0.5 Limits

Def: *if f(x) is approaches to L when x approaches to x_0 from left and right then we say that the limit exist at $x = x_0$.*

or $\lim_{x\to x_0} f(x) = L$

If $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = L$

Theorem: if $\lim_{x\to a} f(x) = L_1$ & $\lim_{x\to a} g(x) = L_2$ and k is constant then:

1) $\lim_{x\to a} \left(f(x) \pm g(x)\right) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L_1 \pm L_2$
2) $\lim_{x\to a} k\, f(x) = k \lim_{x\to a} f(x) = k\, L_1$
3) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}$ Provided $L_2 \neq 0$
4) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$ Provided $L_1 \geq 0$

Ex: ***Find*** $\lim_{x\to 5}(x^2 - 4x + 3)$

$= \lim_{x\to 5} x^2 - 4 \lim_{x\to 5} x + \lim_{x\to 5} 3$

$= (5)^2 - 4(5) + (3) = 8$

Ex: ***Find*** $\lim_{x\to 2} \frac{x^3-8}{x-2}$

$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)} = 12$

Ex: ***Find*** $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

$= \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} * \frac{\sqrt{x}+1}{\sqrt{x}+1}$

$= \lim_{x\to 1} \frac{(x-1)}{(x-1)(\sqrt{x}+1)} = \frac{1}{2}$

Ex: ***Find*** $\lim_{x\to 0} \frac{x}{|x|}$

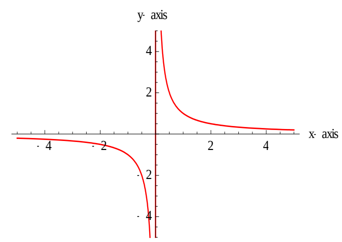
$= \lim_{x\to 0} \frac{x}{|x|} = \begin{cases} \frac{x}{x} & x \geq 0 \\ \frac{x}{-x} & x < 0 \end{cases} = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

and the limit does not exist

Note :

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$



$$\lim_{x\to \infty} \frac{1}{x} = 0$$

$$\lim_{x\to -\infty} \frac{1}{x} = 0$$

Ex: find $\lim\limits_{x\to -\infty} \frac{x-2}{x^2+2x+1}$

$$= \lim_{x\to -\infty} \frac{\frac{1}{x}-\frac{2}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}} = \frac{0}{1} = \mathbf{0}$$

Ex: find $\lim\limits_{x\to \infty} \frac{5x^2-2}{3x^2+2x+1} = \lim\limits_{x\to -\infty} \frac{5-\frac{2}{x^2}}{3+\frac{2}{x}+\frac{1}{x^2}} = \frac{5}{3}$

Ex: find $\lim\limits_{x\to \infty} \frac{x^3-2}{x^2+2x+1} = \lim\limits_{x\to \infty} \frac{1-\frac{2}{x^3}}{\frac{1}{x}+\frac{2}{x^2}+\frac{1}{x^3}} = \frac{1}{0} = \infty$ and the limit does not exist

Ex: find $\lim\limits_{x\to \infty} \frac{2-x}{\sqrt{7+6x^2}} = \lim\limits_{x\to \infty} \frac{\frac{2}{x}-1}{\sqrt{\frac{7}{x^2}+6}} = \frac{-1}{\sqrt{6}}$

Ex: find $\lim\limits_{x\to 1} \frac{x^3+x^2-5x+3}{x^3-3x+2} = \lim\limits_{x\to 1} \frac{(x-1)(x^2+2x-3)}{(x-1)(x^2+x-2}$

$$= \lim_{x\to 1} \frac{(x-1)(x+3)}{(x-1)(x+2)} = \frac{4}{3}$$

$$\begin{array}{r|l} & x^2+2x-3 \\ (x-1) & x^3+x^2-5x+3 \\ & \mp x^3 \pm x^2 \\ \hline & 2x^2-5x+3 \\ & \mp 2x^2 \pm 2x \\ \hline & -3x+3 \\ & \pm 3x \mp 3 \\ \hline & 0+0 \end{array}$$

$$\begin{array}{r|l} & x^2+x-2 \\ (x-1) & x^3-3x+2 \\ & \mp x^3 \pm x^2 \\ \hline & x^2-3x+2 \\ & \mp x^2 \pm x \\ \hline & -2x+2 \\ & \pm 2x \mp 2 \\ \hline & 0+0 \end{array}$$

Theorem : 1. $\lim_{x\to 0}\frac{\sin ax}{ax} = 1 \ , a \neq 0$

2. $\lim_{x\to 0}\sin x = 0$

3. $\lim_{x\to 0}\cos x = 1$

4. $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

5. $\lim_{x\to \infty}\frac{\sin x}{x} = 0$

6. $\lim_{x\to \infty}\frac{\cos x}{x} = 0$

Ex: Find $\lim_{x\to 0}\frac{\sin 6x}{\sin 8x}$

$$= \frac{6\lim_{x\to 0}\frac{\sin 6x}{6x}}{8\lim_{x\to 0}\frac{\sin 8x}{8x}} = \frac{6}{8} = \frac{3}{4}$$

Ex: find $\lim_{x\to 0}\frac{\sin x}{1-\cos x}$

$$= \lim_{x\to 0}\frac{\sin x}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x(1+\cos x)}{1-\cos^2 x} = \lim_{x\to 0}\frac{\sin x(1+\cos x)}{\sin^2 x}$$

$$= \lim_{x\to 0}\frac{(1+\cos x)}{\sin x} = \frac{2}{0} = \infty$$ and the limit does not exist

Ex: find $\lim_{x\to \infty}\sin\left(\frac{\pi x}{2-3x}\right)$

$$= \sin \lim_{x\to \infty}\left(\frac{\pi x}{2-3x}\right)$$

$$= \sin \lim_{x\to \infty}\left(\frac{\pi}{\frac{2}{x}-3}\right) = \sin\left(-\frac{\pi}{3}\right) = \frac{-\sqrt{3}}{2}$$

Ex: find $\lim_{x\to\infty} x\left(1-\cos\frac{1}{x}\right)$

Let $t=\frac{1}{x} \quad \to \quad x=\frac{1}{t} \qquad if\ \ x\to\infty\ \ then\ t\to 0$

$$=\lim_{t\to 0}\frac{(1-\cos t)}{t}=0$$

Ex: find $\lim_{x\to 2}\frac{\cos\left(\frac{\pi}{x}\right)}{x-2}$

$$=\lim_{x\to 2}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{x}\right)}{x-2}$$

$$=\lim_{x\to 2}\frac{\sin\left(\frac{\pi(x-2)}{2x}\right)}{x-2}$$

$$=\lim_{x\to 2}\frac{\sin\left(\frac{\pi(x-2)}{2x}\right)}{\frac{2x}{\pi}\frac{\pi\,(x-2)}{2x}}$$

$$=\lim_{x\to 2}\frac{\pi}{2x}\cdot\boxed{\lim_{x\to 2}\frac{\sin\left(\frac{\pi(x-2)}{2x}\right)}{\frac{\pi\,(x-2)}{2x}}}=\frac{\pi}{4}$$

=1

Ex: find $\lim_{x\to 1}\frac{\sin(\pi x)}{x-1}$

$$=\lim_{x\to 1}\frac{\sin(\pi-\pi x)}{x-1}$$

$$=\lim_{x\to 1}\frac{-\sin(\pi x-\pi)}{x-1}$$

$$=-\pi\boxed{\lim_{x\to 1}\frac{\sin(\pi(x-1))}{\pi(x-1)}}=-\pi$$

=1

Ex: find $\displaystyle \lim_{x\to \pi/4} \frac{\tan(x)-1}{x-\pi/4}$

Let $t = x-\pi/4 \;\to\; x = t+\pi/4$, $if\;\; x \to \pi/4$ then t $\to 0$

$$= \lim_{t\to 0} \frac{\tan(t+\pi/4)-1}{t}$$

$$= \lim_{t\to 0} \frac{\frac{\tan t+\tan(\pi/4)}{1-\tan t\tan(\pi/4)}-1}{t}$$

$$= \lim_{t\to 0} \frac{\frac{\tan t+1}{1-\tan t}-1}{t} = \lim_{t\to 0} \frac{\frac{\tan t+1-(1-\tan t)}{1-\tan t}}{t}$$

$$= \lim_{t\to 0} \frac{2\tan t}{t\,(1-\tan t)} = 2\underbrace{\lim_{t\to 0}\frac{\tan t}{t}}_{=1} \cdot \lim_{t\to 0}\frac{1}{(1-\tan t)} = 2$$

Ex: find $\displaystyle \lim_{x\to\infty} \frac{2x+x\sin x}{5x^2-2x+1}$

$$= \lim_{x\to\infty} \frac{\frac{2x}{x^2}+\frac{x\sin x}{x^2}}{\frac{5x^2}{x^2}-\frac{2x}{x^2}+\frac{1}{x^2}} = \frac{\overbrace{\lim_{x\to\infty}\frac{2}{x}}^{0=}+\overbrace{\lim_{x\to\infty}\frac{\sin x}{x}}^{=0}}{\underbrace{\lim_{x\to\infty}5}_{=5}-\underbrace{\lim_{x\to\infty}\frac{2}{x}}_{=0}+\underbrace{\lim_{x\to\infty}\frac{1}{x^2}}_{=0}} = \frac{0}{5} = 0$$