 1.76 \times 60 \cos 15° = 102 \text{ kips}$$

which is well below the specified maximum.

The first hoop will be placed 2 in. from the end of the member, with the others spaced at 8 in., as shown in Fig. 4.25*c*. Also shown in Fig. 4.25*d* are four No. 5 (No. 16) bars that will provide anchorage for the hoop steel.

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PROBLEMS

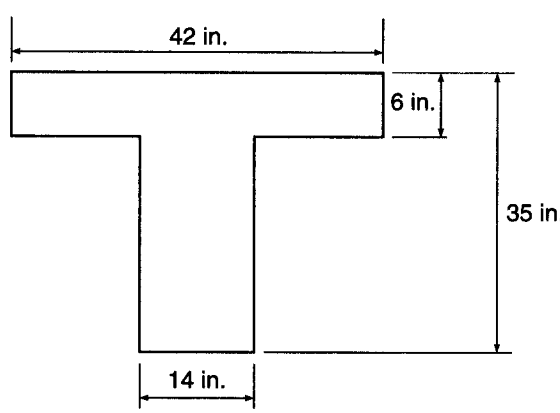
4.1. A beam is to be designed for loads causing a maximum factored shear of 60.0 kips, using concrete with $f_c' = 5000$ psi. Proceeding on the basis that the concrete dimensions will be determined by diagonal tension, select the appropriate width and effective depth (*a*) for a beam in which no web reinforcement is to be used, (*b*) for a beam in which only the minimum web reinforcement is provided, as given by Eq. (4.13), and (*c*) for a beam in which web reinforcement provides shear strength $V_s = 2V_c$. Follow the ACI Code requirements, and let $d = 2b$ in each case. Calculations may be based on the more approximate value of V_c given by Eq. (4.12*b*).

4.2. A rectangular beam having $b = 10$ in. and $d = 17.5$ in. spans 15 ft face to face of simple supports. It is reinforced for flexure with three No. 9 (No. 29) bars that continue uninterrupted to the ends of the span. It is to carry service dead load $D = 1.27$ kips/ft (including self-weight) and service live load $L = 3.70$ kips/ft, both uniformly distributed along the span. Design the shear reinforcement, using No. 3 (No. 10) vertical U stirrups. The more approximate Eq. (4.12*b*) for V_c may be used. Material strengths are $f_c' = 4000$ psi and $f_y = 60{,}000$ psi.

4.3. Redesign the shear reinforcement for the beam of Problem 4.2, basing V_c on the more accurate Eq. (4.12*a*). Comment on your results, with respect to design time and probable construction cost difference.

4.4. Design the shear reinforcement, using No. 4 (No. 13) vertical U stirrups for the independent T beam shown in Fig. P4.4. The beam spans 24 ft face to face between simple supports, has an effective depth $d = 31$ in., and is reinforced for flexure with six No. 10 (No. 32) bars in two layers that continue uninterrupted to the ends of the span. It is to carry service dead load $D = 2.67$ kips/ft (including self-weight) and service live load $L = 5.36$ kips/ft, both uniformly distributed along the span. The more approximate Eq. (4.12*b*) for V_c may be used. Material strengths are $f_c' = 5000$ psi and $f_y = 60{,}000$ psi.

FIGURE P4.4



4.5. A beam of 11 in. width and effective depth of 16 in. carries a factored uniformly distributed load of 5.3 kips/ft, including its own weight, in addition to a central, concentrated factored load of 12 kips. It spans 18 ft, and restraining end moments at full factored load are 137 ft-kips at each support. It is reinforced with three No. 9 (No. 29) bars for both positive and negative bending. If $f_c' =$ 4000 psi, through what part of the beam is web reinforcement theoretically required (*a*) if Eq. (4.12*b*) is used and (*b*) if Eq. (4.12*a*) is used? Comment.

4.6. What effect would an additional clockwise moment of 176 ft-kips at the right support have on the requirement for shear reinforcement determined in part (*a*) of Problem 4.5?

4.7. Design the web reinforcement for the beam of Problem 4.5, with V_c determined by the more approximate ACI equation, using No. 3 (No. 10) vertical stirrups with $f_y = 60{,}000$ psi.

4.8. Design the web reinforcement for the beam of Problem 4.6, with V_c determined by the more approximate ACI equation, using No. 3 (No. 10) vertical stirrups with $f_y = 60{,}000$ psi.

4.9. The beam of Problem 4.2 will be subjected to a factored axial compression load of 88 kips on the 10×20 in. gross cross section, in addition to the loads described earlier. What is the effect on concrete shear strength V_c (*a*) by the more accurate ACI equation and (*b*) by the more approximate ACI equation?

4.10. The beam of Problem 4.2 will be subjected to a factored axial tension load of 44 kips on the 10×20 in. gross cross section, in addition to the loads described earlier. What is the effect on concrete shear strength V_c (*a*) by the more accurate ACI equation and (*b*) by the more conservative ACI approach?

4.11. Redesign the shear reinforcement for the beam of Problem 4.2, using the modified compression field theory with (*a*) $\phi_{\text{shear}} = 0.90$ and (*b*) $\phi_{\text{shear}} = 0.75$.

4.12. Redesign the shear reinforcement for the beam of Problem 4.4, using the modified compression field theory with (*a*) $\phi_{\text{shear}} = 0.90$ and (*b*) $\phi_{\text{shear}} = 0.75$.

4.13. A precast concrete beam having cross-sectional dimensions $b = 10$ in. and $h = 24$ in. is designed to act in a composite sense with a cast-in-place top slab having depth $h_f = 5$ in. and width 48 in. At factored loads, the maximum compressive stress in the flange at midspan is 2400 psi; at the supports of the 28 ft simple span the flange force must be zero. Vertical U stirrups provided for flexural shear will be extended into the slab and suitably anchored to provide also for transfer of the flange force by shear friction. Find the minimum number of No. 4 (No. 13) stirrups that must be provided, based on shear-friction requirements. Concrete in both precast and cast-in-place parts will have $f_c' = 4000$ psi and $f_y = 60{,}000$ psi. The top surface of the precast web will be intentionally roughened according to the ACI Code definition.

4.14. Redesign the beam-end reinforcement of Example 4.6, given that a roller support will be provided so that $N_u = 0$.

5 Bond, Anchorage, and Development Length

5.1 FUNDAMENTALS OF FLEXURAL BOND

If the reinforced concrete beam of Fig. 5.1*a* were constructed using plain round reinforcing bars, and, furthermore, if those bars were to be greased or otherwise lubricated before the concrete were cast, the beam would be very little stronger than if it were built of plain concrete, without reinforcement. If a load were applied, as shown in Fig. 5.1*b*, the bars would tend to maintain their original length as the beam deflected. The bars would slip longitudinally with respect to the adjacent concrete, which would experience tensile strain due to flexure. Proposition 2 of Section 1.8, the assumption that the strain in an embedded reinforcing bar is the same as that in the surrounding concrete, would not be valid. For reinforced concrete to behave as intended, it is essential that *bond forces* be developed on the interface between concrete and steel, such as to prevent significant slip from occurring at that interface.

Figure 5.1*c* shows the bond forces that act on the concrete at the interface as a result of bending, while Fig. 5.1*d* shows the equal and opposite bond forces acting on the reinforcement. It is through the action of these interface bond forces that the slip indicated in Fig. 5.1*b* is prevented.

Some years ago, when plain bars without surface deformations were used, initial bond strength was provided only by the relatively weak chemical adhesion and mechanical friction between steel and concrete. Once adhesion and static friction were overcome at larger loads, small amounts of slip led to interlocking of the natural roughness of the bar with the concrete. However, this natural bond strength is so low that in beams reinforced with plain bars, the bond between steel and concrete was frequently broken. Such a beam will collapse as the bar is pulled through the concrete. To prevent this, end anchorage was provided, chiefly in the form of hooks, as in Fig. 5.2. If the anchorage is adequate, such a beam will not collapse, even if the bond is broken over the entire length between anchorages. This is so because the member acts as a tied arch, as shown in Fig. 5.2, with the uncracked concrete shown shaded representing the arch and the anchored bars the tie-rod. In this case, over the length in which the bond is broken, bond forces are zero. This means that over the entire unbonded length the force in the steel is constant and equal to $T = M_{max}/jd$. As a consequence, the total steel elongation in such beams is larger than in beams in which bond is preserved, resulting in larger deflections and greater crack widths.

To improve this situation, deformed bars are now universally used in the United States and many other countries (see Section 2.14). With such bars, the shoulders of the projecting deformations bear on the surrounding concrete and result in greatly increased bond strength. It is then possible in most cases to dispense with special anchorage devices such as hooks. In addition, crack widths as well as deflections are reduced.

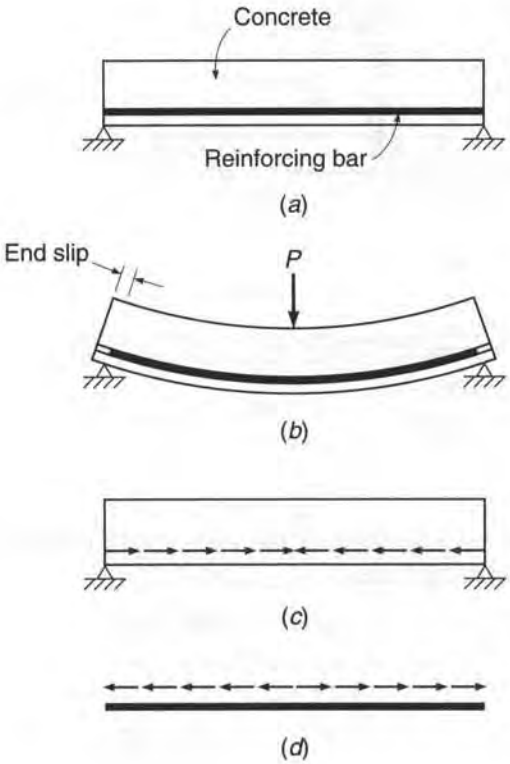


FIGURE 5.1
Bond forces due to flexure: (*a*) beam before loading; (*b*) unrestrained slip between concrete and steel; (*c*) bond forces acting on concrete; (*d*) bond forces acting on steel.

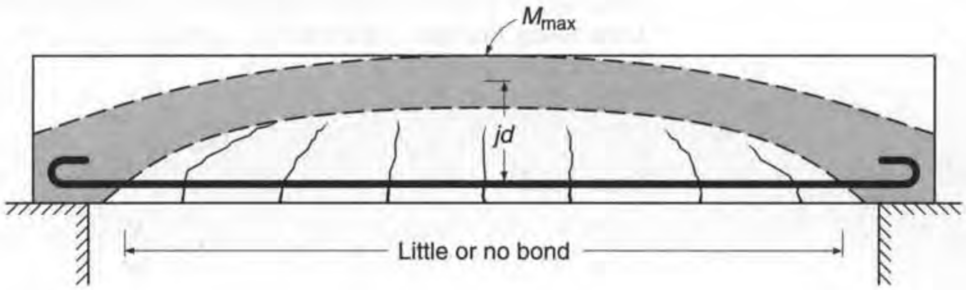


FIGURE 5.2
Tied-arch action in a beam with little or no bond.

a. Bond Force Based on Simple Cracked Section Analysis

In a short piece of a beam of length dx, such as shown in Fig. 5.3*a*, the moment at one end will generally differ from that at the other end by a small amount dM. If this piece is isolated, and if one assumes that, after cracking, the concrete does not resist any tension stresses, the internal forces are those shown in Fig. 5.3*a*. The change in bending moment dM produces a change in the bar force

$$dT = \frac{dM}{jd} \qquad (a)$$

where jd is the internal lever arm between tensile and compressive force resultants. Since the bar or bars must be in equilibrium, this change in bar force is resisted at the contact surface between steel and concrete by an equal and opposite force produced by bond, as indicated by Fig. 5.3*b*.

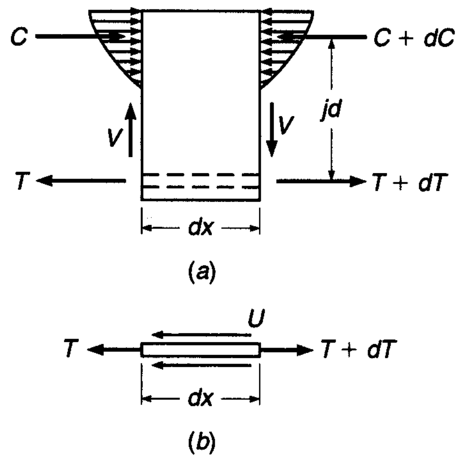


FIGURE 5.3
Forces acting on elemental length of beam: (*a*) free-body sketch of reinforced concrete element; (*b*) free-body sketch of steel element.

If U is the magnitude of the local bond force per unit length of bar, then, by summing horizontal forces

$$U\,dx = dT \qquad (b)$$

Thus

$$U = \frac{dT}{dx} \qquad (5.1)$$

indicating that the local unit bond force is proportional to the rate of change of bar force along the span. Alternatively, substituting Eq. (*a*) in Eq. (5.1), the unit bond force can be written as

$$U = \frac{1}{jd}\frac{dM}{dx} \qquad (c)$$

from which

$$U = \frac{V}{jd} \qquad (5.2)$$

Equation (5.2) is the "elastic cracked section equation" for flexural bond force, and it indicates that the bond force per unit length is proportional to the shear at a particular section, i.e., to the rate of change of bending moment.

Note that Eq. (5.2) applies to the *tension* bars in a concrete zone that is assumed to be fully cracked, with the concrete resisting no tension. It applies, therefore, to the tensile bars in simple spans, or, in continuous spans, either to the bottom bars in the positive bending region between inflection points or to the top bars in the negative bending region between the inflection points and the supports. It does not apply to compression reinforcement, for which it can be shown that the flexural bond forces are very low.

b. Actual Distribution of Flexural Bond Force

The actual distribution of bond force along deformed reinforcing bars is much more complex than that represented by Eq. (5.2), and Eq. (5.1) provides a better basis for understanding beam behavior. Figure 5.4 shows a beam segment subject to pure

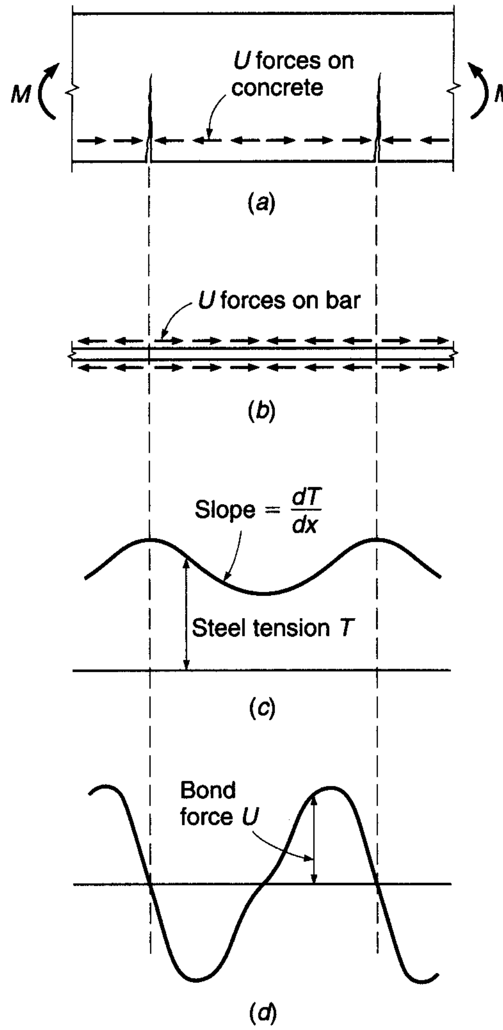


FIGURE 5.4
Variation of steel and bond forces in a reinforced concrete member subject to pure bending: (*a*) cracked concrete segment; (*b*) bond forces acting on reinforcing bar; (*c*) variation of tensile force in steel; (*d*) variation of bond force along steel.

bending. The concrete fails to resist tensile stresses only where the actual crack is located; there the steel tension is maximum and has the value predicted by simple theory: $T = M/jd$. Between cracks, the concrete *does* resist moderate amounts of tension, introduced by bond forces acting along the interface in the direction shown in Fig. 5.4*a*. This reduces the tensile force in the steel, as illustrated by Fig. 5.4*c*. From Eq. (5.1), it is clear that U is proportional to the rate of change of bar force, and thus will vary as shown in Fig. 5.4*d*; unit bond forces are highest where the slope of the steel force curve is greatest and are zero where the slope is zero. Very high local bond forces adjacent to cracks have been measured in tests (Refs. 5.1 and 5.2). They are so high that inevitably some slip occurs between concrete and steel adjacent to each crack.

Beams are seldom subject to pure bending moment; they generally carry transverse loads producing shear and moment that vary along the span. Figure 5.5*a* shows a beam carrying a distributed load. The cracking indicated is typical. The steel force T predicted by simple cracked section analysis is proportional to the moment diagram and is as shown by the dashed line in Fig. 5.5*b*. However, the actual value of

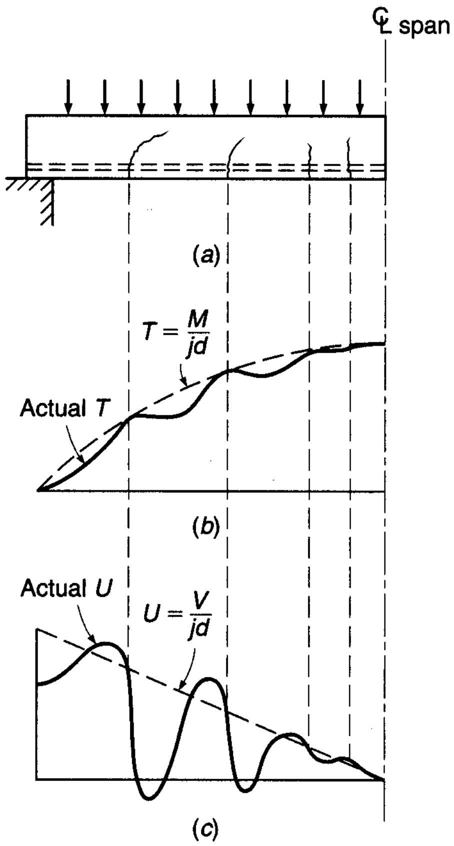


FIGURE 5.5
Effect of flexural cracks on bond forces in beam: (*a*) beam with flexural cracks; (*b*) variation of tensile force T in steel along span; (*c*) variation of bond force per unit length U along span.

T is less than that predicted by the simple analysis everywhere except at the actual crack locations. The actual variation of T is shown by the solid line of Fig. 5.5*b*. In Fig. 5.5*c*, the bond forces predicted by the simplified theory are shown by the dashed line, and the actual variation is shown by the solid line. Note that the value of U is equal to that given by Eq. (5.2) only at those locations where the slope of the steel force diagram equals that of the simple theory. Elsewhere, if the slope is greater than assumed, the local bond force is greater; if the slope is less, local bond force is less. Just to the left of the cracks, for the present example, U is much higher than predicted by Eq. (5.2), and in all probability will result in local bond failure. Just to the right of the cracks, U is much lower than predicted and in fact is generally negative very close to the crack; i.e., the bond forces act in the reverse direction.

It is evident that actual bond forces in beams bear very little relation to those predicted by Eq. (5.2), except in the general sense that they are highest in the regions of high shear.

5.2 BOND STRENGTH AND DEVELOPMENT LENGTH

For reinforcing bars in tension, two types of bond failure have been observed. The first is *direct pullout* of the bar, which occurs when ample confinement is provided by the surrounding concrete. This could be expected when relatively small-diameter bars are used with sufficiently large concrete cover distances and bar spacing. The second type of failure is *splitting* of the concrete along the bar when cover, confinement, or bar spacing is insufficient to resist the lateral concrete tension resulting from the wedging effect of the bar deformations. Present-day design methods require that both possible failure modes be accounted for.

a. Bond Strength

If the bar is sufficiently confined by a mass of surrounding concrete, then as the tensile force on the bar is increased, adhesive bond and friction are overcome, the concrete eventually crushes locally ahead of the bar deformations, and bar pullout results. The surrounding concrete remains intact, except for the crushing that takes place ahead of the ribs immediately adjacent to the bar interface. For modern deformed bars, adhesion and friction are much less important than the mechanical interlock of the deformations with the surrounding concrete.

Bond failure resulting from splitting of the concrete is more common in beams than direct pullout. Such splitting comes mainly from wedging action when the ribs of the deformed bars bear against the concrete (Refs. 5.3 and 5.4). It may occur either in a vertical plane as in Fig. 5.6*a* or horizontally in the plane of the bars as in Fig. 5.6*b*. The horizontal type of splitting of Fig 5.6*b* frequently begins at a diagonal crack. In this case, as discussed in connection with Fig. 4.7*b* and shown in Fig. 4.1, dowel action increases the tendency toward splitting. This indicates that shear and bond failures are often intricately interrelated.

When pullout resistance is overcome or when splitting has spread all the way to the end of an unanchored bar, complete bond failure occurs. Sliding of the steel relative to the concrete leads to immediate collapse of the beam.

If one considers the large local variations of bond force caused by flexural and diagonal cracks (see Figs. 5.4 and 5.5), it becomes clear that local bond failures immediately adjacent to cracks will often occur at loads considerably below the failure load of the beam. These local failures result in small local slips and some widening of cracks and increase of deflections, but will be harmless as long as failure does not propagate all along the bar, with resultant total slip. In fact, as discussed in connection with Fig. 5.2, when end anchorage is reliable, bond can be severed along the entire length of the bar, excluding the anchorages, without endangering the carrying capacity of the beam. End anchorage can be provided by hooks as suggested by Fig. 5.2 or,

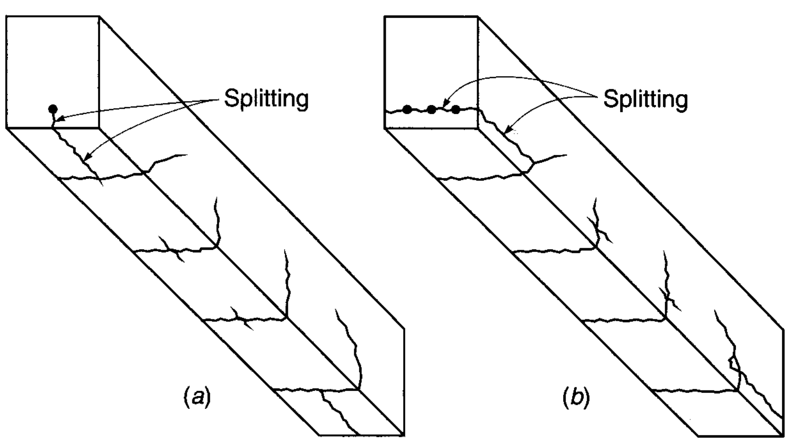
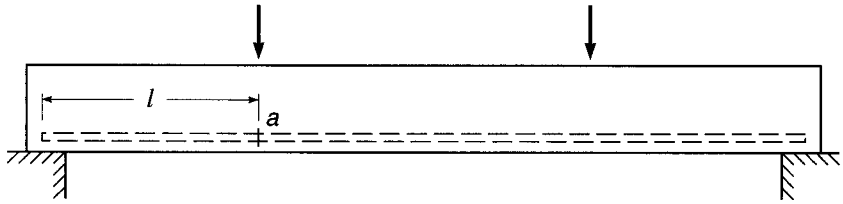


FIGURE 5.6
Splitting of concrete along reinforcement.

FIGURE 5.7
Development length.



much more commonly, by extending the straight bar a sufficient distance from the point of maximum stress.

Extensive testing (Refs. 5.5 to 5.11), using beam specimens, has established limiting values of bond strength. This testing provides the basis for current design requirements.

b. Development Length

The preceding discussion suggests the concept of *development length* of a reinforcing bar. The development length is defined as that length of embedment necessary to develop the full tensile strength of the bar, controlled by either pullout or splitting. With reference to Fig. 5.7, the moment, and therefore the steel stress, is evidentally maximum at point a (neglecting the weight of the beam) and zero at the supports. If the bar stress is f_s at a, then the total tension force $A_b f_s$ must be transferred from the bar to the concrete in the distance l by bond forces. To fully develop the strength of the bar $A_b f_y$, the distance l must be at least equal to the development length of the bar, established by tests. In the beam of Fig. 5.7, if the actual length l is equal to or greater than the development length l_d, no premature bond failure will occur. That is, the beam will fail in bending or shear rather than by bond failure. This will be so even if in the vicinity of cracks local slip may have occurred over small regions along the beam.

It is seen that the main requirement for safety against bond failure is this: the length of the bar, from any point of given steel stress (f_s or at most f_y) to its nearby free end, must be at least equal to its development length. If this requirement is satisfied, the magnitude of the nominal flexural bond force along the beam, as given by Eq. (5.2), is of only secondary importance, since the integrity of the member is ensured even in the face of possible minor local bond failures. However, if the actual available length is inadequate for full development, special anchorage, such as by hooks, must be provided.

c. Factors Influencing Development Length

Experimental research has identified the factors that influence development length, and analysis of the test data has resulted in the empirical equations used in present design practice. The most basic factors will be clear from review of the preceding paragraphs and include concrete tensile strength, cover distance, spacing of the reinforcing bars, and the presence of transverse steel reinforcement.

Clearly, the *tensile strength* of the concrete is important because the most common type of bond failure in beams is the type of splitting shown in Fig. 5.6. Although tensile strength does not appear explicitly in experimentally derived equations for development length (see Section 5.3), the term $\sqrt{f_c'}$ appears in the denominator of those equations and reflects the influence of concrete tensile strength.

As discussed in Section 2.9, the fracture energy of concrete plays an important role in bond failure because a splitting crack must propagate after it has formed. Since fracture energy is largely independent of compressive strength, bond strength increases more slowly than $\sqrt{f_c'}$, and as data for higher-strength concretes have become available, $f_c'^{1/4}$ has been shown to provide a better representation of the effect of concrete strength on bond than $\sqrt{f_c'}$ (Refs. 5.12 to 5.14). This point is recognized by ACI Committee 408, Bond and Development of Reinforcement (Ref. 5.15), in proposed design expressions based on $f_c'^{1/4}$ and within the ACI Code, which sets an upper limit on the value of $\sqrt{f_c'}$ for use in design.

For lightweight concretes, the tensile strength is usually less than for normal-density concrete having the same compressive strength; accordingly, if lightweight concrete is used, development lengths must be increased. Alternatively, if split-cylinder strength is known or specified for lightweight concrete, it can be incorporated in development length equations as follows. For normal concrete, the split-cylinder tensile strength f_{ct} is generally taken as $f_{ct} = 6.7\sqrt{f_c'}$. If the split-cylinder strength f_{ct} is known for a particular lightweight concrete, then $\sqrt{f_c'}$ in the development length equations can be replaced by $f_{ct}/6.7$.

Cover distance—conventionally measured from the *center* of the bar to the nearest concrete face and measured either in the plane of the bars or perpendicular to that plane—also influences splitting. Clearly, if the vertical or horizontal cover is increased, more concrete is available to resist the tension resulting from the wedging effect of the deformed bars, resistance to splitting is improved, and development length is less.

Similarly, Fig. 5.6*b* illustrates that if the *bar spacing* is increased (e.g., if only two instead of three bars are used), more concrete per bar will be available to resist horizontal splitting (Ref. 5.16). In beams, bars are typically spaced about one or two bar diameters apart. On the other hand, for slabs, footings, and certain other types of member, bar spacings are typically much greater, and the required development length is reduced.

Transverse reinforcement, such as that provided by stirrups of the types shown in Fig. 4.8, improves the resistance of tensile bars to both vertical or horizontal splitting failure because the tensile force in the transverse steel tends to prevent opening of the actual or potential crack. The effectiveness of such transverse reinforcement depends on its cross-sectional area and spacing along the development length. Its effectiveness does not depend on its yield strength f_{yt}, because transverse reinforcement rarely yields during a bond failure (Refs. 5.12 to 5.15). The yield strength of the transverse steel f_{yt}, however, is presently used in the bond provisions of the ACI Code.

Based on the results of a statistical analysis of test data (Ref. 5.10), with appropriate simplifications, the length l_d needed to develop stress f_s in a reinforcing bar may be expressed as

$$l_d = \left(\frac{3}{40} \frac{f_s}{\sqrt{f_c'} \left[\dfrac{(c + K_{tr})}{d_b} \right]} \right) d_b \qquad (5.3)$$

where d_b = bar diameter
c = smaller of minimum cover or one-half of bar spacing *measured to center of bar*
$K_{tr} = 40A_{tr}/sn$, which represents effect of confining reinforcement

A_{tr} = area of transverse reinforcement normal to plane of splitting through the bars being developed
s = spacing of transverse reinforcement
n = number of bars developed or spliced at same location

Equation (5.3) captures the effects of concrete strength, concrete cover, and transverse reinforcement on l_d and serves as the basis for design in the 2008 ACI Code. For full development of the bar, f_s is set equal to f_y.

In addition to the factors just discussed, other influences have been identified. The vertical location of horizondal bars relative to beam depth has been found to have an effect (Ref. 5.17). If bars are placed in the forms during construction such that a substantial *depth of concrete is placed below those bars*, there is a tendency for excess water, often used in the mix for workability, and for entrapped air to rise to the top of the concrete during consolidation. Air and water tend to accumulate on the underside of the bars. Tests have shown a significant loss in bond strength for bars with more than 12 in. of fresh concrete cast beneath them, and accordingly the development length must be increased. This effect increases as the slump of the concrete increases and is greatest for bars cast near the upper surface of a concrete placement (Ref. 5.18).

Epoxy-coated reinforcing bars are used regularly in projects where the structure may be subjected to corrosive environmental conditions or deicing chemicals, such as for highway bridge decks and parking garages. Studies have shown that bond strength is reduced because the epoxy coating reduces the friction between the concrete and the bar, and the required development length must be increased substantially (Refs. 5.19 to 5.23). Early evidence showed that if cover and bar spacing were large, the effect of the epoxy coating would not be so pronounced, and as a result, a smaller increase was felt justified under these conditions (Ref. 5.20). Although later research (Ref. 5.12) does not support this conclusion, provisions to allow for a smaller increase remain in the ACI Code. Since the bond strength of epoxy-coated bars is already reduced because of lack of adhesion, an upper limit has been established for the product of development length factors accounting for the depth of concrete cast below horizontal bars and epoxy coating.

Not infrequently, tensile reinforcement somewhat in excess of the calculated requirement will be provided, e.g., as a result of upward rounding A_s when bars are selected or when minimum steel requirements govern. Logically, in this case, the required development length may be reduced by the ratio of steel area required to steel area actually provided. The modification for *excess reinforcement* should be applied only where anchorage or development for the full yield strength of the bar is not required.

Finally, based on bars with very short development lengths (most with values of $l_d/d_b < 15$), it was observed that *smaller-diameter bars* required lower development lengths than predicted by Eq. (5.3). As a result, the required development lengths for No. 6 (No. 19) and smaller bars were reduced below the values required by Eq. (5.3).[†]

Reference 5.15 presents a detailed discussion of the factors that control the bond and development of reinforcing bars in tension. Except as noted, these influences are accounted for in the basic equation for development length in the 2008 ACI Code.

[†] The use of Eq. (5.3) for low values of l_d/d_b greatly underestimates the actual value of bond strength and makes it appear that a lower value of l_d can be used safely. An evaluation of test results for small bars with more realistic development lengths ($l_d/d_b \geq 16$), however, has shown that the special provision in the ACI Code for smaller bars is not justified (Refs. 5.14, 5.15, and 5.24). Because of the unconservative nature of the small bar provision, ACI Committee 408 (Ref. 5.15) recommends that it not be applied in design.

All modification factors for development length are defined explicitly in the Code, with appropriate restrictions. Details are given next.

5.3 ACI CODE PROVISIONS FOR DEVELOPMENT OF TENSION REINFORCEMENT

The approach to bond strength incorporated in the ACI Code follows from the discussion presented in Section 5.2. The fundamental requirement is that the calculated force in the reinforcement at each section of a reinforced concrete member be developed on each side of that section by adequate embedment length, hooks, mechanical anchorage, or a combination of these, to ensure against pullout. Local high bond forces, such as are known to exist adjacent to cracks in beams, are not considered to be significant. Generally, the force to be developed is calculated based on the yield stress in the reinforcement; i.e., the bar strength is to be fully developed.

In the ACI Code, the required development length for deformed bars in tension is based on Eq. (5.3). A single basic equation is given that includes *all* the influences discussed in Section 5.2 and thus appears highly complex because of its inclusiveness. However, it does permit the designer to see the effects of all the controlling variables and allows more rigorous calculation of the required development length when it is critical. The ACI Code also includes simplified equations that can be used for most cases in ordinary design, provided that some restrictions are accepted on bar spacing, cover values, and minimum transverse reinforcement. These alternative equations can be further simplified for normal-density concrete and uncoated bars.†

In the following presentation of development length, the basic ACI equation is given first and its terms are defined and discussed. After this, the alternative equations, also part of the ACI Code, are presented. Note that, in any case, development length l_d must not be less than 12 in.

a. Basic Equation for Development of Tension Bars

According to ACI Code 12.2.3, for deformed bars or deformed wires,

$$l_d = \left(\frac{3}{40} \frac{f_y}{\lambda\sqrt{f_c'}} \frac{\psi_t \psi_e \psi_s}{\left[\dfrac{(c_b + K_{tr})}{d_b} \right]} \right) d_b \tag{5.4}$$

in which the term $(c + K_{tr})/d_b$ shall not be taken greater than 2.5. In Eq. (5.4), terms are defined and values established as follows.

ψ_t = reinforcement location factor

Horizontal reinforcement so placed that more than 12 in. of fresh concrete is cast in the member below the development length or splice:	1.3
Other situations:	1.0

† This two-tier approach to development length corresponds exactly to the ACI Code treatment for V_c, the contribution of concrete in shear calculations. The more detailed calculation by Eq. (4.12*a*) is useful for computerized design or research but is tedious for manual calculations because of the need to recalculate the governing variables at close intervals along the span. For ordinary design, recognizing that overall economy is but little affected, the simpler but more approximate and more conservative Eq. (4.12*b*) is used.

ψ_e = coating factor

Epoxy-coated bars or wires with cover less than $3d_b$ or clear spacing less than $6d_b$: 1.5

All other epoxy-coated bars or wires: 1.2

Uncoated and zinc-coated (galvanized) reinforcement: 1.0

However, the product of $\psi_t \psi_e$ need not be taken greater than 1.7.

ψ_s = reinforcement size factor

No. 6 (No. 19) and smaller bars and deformed wires: 0.8[†]

No. 7 (No. 22) and larger bars: 1.0

λ = lightweight aggregate concrete factor

When lightweight aggregate concrete is used: 0.75

However, when f_{ct} is specified, $\lambda = f_{ct}/(6.7\sqrt{f'_c}) \leq 1.0$.

When normalweight concrete is used: 1.0

c = spacing or cover dimension, in.

Use the smaller of either the distance from the center of the bar to the nearest concrete surface or one-half the center-to-center spacing of the bars being developed.

K_{tr} = transverse reinforcement index: $40A_{tr}/sn$

where A_{tr} = total cross-sectional area of all transverse reinforcement that is within the spacing s and that crosses the potential plane of splitting through the reinforcement being developed, in^2

s = maximum spacing of transverse reinforcement within l_d center to center, in.

n = number of bars or wires being developed along the plane of splitting

As a simplification, the designer is permitted to use $K_{tr} = 0$ even if transverse reinforcement is present.

The limit of 2.5 on $(c + K_{tr})/d_b$ is imposed to avoid pullout failure. With that term taken equal to its limit of 2.5, evaluation of Eq. (5.4) results in $l_d = 0.03d_b f_y/\sqrt{f'_c}$, the experimentally derived limit found in earlier ACI Codes when pullout failure controls. Note that in Eq. (5.4) and in all other ACI Code equations relating to the development length and splices of reinforcement, *values of* $\sqrt{f'_c}$ *are not to be taken greater than 100 psi* because of the lack of experimental evidence on bond strengths obtainable with concretes having compressive strength in excess of 10,000 psi at the time that Eqs. (5.3) and (5.4) were formulated. More recent tests with concrete with values of f'_c to 16,000 psi justify this limitation.

b. Simplified Equations for Development Length

Calculation of required development length (in terms of bar diameter) by Eq. (5.4) requires that the term $(c + K_{tr})/d_b$ be calculated for each particular combination of cover, spacing, and transverse reinforcement. Alternatively, according to the Code, a simplified form of Eq. (5.4) may be used in which $(c + K_{tr})/d_b$ is set equal to 1.5, provided that certain restrictions are placed on cover, spacing, and transverse reinforcement. Two cases of practical importance are:

1. Minimum clear cover of $1.0d_b$, minimum clear spacing of $1.0d_b$, and at least the Code required minimum stirrups or ties (see Section 4.5b) throughout l_d
2. Minimum clear cover of $1.0d_b$ and minimum clear spacing of $2d_b$

[†] ACI Committee 408 recommends a value of 1.0 for all bar sizes based on experimental evidence. The ACI Code value of 0.8, however, will be used in what follows.

TABLE 5.1
Simplified tension development length in bar diameters according to the ACI Code

	No. 6 (No. 19) and Smaller Bars and Deformed Wires†	No. 7 (No. 22) and Larger Bars
Clear spacing of bars being developed or spliced $\geq d_b$, clear cover $\geq d_b$, and stirrups or ties throughout l_d not less than the Code minimum	$l_d = \left(\frac{f_y \psi_t \psi_e}{25\lambda\sqrt{f'_c}}\right) d_b$	$l_d = \left(\frac{f_y \psi_t \psi_e}{20\lambda\sqrt{f'_c}}\right) d_b$
Clear spacing of bars being developed or spliced $\geq 2d_b$, and clear cover $\geq d_b$	Same as above	Same as above
Other cases	$l_d = \left(\frac{3f_y \psi_t \psi_e}{50\lambda\sqrt{f'_c}}\right) d_b$	$l_d = \left(\frac{3f_y \psi_t \psi_e}{40\lambda\sqrt{f'_c}}\right) d_b$

† For reasons discussed in Section 5.3a, ACI Committee 408 recommends that l_d for No. 7 (No. 22) and larger bars be used for all bar sizes.

For either of these common cases, it is easily confirmed from Eq. (5.4) that for No. 7 (No. 22) and larger bars

$$l_d = \left(\frac{f_y \psi_t \psi_e}{20\lambda\sqrt{f'_c}}\right) d_b \tag{5.5a}$$

and for No. 6 (No. 19) bars and smaller (with $\gamma = 0.8$)

$$l_d = \left(\frac{f_y \psi_t \psi_e}{25\lambda\sqrt{f'_c}}\right) d_b \tag{5.5b}$$

If these restrictions on spacing are not met, then, provided that Code-imposed minimum spacing requirements are met (see Section 3.6c), the term $(c + K_{tr})/d_b$ will have a value not less than 1.0 (rather than 1.5 as before) whether or not transverse steel is used. The values given by Eqs. (5.5*a*) and (5.5*b*) are then multiplied by the factor 1.5/1.0.

Thus if the designer accepts certain restrictions on bar cover, spacing, and transverse reinforcement, simplified calculation of development requirements is possible. The simplified equations are summarized in Table 5.1.

Further simplification is possible for the most common condition of normal-density concrete and uncoated reinforcement. Then λ and ψ_e in Table 5.1 take the value 1.0, and the development lengths, in terms of bar diameters, are simply a function of f_y, f'_c, and the bar location factor ψ_t. Thus development lengths are easily tabulated for the usual combinations of material strengths and bottom or top bars and for the restrictions on bar spacing, cover, and transverse steel defined.† Results are given in Table A.10 of Appendix A.

Regardless of whether development length is calculated using the basic Eq. (5.4) or the more approximate Eqs. (5.5*a*) and (5.5*b*), development length may be reduced where reinforcement in a flexural member is in excess of that required by analysis,

† Note that, for convenient reference, the term top bar is used for any horizontal reinforcing bar placed with more than 12 in. of fresh concrete cast below the development length or splice. This definition may require that bars relatively near the bottom of a deep member be treated as top bars.

except where anchorage or development for f_y is specifically required or the reinforcement is designed for a region of high seismic risk. According to the ACI Code, the reduction is made according to the ratio (A_s required/A_s provided).

EXAMPLE 5.1

Development length in tension. Figure 5.8 shows a beam-column joint in a continuous building frame. Based on frame analysis, the negative steel required at the end of the beam is 2.90 in²; two No. 11 (No. 36) bars are used, providing A_s = 3.12 in². Beam dimensions are b = 10 in., d = 18 in., and h = 21 in. The design will include No. 3 (No. 10) stirrups spaced four at 3 in., followed by a constant 5 in. spacing in the region of the support, with 1.5 in. clear cover. Normalweight concrete is to be used, with f_c' = 4000 psi, and reinforcing bars have f_y = 60,000 psi. Find the minimum distance l_d at which the negative bars can be cut off, based on development of the required steel area at the face of the column, (*a*) using the simplified equations of Table 5.1, (*b*) using Table A.10, of Appendix A, and (*c*) using the basic Eq. (5.4).

Solution. Checking for lateral spacing in the No. 11 (No. 36) bars determines that the clear distance between the bars is 10 − 2(1.50 + 0.38 + 1.41) = 3.42 in., or 2.43 times the bar diameter d_b. The clear cover of the No. 11 (No. 36) bars to the side face of the beam is 1.50 + 0.38 = 1.88 in., or 1.33 bar diameters, and that to the top of the beam is 3.00 − 1.41/2 = 2.30 in., or 1.63 bar diameters. These dimensions meet the restrictions stated in the second row of Table 5.1. Then for top bars, uncoated, and with normal-density concrete, we have the values of ψ_t = 1.3, ψ_e = 1.0, and λ = 1.0. From Table 5.1,

$$l_d = \frac{60{,}000 \times 1.3 \times 1.0}{20 \times 1.0\sqrt{4000}} 1.41 = 62 \times 1.41 = 87 \text{ in.}$$

This can be reduced by the ratio of steel required to that provided, so that the final development length is 87 × 2.90/3.12 = 81 in.

Alternatively, from the lower portion of Table A.10, l_d/d_b = 62. The required length to point of cutoff is 62 × 1.41 × 2.90/3.12 = 81 in., as before.

The more accurate Eq. (5.4) will now be used. The center-to-center spacing of the No. 11 (No. 36) bars is 10 − 2(1.50 + 0.38 + 1.41/2) = 4.83, one-half of which is 2.42 in. The side cover to bar centerline is 1.50 + 0.38 + 1.41/2 = 2.59 in., and the top cover is 3.00 in. The smallest of these three distances controls, and c = 2.42 in. Potential splitting would be in the

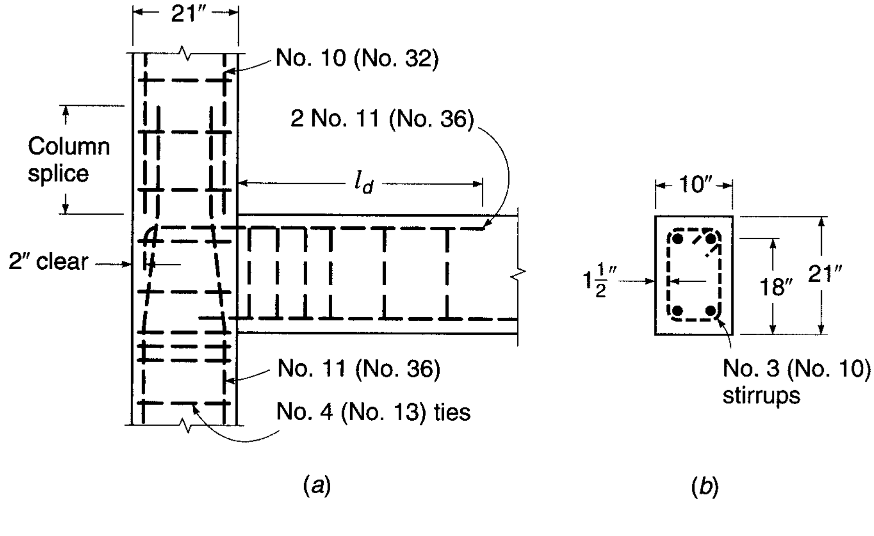


FIGURE 5.8
Bar details at beam-column joint for bar development examples.

horizontal plane of the bars, and in calculating A_{tr} two times the stirrup bar area is used.† Based on the No. 3 (No. 10) stirrups at 5 in. spacing:

$$K_{tr} = \frac{40 \times 0.11 \times 2}{5 \times 2} = 0.88 \qquad \text{and} \qquad \frac{c + K_{tr}}{d_b} = \frac{2.42 + 0.88}{1.41} = 2.34$$

This is less than the limit value of 2.5. Then from Eq. (5.4)

$$l_d = \frac{3 \times 60{,}000 \times 1.3}{40 \times 1.0\sqrt{4000} \times 2.34} 1.41 = 40 \times 1.41 = 55.7 \text{ in.}$$

and the required development length is 55.7 × 2.90/3.12 = 52 in. rather than 81 in. as before. Clearly, the use of the more accurate Eq. (5.4) permits a considerable reduction in development length. Even though its use requires much more time and effort, it is justified if the design is to be repeated many times in a structure.

5.4 ANCHORAGE OF TENSION BARS BY HOOKS

a. Standard Dimensions

In the event that the desired tensile stress in a bar cannot be developed by bond alone, it is necessary to provide special anchorage at the ends of the bar, usually by means of a 90° or a 180° hook or a headed bar (the latter is discussed in Section 5.5). The dimensions and bend radii for hooks have been standardized in ACI Code 7.1 as follows (see Fig. 5.9):

1. A 180° bend plus an extension of at least 4 bar diameters, but not less than $2\frac{1}{2}$ in. at the free end of the bar, or
2. A 90° bend plus an extension of at least 12 bar diameters at the free end of the bar, or
3. For stirrup and tie anchorage only:
   (*a*) For No. 5 (No. 16) bars and smaller, a 90° bend plus an extension of at least 6 bar diameters at the free end of the bar, or
   (*b*) For Nos. 6, 7, and 8 (Nos. 19, 22, and 25) bars, a 90° bend plus an extension of at least 12 bar diameters at the free end of the bar, or
   (*c*) For No. 8 (No. 25) bars and smaller, a 135° bend plus an extension of at least 6 bar diameters at the free end of the bar.

The minimum diameter of bend, measured on the inside of the bar, for standard hooks other than for stirrups or ties in sizes Nos. 3 through 5 (Nos. 10 through 16), should be not less than the values shown in Table 5.2. For stirrup and tie hooks, for bar sizes No. 5 (No. 16) and smaller, the inside diameter of bend should not be less than 4 bar diameters, according to the ACI Code.

When welded wire reinforcement (smooth or deformed wires) is used for stirrups or ties, the inside diameter of bend should not be less than 4 wire diameters for deformed wire larger than D6 and 2 wire diameters for all other wires. Bends with an inside diameter of less than 8 wire diameters should not be less than 4 wire diameters from the nearest welded intersection.

† If the top cover had controlled, the potential splitting plane would be vertical and one times the stirrup bar area would be used in calculating A_{tr}.

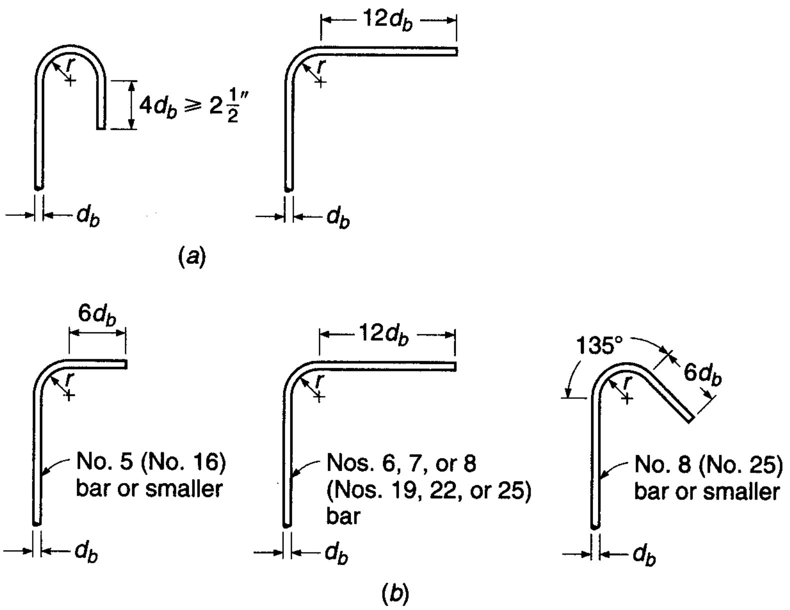


FIGURE 5.9
Standard bar hooks: (*a*) main reinforcement; (*b*) stirrups and ties.

TABLE 5.2
Minimum diameters of bend for standard hooks

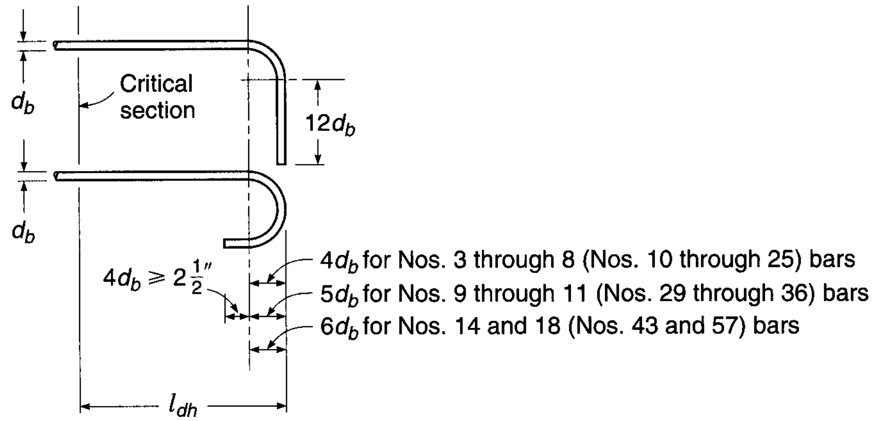
Bar Size	Minimum Diameter
Nos. 3 through 8 (Nos. 10 through 25)	6 bar diameters
Nos. 9, 10, and 11 (Nos. 29, 32, and 36)	8 bar diameters
Nos. 14 and 18 (Nos. 43 and 57)	10 bar diameters

b. Development Length and Modification Factors for Hooked Bars

Hooked bars resist pullout by the combined actions of bond along the straight length of bar leading to the hook and anchorage provided by the hook. Tests indicate that the main cause of failure of hooked bars in tension is splitting of the concrete in the plane of the hook. This splitting is due to the very high stresses in the concrete inside of the hook; these stresses are influenced mainly by the bar diameter d_b for a given tensile force, and the radius of bar bend. Resistance to splitting has been found to depend on the concrete cover for the hooked bar, measured laterally from the edge of the member to the bar perpendicular to the plane of the hook, and measured to the top (or bottom) of the member from the point where the hook starts, parallel to the plane of the hook. If these distances must be small, the strength of the anchorage can be substantially increased by providing confinement steel in the form of closed stirrups or ties.

ACI Code 12.5 provisions for hooked bars in tension are based on research summarized in Refs. 5.8 and 5.9. The Code requirements account for the combined contribution of bond along the straight bar leading to the hook, plus the hooked anchorage. A total development length l_{dh} is defined as shown in Fig. 5.10 and is

FIGURE 5.10
Bar details for development of standard hooks.



measured from the critical section to the farthest point on the bar, parallel to the straight part of the bar. For standard hooks, as shown in Fig. 5.9, the development length is

$$l_{dh} = \left(\frac{0.02\ \psi_e f_y}{\lambda \sqrt{f_c'}} \right) d_b \tag{5.6}$$

with $\psi_e = 1.2$ for epoxy-coated reinforcement and $\lambda = 0.75$ for lightweight aggregate concrete. For other cases, ψ_e and λ are taken as 1.0.

The development length l_{dh} should be multiplied by certain applicable modifying factors, summarized in Table 5.3. These factors are combined as appropriate; e.g., if side cover of at least $2\frac{1}{2}$ in. is provided for a 180° hook and if, in addition, ties are provided, the development length is multiplied by the product of 0.7 and 0.8. In any case, the length l_{dh} is not to be less than 8 bar diameters and not less than 6 in.

Transverse confinement steel is essential if the full bar strength must be developed with minimum concrete confinement, such as when hooks may be required at the ends of a simply supported beam or where a beam in a continuous structure frames into an end column and does not extend past the column or when bars must be anchored in a short cantilever, as shown in Fig. 5.11 (Ref. 5.11). According to ACI Code 12.5.4, for bars hooked at the discontinuous ends of members with both side

FIGURE 5.11
Transverse reinforcement requirements at discontinuous ends of members with small cover distances.

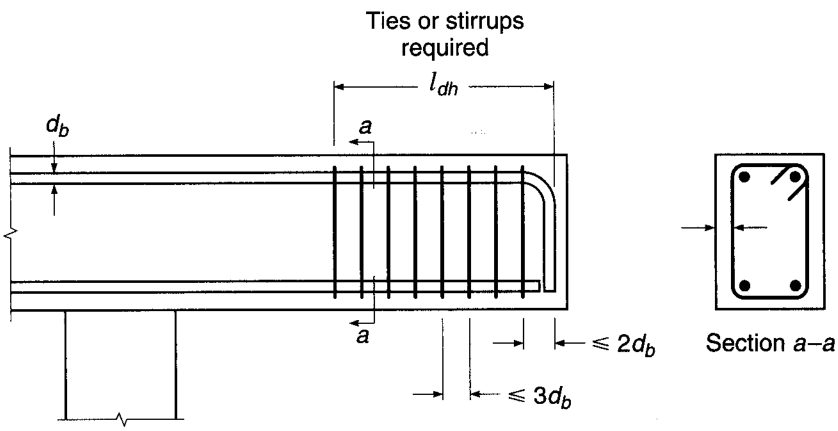


TABLE 5.3
Development lengths for hooked deformed bars in tension

A. Development length l_{dh} for hooked bars	$\left(\frac{0.02\psi_e f_y}{\lambda\sqrt{f'_c}}\right) d_b$
B. Modification factors applied to l_{dh}	
For No. 11 (No. 36) and smaller bar hooks with side cover (normal to plane of hook) not less than $2\frac{1}{2}$ in., and for 90° hooks with cover on bar extension beyond hook not less than 2 in.	0.7
For 90° hooks of No. 11 (No. 36) and smaller bars that are either enclosed within ties or stirrups perpendicular to the bar being developed, spaced not greater than $3d_b$ along the development length l_{dh} of the hook; or enclosed within ties or stirrups parallel to the bar being developed, spaced not greater than $3d_b$ along the length of the tail extension of the hook plus bend	0.8
For 180° hooks of No. 11 (No. 36) and smaller bars that are enclosed within ties or stirrups perpendicular to the bar being developed, spaced not greater than $3d_b$ along the development length l_{dh} of the hook	0.8
Where anchorage or development for f_y is not specifically required, reinforcement in excess of that required by analysis	$\frac{A_s \text{ required}}{A_s \text{ provided}}$
ψ_e:	
For epoxy-coated bars	1.2
For other bars	1.0
λ:	
For lightweight concrete	0.75
For normalweight concrete	1.0

cover and top or bottom cover less than $2\frac{1}{2}$ in., hooks *must* be enclosed with closed stirrups or ties along the full development length, as shown in Fig. 5.11. The spacing of the confinement steel must not exceed 3 times the diameter of the hooked bar d_b, and the first stirrup or tie must enclose the bent portion of the hook within a distance equal to $2d_b$ of the outside of the bend. In such cases, the factor 0.8 of Table 5.3 does not apply.

EXAMPLE 5.2 **Development of hooked bars in tension.** Referring to the beam-column joint shown in Fig. 5.8, the No. 11 (No. 36) negative bars are to be extended into the column and terminated in a standard 90° hook, keeping 2 in. clear to the outside face of the column. The column width in the direction of beam width is 16 in. Find the minimum length of embedment of the hook past the column face, and specify the hook details.

Solution. The development length for hooked bars, measured from the critical section along the bar to the far side of the vertical hook, is given by Eq. (5.6):

$$l_{dh} = \frac{0.02 \times 1.0 \times 60{,}000}{1.0 \times \sqrt{4000}} 1.41 = 27 \text{ in.}$$

In this case, side cover for the No. 11 (No. 36) bars exceeds 2.5 in. and cover beyond the bent bar is adequate, so a modifying factor of 0.7 can be applied. The only other factor applicable is for excess reinforcement, which is 0.93 as for Example 5.1. Accordingly, the minimum development length for the hooked bars is

$$l_{dh} = 27 \times 0.7 \times 0.93 = 18 \text{ in.}$$

With $21 - 2 = 19$ in. available, the required length is contained within the column. The hook will be bent to a minimum diameter of $8 \times 1.41 = 11.28$ in. The bar will continue for 12 bar diameters, or 17 in. past the end of the bend in the vertical direction.

5.5 ANCHORAGE IN TENSION USING HEADED BARS

a. Requirements for Headed Bars

Headed bars provide an alternative to hooks when the desired tensile stress in the bar cannot be developed by bond alone. ACI Code 3.5.9 requires that headed deformed bars conform to ASTM A970 and, in addition, that obstructions or interruptions of the bar deformations not extend more than 2 bar diameters from the bearing face of the head, as shown in Fig. 5.12. While heads come in many configurations and sizes, ACI Code 12.6.1 requires that the bearing area of the head A_{brg} be equal to at least 4 times the area of the bar A_b.

Obstructions, such as shown in Fig. 5.12, are not counted as part of the bearing area according to the Commentary to ACI Code 3.5.9, and thus, the net bearing area of the head may be less than the gross area of the head minus the area of the bar.

b. Development Length and Modification Factors for Headed Bars

Differences between the mode of failure of headed bars loaded in tension and those exhibited by straight bars and hooks, coupled with the fact that only limited test data are available for headed bars, have resulted in the ACI Code adding restrictions to the design criteria for headed bars.

The bond strength of headed bars results from a combination of bond along the length of the bar and bearing at the face of the head. Prior to failure, the bond force along the bar increases and then decreases as slip occurs, while the bearing force on

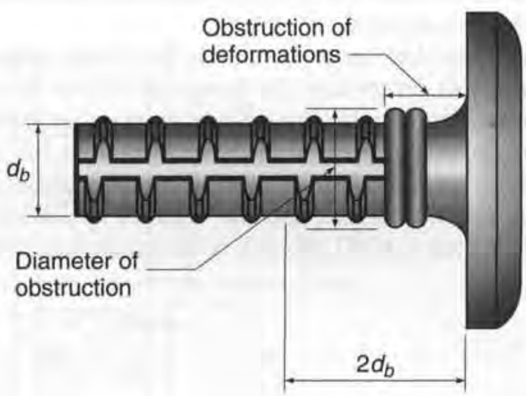


FIGURE 5.12
Headed deformed reinforcing bar with an obstruction of the deformations that extends less than 2 bar diameters from the bearing face of the head.

FIGURE 5.13
Headed deformed bars showing conical concrete wedges. (*Photograph courtesy of Michael Keith Thompson.*)

the head increases. In some cases, the contribution of bond along the length of the bar may become negligible prior to failure. Unlike straight reinforcing bars, which tend to fail in bond due to the formation of splitting cracks between bars or between the bar and the surface of the concrete, and hooks, which tend to fail in bond by cracking in the plane of the hook, headed bars fail in bond due to the formation of a conical wedge, as shown in Fig. 5.13, which causes radial splitting cracks in the concrete. In addition to radial splitting, failure can also occur due to the formation of a flat concrete cone near the surface (shallow pullout), if the development is relatively short, or a breakout cone, if the development length is long, and due to spalling or side-face blowout, if the side cover is low (Refs. 5.25 to 5.28). Transverse confining reinforcement, such as stirrups and ties, which increase the bond strength of both straight and hooked bars, provides little additional capacity to headed bars and is, thus, not considered when calculating the development length of headed bars. Because transverse reinforcement limits the width of splitting cracks, however, its use is still recommended when headed bars are used.

Because test data are not available for a wide range of concrete properties, bar sizes, and member geometries, the design provisions are restricted to No. 11 (No. 36) and smaller bars with yield strengths not greater than 60,000 psi. The bars, as distinct from the heads, must have a clear cover of at least $2d_b$ and a clear spacing between bars of at least $4d_b$. In addition, headed bars are restricted by ACI Code 12.6 to use with normalweight concrete, and the value of f'_c used to calculate the development length l_{dt} is limited to 6000 psi. The development length for headed deformed bars in tension is

$$l_{dt} = \left(\frac{0.016\psi_e f_y}{\sqrt{f'_c}}\right)d_b \tag{5.7}$$

where $\psi_e = 1.2$ for epoxy-coated reinforcement and 1.0 for other cases.

TABLE 5.4
Development lengths for headed deformed bars in tension

A. Development length l_{dt} for headed bars	$\left(\frac{0.016\psi_e f_y}{\sqrt{f'_c}}\right) d_b$
B. Modification factors applied to l_{dt}	
Where anchorage or development for f_y is not specifically required, reinforcement in excess of that required by analysis	$\frac{A_s \text{ required}}{A_s \text{ provided}}$
ψ_e	
For epoxy-coated bars	1.2
For other bars	1.0

Where the reinforcement provided exceeds that required by analysis, except when development of the yield strength f_y is specifically required, the value of l_{dt} in Eq. (5.7) may be multiplied by the factor (A_s required)/(A_s provided). Under any circumstances, l_{dt} may not be less than 8 bar diameters or less than 6 in. Calculation of the development length l_{dt} and the applicable modifying factors are summarized in Table 5.4.

The development length l_{dt} should be measured from the bearing face of the head to the critical section, as shown in Fig. 5.14. When headed bars from a flexural member, such as a beam or a slab, terminate in a supporting member, such as the column shown in Fig. 5.15, the commentary to ACI Code 12.6 recommends that the bar be extended "through the joint to the far face of the confined core of the supporting member, allowing for cover and avoidance of interference with column reinforcement," even if the resulting anchorage length is greater than l_{dt}. Doing so helps to adequately anchor

FIGURE 5.14
Development length of headed deformed bars.

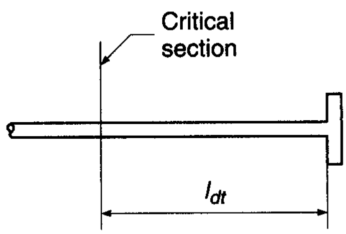
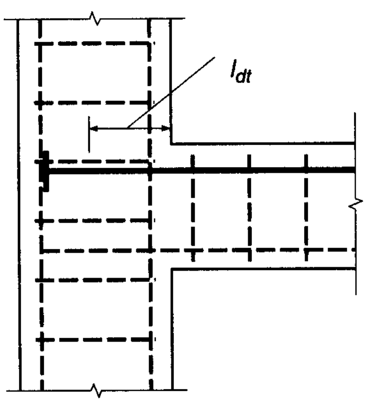


FIGURE 5.15
Headed deformed bar extended to far side of column with anchorage length that exceeds l_{dt}.



the compressive forces that are developed at the face of the head and improves the performance of the beam-column connection.

c. Mechanical Anchorage

In cases where headed bars do not meet the requirements specified in Section 12.6 or in cases where bars are terminated by mechanisms such as welded plates or other manufactured devices, ACI Code 12.6.4 allows such devices to be used to develop the reinforcement if the adequacy of the devices is established by tests. In such cases, the development of the reinforcement may consist of the combined contributions of bond along the length of the bar leading to the critical section, plus that of the mechanical anchorage, much in the way that the total resistance of headed bars is provided.

EXAMPLE 5.3 **Development of headed deformed bars in tension.** Three No. 7 (No. 22) bars serve as top reinforcement for a bracket framing into a 16 × 16 in. column (Fig. 5.16). The bracket projects 15 in. from the column and is the same width as the column. The top cover to the center of the bars is 3 in., and the side cover to the center of the bars is 3.5 in. The bars are spaced laterally at 4.5 in. These dimensions are inadequate for straight development length or for standard hooks. Based on other reinforcement, cover requirements, and head thickness, total development lengths for headed bars of 12 in. in the column and 12.5 in. in the bracket are available. The reinforcing bars have $f_y = 60{,}000$ psi, and the concrete is normalweight with $f_c' = 5000$ psi. Determine if a bar with heads at both ends can be used in this application.

SOLUTION. The minimum head size is $A_{brg} = 4A_b = 2.4\text{ in}^2$. The smaller available anchorage length in the column governs. Assuming that the bars will be used at the full yield strength, the development length l_{dt} calculated using Eq. (5.7) is

$$l_{dt} = \left(\frac{0.016\psi_e f_y}{\sqrt{f_c'}}\right)d_b = \left(\frac{0.016 \times 1.0 \times 60{,}000}{\sqrt{5000}}\right)0.875 = 11.9\text{ in.}$$

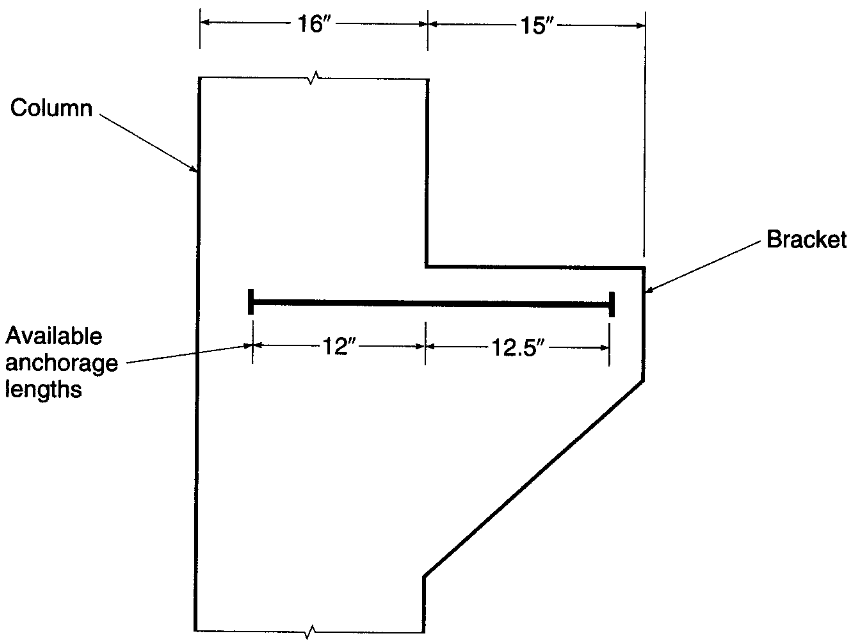


FIGURE 5.16 Column and bracket for headed deformed bar development example.

which must be checked against the minimum values for l_{dt}, which are

$$l_{dt} \geq 8d_b = 7 \text{ in.}$$
$$l_{dt} \geq 6 \text{ in.}$$

Thus, the value of l_{dt} obtained using Eq. (5.7) governs and is less than the available anchorage length. Thus, a bar with heads at both ends can be used, with a distance between heads of 24.5 in., as shown in Fig. 5.16.

5.6 ANCHORAGE REQUIREMENTS FOR WEB REINFORCEMENT

Stirrups should be carried as close as possible to the compression and tension faces of a beam, and special attention must be given to proper anchorage. The truss model (see Section 4.8 and Fig. 4.19) for design of shear reinforcement indicates the development of diagonal compressive struts, the thrust from which is equilibrated, near the top and bottom of the beam, by the tension web members (i.e., the stirrups). Thus, at the factored load, the tensile strength of the stirrups must be developed for almost their full height. Clearly, it is impossible to do this by development length. For this reason, stirrups normally are provided with 90° or 135° hooks at their upper end (see Fig. 5.9*b* for standard hook details) and at their lower end are bent 90° to pass around the longitudinal reinforcement. In simple spans, or in the positive bending region of continuous spans, where no top bars are required for flexure, stirrup support bars must be used. These are usually about the same diameter as the stirrups themselves, and they not only provide improved anchorage of the hooks but also facilitate fabrication of the reinforcement cage, holding the stirrups in position during placement of the concrete.

ACI Code 12.13 includes special provisions for anchorage of web reinforcement. The ends of single-leg, simple-U, or multiple-U stirrups are to be anchored by one of the following means:

1. For No. 5 (No. 16) bars and smaller, and for Nos. 6, 7, and 8 (Nos. 19, 22, and 25) bars with f_{yt} of 40,000 psi or less, a standard hook around longitudinal reinforcement, as shown in Fig. 5.17*a*.
2. For Nos. 6, 7, and 8 (Nos. 19, 22, and 25) stirrups with f_{yt} greater than 40,000 psi, a standard hook around a longitudinal bar, plus an embedment between midheight of the member and the outside end of the hook equal to or greater than $0.014d_b f_{yt}/\lambda\sqrt{f'_c}$ in., as shown in Fig. 5.17*b*.

ACI Code 12.13 specifies further that, between anchored ends, each bend in the continuous portion of a simple-U or multiple-U stirrup shall enclose a longitudinal bar, as in Fig. 5.17*c*. Longitudinal bars bent to act as shear reinforcement, if extended into a region of tension, shall be continuous with longitudinal reinforcement and, if extended into a region of compression, shall be anchored beyond middepth $d/2$ as specified for development length. Pairs of U stirrups or ties so placed as to form a closed unit shall be considered properly spliced when length of laps are $1.3l_d$ as in Fig. 5.17*d*. In members at least 18 in. deep, such splices are considered adequate if $A_b f_{yt} \leq 9000$ lb and the stirrup legs extend the full depth of the member. As will be discussed in Section 5.11, pairs of U stirrups may not be used in perimeter beams.

Other provisions are contained in the ACI Code relating to the use of welded wire reinforcement, which is sometimes used for web reinforcement in precast and prestressed concrete beams.

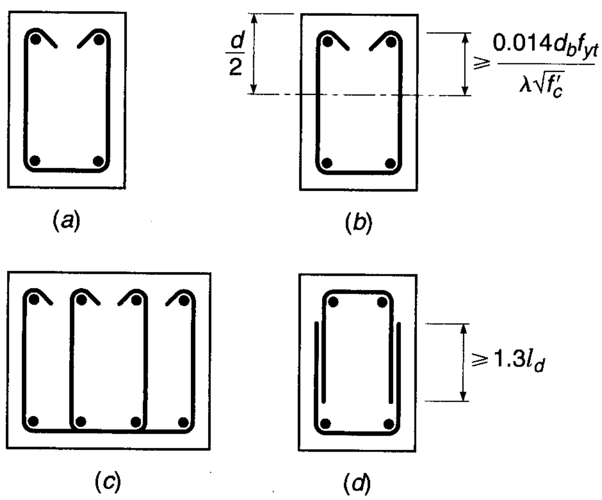


FIGURE 5.17
ACI requirements for stirrup anchorage: (*a*) No. 5 (No. 16) stirrups and smaller, and Nos. 6, 7, and 8 (Nos. 19, 22, and 25) stirrups with yield stress not exceeding 40,000 psi; (*b*) Nos. 6, 7, and 8 (Nos. 19, 22, and 25) stirrups with yield stress exceeding 40,000 psi; (*c*) wide beam with multiple-leg U stirrups; (*d*) pairs of U stirrups forming a closed unit. See Fig. 5.9 for alternative standard hook details.

5.7 WELDED WIRE REINFORCEMENT

Tensile steel consisting of welded wire reinforcement (often referred to as welded wire fabric), with either deformed or smooth wires, is commonly used in one-way and two-way slabs and certain other types of members (see Section 2.15). For *deformed* wire reinforcement, some of the development is assigned to the welded cross wires and some to the embedded length of the deformed wire. According to ACI Code 12.7, the development length of welded deformed wire reinforcement measured from the point of the critical section to the end of the wire is computed as the product of the development length l_d from Table 5.1 or from the more accurate Eq. (5.4) and the appropriate modification factor or factors related to those equations, except that the development length is not to be less than 8 in. For welded deformed wire reinforcement with at least one cross wire within the development length and not less than 2 in. from the point of the critical section, a *deformed wire factor* ψ_w equal to the greater of

$$\frac{f_y - 35{,}000}{f_y} \tag{5.8a}$$

or

$$\frac{5d_b}{s} \tag{5.8b}$$

is applied, where s is the lateral spacing of the wires being developed; but this factor need not exceed 1.0. When ψ_w from Eq. (5.8*a*) or (5.8*b*) is used, the epoxy coating factor ψ_e is taken as 1.0. For welded wire deformed reinforcement with no cross wires within the development length or with a single cross wire less than 2 in. from the point of the critical section, the wire fabric factor is taken to be equal to 1.0 and the development length determined as for the deformed wire.

For welded *plain* wire reinforcement, development is considered to be provided by embedment of two cross wires, with the closer wire not less than 2 in. from the critical section. However, the development length measured from the critical section to the outermost cross wire is not to be less than

$$l_d = 0.27 \frac{A_b}{s} \frac{f_y}{\lambda\sqrt{f'_c}} \tag{5.9}$$

according to ACI Code 12.8, where A_b is the cross-sectional area of an individual wire to be developed or spliced. The modification factor for excess reinforcement may be applied, but l_d is not to be less than 6 in. for the welded plain wire reinforcement.†

5.8 DEVELOPMENT OF BARS IN COMPRESSION

Reinforcement may be required to develop its compressive strength by embedment under various circumstances, e.g., where bars transfer their share of column loads to a supporting footing or where lap splices are made of compression bars in column (see Section 5.13). In the case of bars in compression, a part of the total force is transferred by bond along the embedded length, and a part is transferred by end bearing of the bars on the concrete. Because the surrounding concrete is relatively free of cracks and because of the beneficial effect of end bearing, shorter basic development lengths are permissible for compression bars than for tension bars. If transverse confinement steel is present, such as spiral column reinforcement or special spiral steel around an individual bar, the required development length is further reduced. Hooks and heads such as are shown in Figs. 5.9 and 5.12 are *not* effective in transferring compression from bars to concrete, and, if present for other reasons, should be disregarded in determining required embedment length.

According to ACI Code 12.3, the development length in compression is the greater of

$$l_{dc} = \left(\frac{0.02 f_y}{\lambda\sqrt{f'_c}}\right) d_b \tag{5.10a}$$

and

$$l_{dc} = 0.0003 f_y d_b \tag{5.10b}$$

Modification factors summarized in part *B* of Table 5.5, as applicable, are applied to the development length in compression to obtain the value of development length l_{dc} to be used in design. In no case is l_d to be less than 8 in., according to the ACI Code. Basic and modified compressive development lengths are given in Table A.11 of Appendix A.

5.9 BUNDLED BARS

It was pointed out in Section 3.6c that it is sometimes advantageous to "bundle" tensile reinforcement in large beams, with two, three, or four bars in contact, to provide for improved placement of concrete around and between bundles of bars. Bar bundles are typically triangular or L-shaped for three bars, and square for four. When bars are cut off in a bundled group, the cutoff points must be staggered at least 40 diameters.

† The ACI Code offers no explanation as to why $l_{d,\min}$ = 6 in. for welded plain wire reinforcement, but 8 in. for welded deformed wire reinforcement.

TABLE 5.5
Development lengths for deformed bars in compression

A. Basic development length l_{dc}	$\begin{cases} \geq \left(\dfrac{0.02 f_y}{\lambda \sqrt{f'_c}}\right) d_b \\ \geq 0.0003 f_y d_b \end{cases}$
B. Modification factors to be applied to l_{dc}	
Reinforcement in excess of that required by analysis	$\dfrac{A_s \text{ required}}{A_s \text{ provided}}$
Reinforcement enclosed within spiral reinforcement not less than $\frac{1}{4}$ in. diameter and not more than 4 in. pitch or within No. 4 (No. 13) ties spaced at not more than 4 in. on centers	0.75

According to ACI Code 12.4, the development length of individual bars within a bundle, for both tension and compression, is that of the individual bar increased by 20 percent for a three-bar bundle and by 33 percent for a four-bar bundle, to account for the probable deficiency of bond at the inside of the bar group.

For bundled bars, to determine the appropriate spacing and cover values (1) for use in Table 5.1, (2) when calculating the confinement term K_{tr} in Eq. (5.4), or (3) when selecting the epoxy coating factor ψ_e, the unit of bundled bars is treated as a single bar with a diameter derived from the equivalent total area and having a centroid that coincides with that of the bar group.

5.10 BAR CUTOFF AND BEND POINTS IN BEAMS

Chapter 3 dealt with moments, flexural stresses, concrete dimensions, and longitudinal bar areas at the critical moment sections of beams. These critical moment sections are generally at the face of the supports (negative bending) and near the middle of the span (positive bending). Occasionally, haunched members having variable depth or width are used so that the concrete flexural capacity will agree more closely with the variation of bending moment along a span or series of spans. Usually, however, prismatic beams with constant concrete cross-sectional dimensions are used to simplify formwork and thus to reduce cost.

The steel requirement, on the other hand, is easily varied in accordance with requirements for flexure, and it is common practice either to cut off bars where they are no longer needed to resist stress or, sometimes in the case of continuous beams, to bend up the bottom steel (usually at 45°) so that it provides tensile reinforcement at the top of the beam over the supports.

a. Theoretical Points of Cutoff or Bend

The tensile force to be resisted by the reinforcement at any cross section is

$$T = A_s f_s = \frac{M}{z}$$

where M is the value of bending moment at that section and z is the internal lever arm of the resisting moment. The lever arm z varies only within narrow limits and is never

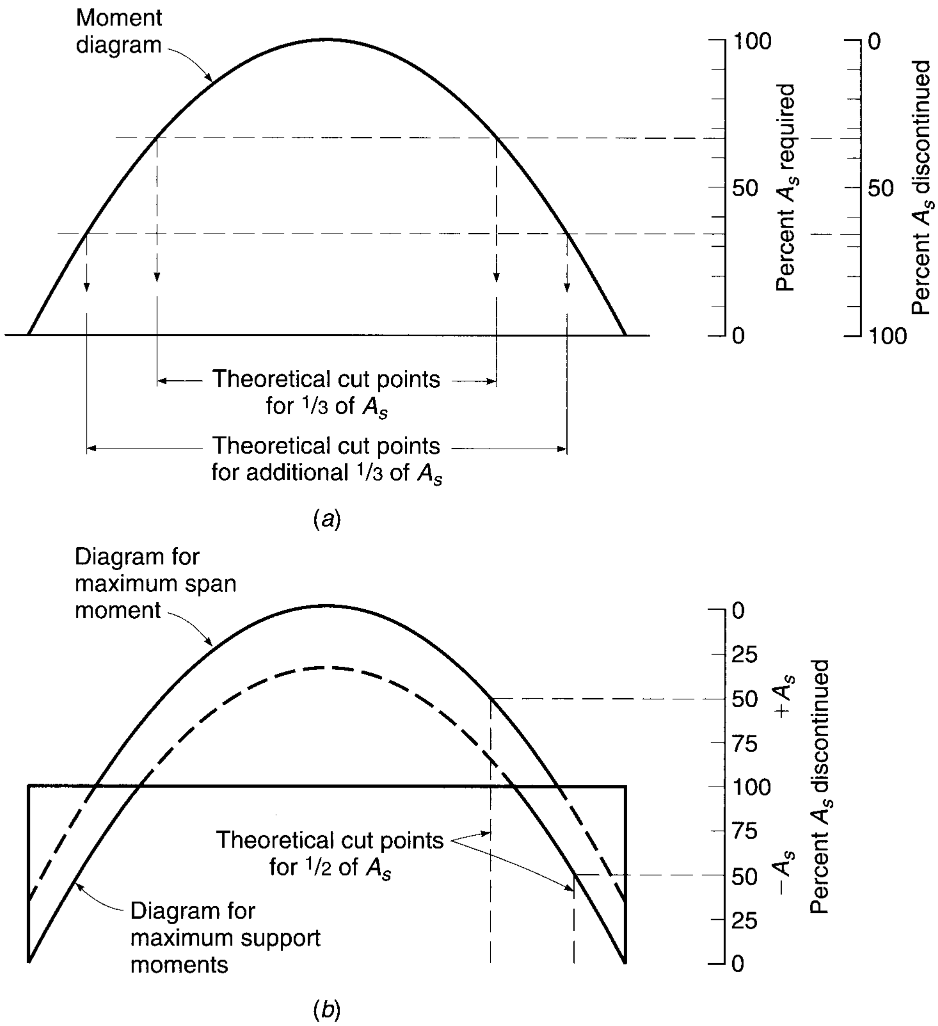


FIGURE 5.18
Bar cutoff points from moment diagrams.

less than the value at the maximum-moment section. Consequently, the tensile force can be taken with good accuracy directly proportional to the bending moment. Since it is desirable to design so that the steel everywhere in the beam is as nearly fully stressed as possible, it follows that the required steel area is very nearly proportional to the bending moment.

To illustrate, the moment diagram for a uniformly loaded simple-span beam shown in Fig. 5.18*a* can be used as a steel requirement diagram. At the maximum-moment section, 100 percent of the tensile steel is required (0 percent can be discontinued or bent), while at the supports, 0 percent of the steel is theoretically required (100 percent can be discontinued or bent). The percentage of bars that could be discontinued elsewhere along the span is obtainable directly from the moment diagram, drawn to scale. To facilitate the determination of cutoff or bend points for simple spans, Graph A.2 of Appendix A has been prepared. It represents a half-moment diagram for a uniformly loaded simple span.

To determine cutoff or bend points for continuous beams, the moment diagrams resulting from loading for maximum span moment and maximum support moment are drawn. A moment envelope results that defines the range of values of moment at any section. Cutoff or bend points can be found from the appropriate moment curve as for simple spans. Figure 5.18*b* illustrates, for example, a continuous beam with moment envelope resulting from alternate loadings to produce maximum span and maximum support moments. The locations of the points at which 50 percent of the bottom and top steel may theoretically be discontinued are shown.

According to ACI Code 8.3, uniformly loaded, continuous reinforced concrete beams of fairly regular span may be designed using moment coefficients (see Table 12.1). These coefficients, analogous to the numerical constant in the expression $\frac{1}{8}wL^2$ for simple-beam bending moment, give a conservative approximation of span and support moments for continuous beams. When such coefficients are used in design, cutoff and bend points may conveniently be found from Graph A.3 of Appendix A. Moment curves corresponding to the various span and support-moment coefficients are given at the top and bottom of the chart, respectively.

Alternatively, if moments are found by frame analysis rather than from ACI moment coefficients, the location along the span where bending moment reduces to any particular value (e.g., as determined by the bar group after some bars are cut off), or to zero, is easily computed by statics.

b. Practical Considerations and ACI Code Requirements

Actually, in no case should the tensile steel be discontinued exactly at the theoretically described points. As described in Section 4.3 and shown in Fig. 4.7, when diagonal tension cracks form, an internal redistribution of forces occurs in a beam. Prior to cracking, the steel tensile force at any point is proportional to the moment at a vertical section passing through the point. However, after the crack has formed, the tensile force in the steel at the crack is governed by the moment at a section nearer midspan, which may be much larger. Furthermore, the actual moment diagram may differ from that used as a design basis, due to approximation of the real loads, approximations in the analysis, or the superimposed effect of settlement or lateral loads. In recognition of these facts, ACI Code 12.10 requires that every bar be continued at least a distance equal to the effective depth of the beam or 12 bar diameters (whichever is larger) beyond the point at which it is theoretically no longer required to resist stress, except at supports of simple spans and at the free end of cantilevers.

In addition, it is necessary that the calculated stress in the steel at each section be developed by adequate embedded length or end anchorage, or a combination of the two. For the usual case, with no special end anchorage, this means that the full development length l_d must be provided beyond critical sections at which peak stress exists in the bars. These critical sections are located at points of maximum moment and at points where adjacent terminated reinforcement is no longer needed to resist bending.[†]

Further reflecting the possible change in peak stress location, ACI Code 12.11 requires that at least one-third of the positive-moment steel (one-fourth in continuous spans) be continued uninterrupted along the same face of the beam a distance at least 6 in. into the support. When a flexural member is a part of a primary lateral load resisting system, positive-moment reinforcement required to be extended into the support must be anchored to develop the yield strength of the bars at the face of support to account for

[†] The ACI Code is ambiguous as to whether or not the extension length d or $12d_b$ is to be added to the required development length l_d. The Code Commentary presents the view that these requirements need not be superimposed, and Fig. 5.19 has been prepared on that basis. However, the argument just presented regarding possible shifts in moment curves or steel stress distribution curves leads to the conclusion that these requirements should be superimposed. In such cases, each bar should be continued a distance l_d plus the greater of d or $12d_b$ beyond the peak stress location.

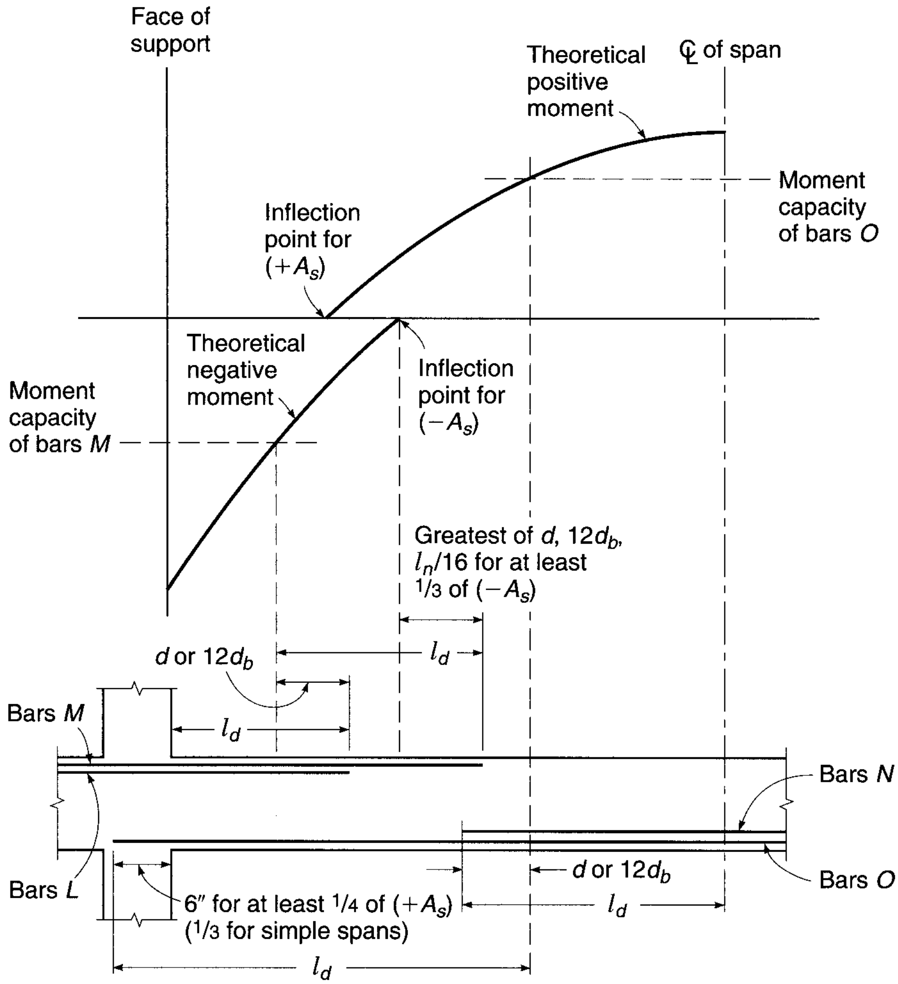


FIGURE 5.19
Bar cutoff requirements of the ACI Code.

the possibility of reversal of moment at the supports. According to ACI Code 12.12, at least one-third of the total reinforcement provided for negative moment at the support must be extended beyond the extreme position of the point of inflection a distance not less than one-sixteenth the clear span, or d, or $12d_b$, whichever is greatest.

Requirements for bar cutoff or bend point locations are summarized in Fig. 5.19. If negative bars L are to be cut off, they must extend a full development length l_d beyond the face of the support. In addition, they must extend a distance d or $12d_b$ beyond the theoretical point of cutoff defined by the moment diagram. The remaining negative bars M (at least one-third of the total negative area) must extend at least l_d beyond the theoretical point of cutoff of bars L and in addition must extend d, $12d_b$, or $l_n/16$ (whichever is greatest) past the point of inflection of the negative-moment diagram.

If the positive bars N are to be cut off, they must project l_d past the point of theoretical maximum moment, as well as d or $12d_b$ beyond the cutoff point from the positive-moment diagram. The remaining positive bars O must extend l_d past the theoretical point of cutoff of bars N and must extend at least 6 in. into the face of the support.

When bars are cut off in a tension zone, there is a tendency toward the formation of premature flexural and diagonal tension cracks in the vicinity of the cut end. This may result in a reduction of shear capacity and a loss in overall ductility of the beam.

ACI Code 12.10 requires special precautions, specifying that no flexural bar shall be terminated in a tension zone unless *one* of the following conditions is satisfied:

1. The shear is not over two-thirds of the design strength ϕV_n.
2. Stirrups in excess of those normally required are provided over a distance along each terminated bar from the point of cutoff equal to $\frac{3}{4}d$. These "binder" stirrups shall provide an area $A_v \geq 60 b_w s / f_{yt}$. In addition, the stirrup spacing shall not exceed $d/8\beta_b$, where β_b is the ratio of the area of bars cut off to the total area of bars at the section.
3. The continuing bars, if No. 11 (No. 36) or smaller, provide twice the area required for flexure at that point, and the shear does not exceed three-quarters of the design strength ϕV_n.

As an alternative to cutting off the steel, tension bars may be anchored by bending them across the web and making them continuous with the reinforcement on the opposite face. Although this leads to some complication in detailing and placing the steel, thus adding to construction cost, some engineers prefer the arrangement because added insurance is provided against the spread of diagonal tension cracks. In some cases, particularly for relatively deep beams in which a large percentage of the total bottom steel is to be bent, it may be impossible to locate the bend-up point for bottom bars far enough from the support for the same bars to meet the requirements for top steel. The theoretical points of bend should be checked carefully for both bottom and top steel.

Because the determination of cutoff or bend points may be rather tedious, particularly for frames that have been analyzed by elastic methods rather than by moment coefficients, many designers specify that bars be cut off or bent at more or less arbitrarily defined points that experience has proved to be safe. For nearly equal spans, uniformly loaded, in which not more than about one-half the tensile steel is to be cut off or bent, the locations shown in Fig. 5.20 are satisfactory. Note, in Fig. 5.20, that the beam at the exterior support at the left is shown to be simply supported. If the beam is monolithic with exterior columns or with a concrete wall at that end, details for a typical interior span could be used for the end span as well.

c. Special Requirements near the Point of Zero Moment

While the basic requirement for flexural tensile reinforcement is that a full development length l_d be provided beyond the point where the bar is assumed fully stressed to f_y, this requirement may *not* be sufficient to ensure safety against bond distress. Figure 5.21 shows the moment and shear diagram representative of a uniformly loaded continuous beam. Positive bars provided to resist the maximum moment at c are required to have a full development length beyond the point c, measured in the direction of decreasing moment. Thus l_d in the limiting case could be exactly equal to the distance from point c to the point of inflection. However, if that requirement were exactly met, then at point b, halfway from c to the point of inflection, those bars would have only one-half their development length remaining, whereas the moment would be three-quarters of that at point c, and three-quarters of the bar force must yet be developed. This situation arises whenever the moments over the development length are greater than those corresponding to a linear reduction to zero. Therefore, the problem is a concern in the positive-moment region of continuous uniformly loaded spans, but not in the negative-moment region.

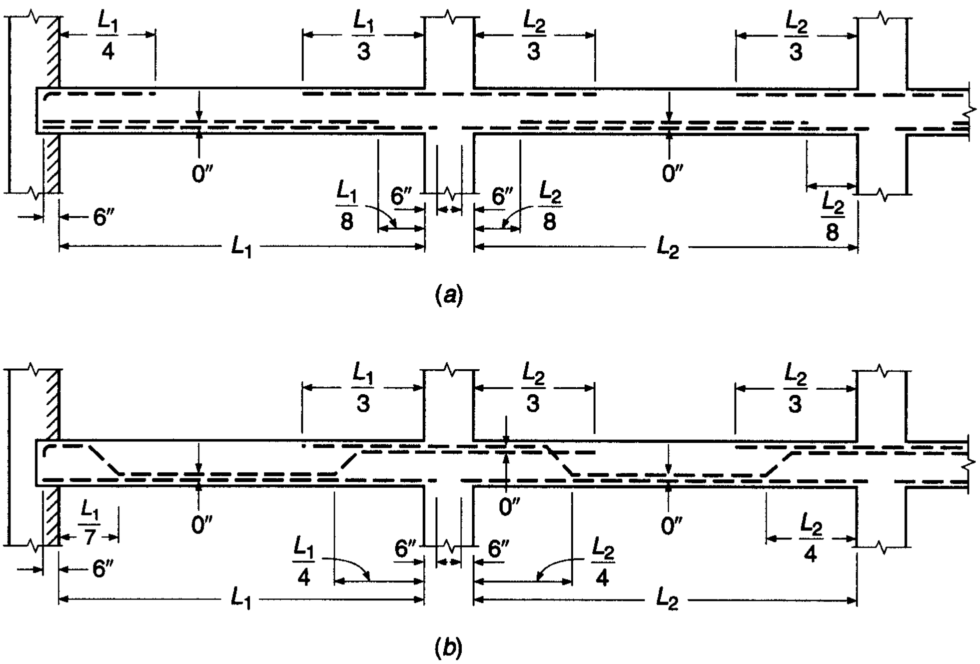


FIGURE 5.20
Cutoff or bend points for bars in approximately equal spans with uniformly distributed loads.

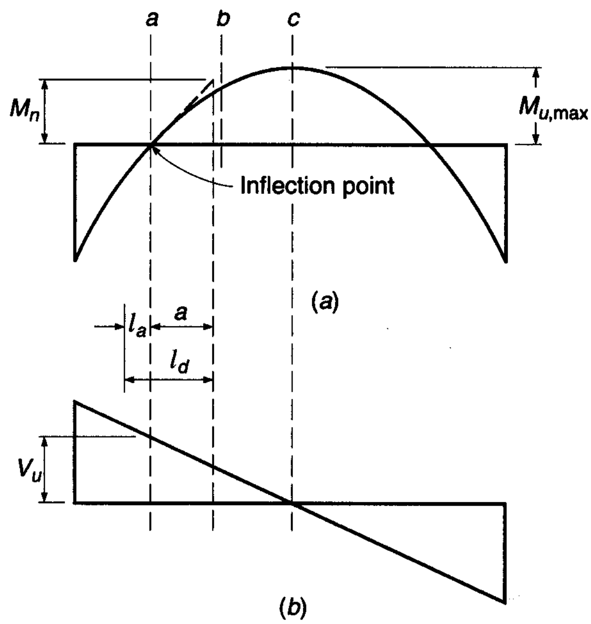


FIGURE 5.21
Development length requirement at point of inflection.

The bond force U per unit length along the tensile reinforcement in a beam is $U = dT/dx$, where dT is the change in bar tension in the length dx. Since $dT = dM/z$, this can be written

$$U = \frac{dM}{z\,dx} \tag{a}$$

that is, the bond force per unit length of bar, generated by bending, is proportional to the slope of the moment diagram. In reference to Fig. 5.21*a*, the maximum bond force U in the positive-moment region would therefore be at the point of inflection, and U would gradually diminish along the beam toward point c. Clearly, a conservative approach in evaluating adequacy in bond for those bars that are continued as far as the point of inflection (not necessarily the full A_s provided for M_u at point c) would be to require that the bond resistance, which is assumed to increase linearly along the bar from its end, be governed by the maximum rate of moment increase, i.e., the maximum slope dM/dx of the moment diagram, which for positive bending is seen to occur at the inflection point.

From elementary mechanics, it is known that the slope of the moment diagram at any point is equal to the value of the shear force at that point. Therefore, with reference to Fig. 5.21, the slope of the moment diagram at the point of inflection is V_u. A dashed line may therefore be drawn tangent to the moment curve at the point of inflection having the slope equal to the value of shear force V_u. Then if M_n is the nominal flexural strength provided by those bars that extend to the point of inflection, and if the moment diagram were conservatively assumed to vary linearly along the dashed line tangent to the actual moment curve, from the basic relation that $M_n/a = V_u$, a distance a is established:

$$a = \frac{M_n}{V_u} \tag{b}$$

If the bars in question were fully stressed at a distance a to the right of the point of inflection, and if the moments diminished linearly to the point of inflection, as suggested by the dashed line, then bond failure would not occur if the development length l_d did not exceed the distance a. The actual moments are less than indicated by the dashed line, so the requirement is on the safe side.

If the bars extend past the point of inflection toward the support, as is always required, then the extension can be counted as contributing toward satisfying the requirement for embedded length. Arbitrarily, according to ACI Code 12.11, a length past the point of inflection not greater than the larger of the beam depth d or 12 times the bar diameter d_b may be counted toward satisfying the requirement. Thus, the requirement for tensile bars at the point of inflection is that

$$l_d \le \frac{M_n}{V_u} + l_a \tag{5.11}$$

where M_n = nominal flexural strength assuming all reinforcement at section to be stressed to f_y

V_u = factored shear force at section

l_a = embedded length of bar past point of zero moment, but not to exceed the greater of d or $12d_b$

A corresponding situation occurs near the supports of simple spans carrying uniform loads, and similar requirements must be imposed. However, because of the beneficial effect of vertical compression in the concrete at the end of a simply supported span, which tends to prevent splitting and bond failure along the bars, the value M_n/V_u may be increased 30 percent for such cases, according to ACI Code 12.11. Thus, at the ends of a simply supported span, the requirement for tension reinforcement is

$$l_d \le 1.3\frac{M_n}{V_u} + l_a \tag{5.12}$$

The consequence of these special requirements at the point of zero moment is that, in some cases, smaller bar sizes must be used to obtain smaller l_d, even though requirements for development past the point of maximum stress are met.

It may be evident from review of Sections 5.10b and 5.10c that the determination of cutoff or bend points in flexural members is complicated and can be extremely time-consuming in design. It is important to keep the matter in perspective and to recognize that the overall cost of construction will be increased very little if some bars are slightly longer than absolutely necessary, according to calculation, or as dictated by ACI Code provisions. In addition, simplicity in construction is a desired goal, and can, in itself, produce compensating cost savings. Accordingly, many engineers in practice continue *all* positive reinforcement into the face of the supports the required 6 in. and extend *all* negative reinforcement the required distance past the points of inflection, rather than using staggered cutoff points.

5.11 STRUCTURAL INTEGRITY PROVISIONS

Experience with structures that have been subjected to damage to a major supporting element, such as a column, owing to accident or abnormal loading has indicated that total collapse can be prevented through relatively minor changes in bar detailing. If some reinforcement, properly confined, is carried continuously through a support, then even if that support is damaged or destroyed, catenary action of the beams can prevent total collapse. In general, if beams have bottom and top steel meeting or exceeding the requirements summarized in Sections 5.10b and 5.10c, and if binding steel is provided in the form of properly detailed stirrups, then that catenary action can usually be ensured.

According to ACI Code 7.13.2, beams at the perimeter of the structure (spandrel beams) must have continuous reinforcement passing through the region bounded by the longitudinal reinforcement of the columns consisting of at least one-sixth of the tension reinforcement required for negative moment at the support, but not less than two bars, and at least one-quarter of the tension reinforcement required for positive moment at midspan, but not less than two bars. At noncontinuous supports, the reinforcement must be anchored using a standard hook or a headed deformed bar to develop f_y at the face of the support. The continuous reinforcement must be enclosed by closed stirrups or closed ties perpendicular to the axis of the member, a closed cage of welded wire reinforcement with transverse wires perpendicular to the axis of the member, or spiral reinforcement (see Fig. 1.15). This transverse reinforcement must be anchored by a 135° standard hook (Fig. 5.9*b*) or a seismic hook (see Section 20.4) around a longitudinal bar, or where the concrete surrounding the anchorage is restrained against spalling by a flange or slab, by either a 90° or 135° standard hook around a longitudinal bar, as shown in Fig. 5.17*a* and *b*.

Figure 5.22 shows a two-piece stirrup that meets the requirements of ACI Code 7.13.2. Although the spacing of these stirrups is not specified, the requirements for minimum shear steel given in Section 4.5b provide guidance in regions where shear does not require closer spacing. The stirrups need not be extended through the joints. Overlapping pairs of U stirrups of the type shown in Fig. 5.17*d* are not permitted in perimeter beams because damage to the side cover concrete may cause both the stirrups and top longitudinal reinforcement to tear out of the concrete, thus preventing the longitudinal reinforcement from acting as a catenary.

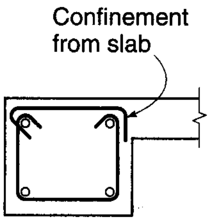


FIGURE 5.22
Two-piece stirrup meeting the requirements of ACI Code 7.13.2 for confinement of longitudinal integrity reinforcement in perimeter beams. The 90 degree hook must be placed adjacent to the slab.

The required continuity of longitudinal steel can be provided using top reinforcement spliced at midspan and bottom reinforcement spliced at or near the supports using Class B tension splices, or mechanical or welded splices (see Section 5.13).

In other than perimeter beams, when stirrups of the type shown in Fig. 5.22 are not provided, at least one-quarter of the positive-moment reinforcement required at midspan, but not less than two bars, must pass through the columns' longitudinal reinforcement and must be continuous. The requirements for anchoring this longitudinal reinforcement at noncontinuous supports and for splicing the bars to provide continuity are the same as for perimeter beams.

Note that these provisions require very little additional steel in the structure. At least one-quarter of the bottom bars must be extended 6 in. into the support by other ACI Code provisions; the structural integrity provisions merely require that these bars be made continuous or spliced. Similarly, other ACI Code provisions require that at least one-third of the negative bars be extended a certain minimum distance past the point of inflection; the structural integrity provisions for perimeter beams require only that one-half of those bars be further extended and spliced at midspan.

5.12 INTEGRATED BEAM DESIGN EXAMPLE

In this and in the preceding chapters, the several aspects of the design of reinforced concrete beams have been studied more or less separately: first the flexural design, then design for shear, and finally for bond and anchorage. The following example is presented to show how the various requirements for beams, which are often in some respects conflicting, are satisfied in the overall design of a representative member.

EXAMPLE 5.4

Integrated design of T beam. A floor system consists of single-span T beams 8 ft on centers, supported by 12 in. masonry walls spaced at 25 ft between inside faces. The general arrangement is shown in Fig. 5.23*a*. A 5 in. monolithic slab carries a uniformly distributed service live load of 165 psf. The T beams, in addition to the slab load and their own weight, must carry two 16,000 lb equipment loads applied over the stem of the T beam 3 ft from the span centerline as shown. A complete design is to be provided for the T beams, using concrete of 4000 psi strength and bars with 60,000 psi yield stress. (*Note*: Because normalweight concrete is used, $\lambda = 1.0$ and, as such, will be dropped from the calculations for shear and bond.)

Solution. According to the ACI Code, the span length is to be taken as the clear span plus the beam depth, but need not exceed the distance between the centers of supports. The latter

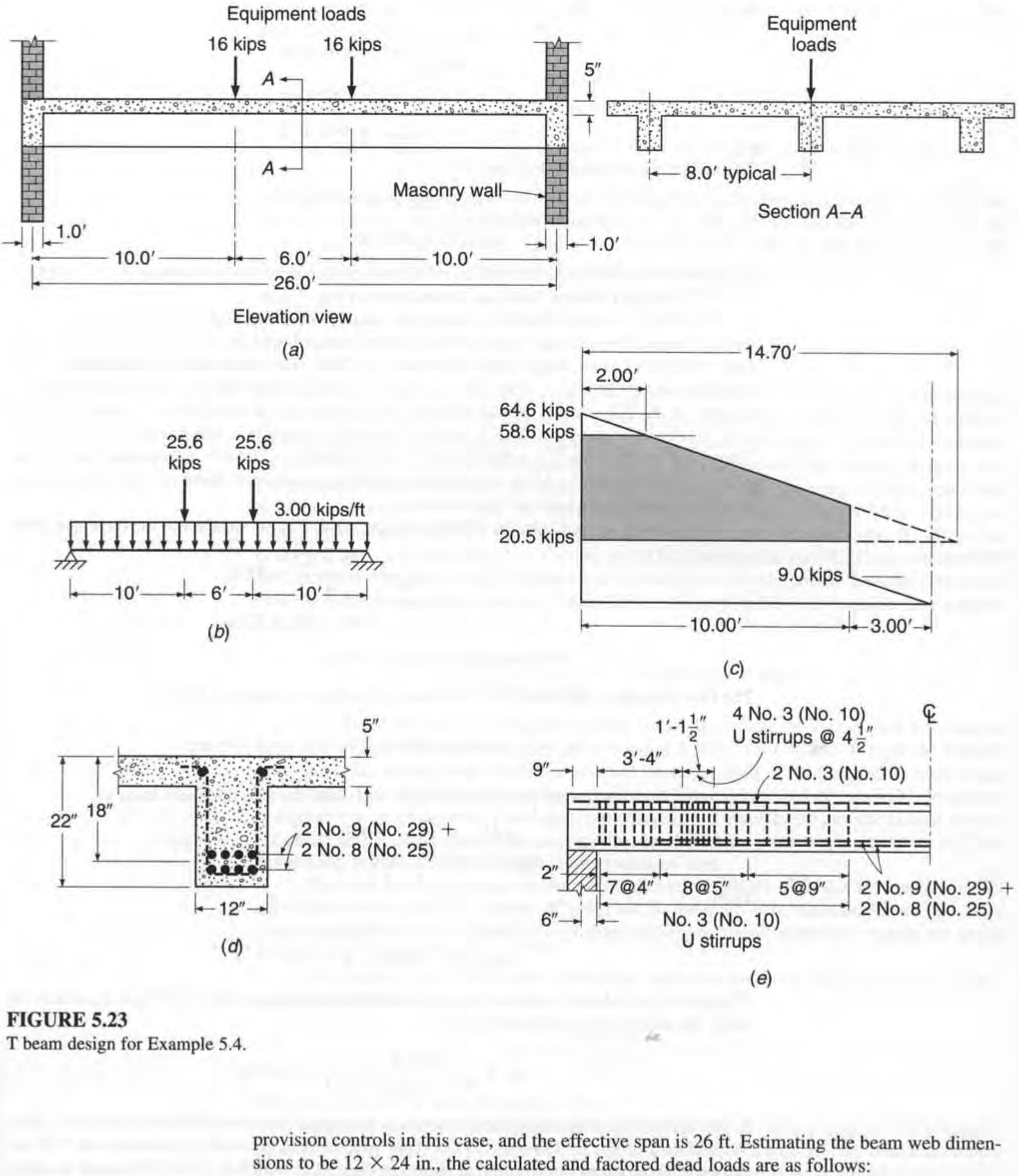


FIGURE 5.23
T beam design for Example 5.4.

provision controls in this case, and the effective span is 26 ft. Estimating the beam web dimensions to be 12 × 24 in., the calculated and factored dead loads are as follows:

Slab:

$$\frac{5}{12} \times 150 \times 7 = 440 \text{ lb/ft}$$