0.6 Continuity

Def : *a fun. f(x) is said to be continuous at a if it satisfy*

1) $f(x)$ is defined at a
2) $\lim_{x \to a} f(x)$ exist
3) $\lim_{x \to a} f(x) = f(a)$

Ex: ***is the fun.*** $f(x)= \begin{cases} 2x+3 & x \le 4 \\ 7+\frac{16}{x} & x > 4 \end{cases}$ ***continuous at*** $x = 4$

1) $f(4) = 2(4)+3 = 11$ define at 4

2) $\lim_{x \to 4^+} f(x) = \lim_{x \to 4} 7+\frac{16}{x} = 11$ exist

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4} (2x+3) = 11$

3) $\lim_{x \to 4} f(x) = f(4) = 11$

$\therefore f$ is continuous at $x = 4$

Ex: *where is the function* $f(x) = \frac{x^2-9}{x^2-5x+6}$ *continuous.*

$x^2-5x+6=0$

$(x-2)(x-3)=0$

$x=2$ and $x=3$

$\therefore$ $f(x)$ is continuous at every where except $x = 2$ and $x = 3$

or the points of discontinuity at $x = 2$ and $x = 3$

: find the value of k that make the function.

$f(x)= \begin{cases} kx^2 & x \le 2 \\ 2x+k & x>2 \end{cases}$ *continuous at* $x=2$

$\lim_{x \to 2} f(x) = f(2)$

$\lim_{x \to 2} 2x+k = k(2^2)$

$4+k = 4k \rightarrow 3k = 4 \quad \rightarrow k = \frac{4}{3}$

Ex: *find* $lim_{x\to\infty}\left[cos\left(\frac{\pi x^2+1}{x^2+3}\right)\right]$

$$\therefore \lim_{x\to\infty}\left[cos\left(\frac{\pi x^2+1}{x^2+3}\right)\right] = cos\left[\lim_{x\to\infty}\left(\frac{\pi x^2+1}{x^2+3}\right)\right]$$

$$= cos\left[\lim_{x\to\infty}\frac{\pi+\frac{1}{x^2}}{1+\frac{3}{x^2}}\right] = \cos\pi = -1$$

Ex: *find* $\lim_{x\to 0}\frac{x\,sinx}{1-cosx}$

$$= \lim_{x\to 0}\frac{x\,sinx}{1-cosx} * \frac{1+cosx}{1+cosx}$$

$$= \lim_{x\to 0}\frac{xsinx(1+cosx)}{1-cos^2x}$$

$$= \lim_{x\to 0}\frac{x\,sinx\,(1+cosx)}{sin^2x} = \lim_{x\to 0}\frac{x\,(1+cosx)}{\sin x} = \frac{\lim_{x\to 0}(1+cosx)}{\lim_{x\to 0}\sin x/x} = \frac{2}{1}$$

Chapter 1

1.1 Differentiation

Def.: Let *y= f(x)* be a function, then the derivative of *f* with respect to x is defined by $\frac{dy}{dx} = \acute{y} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Ex: use definition of derivative to find the derivative of *f(x) = x²*

$$\frac{dy}{dx} = \acute{y} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0} 2x+h = 2x$$

Theorem: if *f* and *g* are differentiable functions at *x* and *k* is constant then:

1. $\frac{d}{dx}(k) = 0$
2. $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$
3. $\frac{d}{dx}\left(f(x).g(x)\right) \quad = f(x).\frac{d}{dx}g(x) + g(x).\frac{d}{dx}f(x)$
4. $\frac{d}{dx}\left(kf(x)\right) \quad = k\frac{d}{dx}f(x)$
5. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \quad = \frac{g(x).\frac{d}{dx}f(x) - f(x).\frac{d}{dx}g(x).}{g(x)^2}$ $\quad provided\ g(x) \neq 0$
6. $\frac{d}{dx}(x^n) \quad = n\,x^{n-1}$

Higher Derivatives:

If f is differentiable function at x then the higher derivatives define:

$$f''(x) = \frac{d^2 f}{dx^2}$$

$$f'''(x) = \frac{d^3 f}{dx^3}$$

$$f^{(4)}(x) = \frac{d^4(x)}{dx^4}$$

Ex: Let $y = \left(\frac{x-1}{x+1}\right)(2x^7 - x^2)$ **find** $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \left(\frac{x-1}{x+1}\right)(14x^6 - 2x) + (2x^7 - x^2)\frac{(x+1).1-(x-1).1}{(x+1)^2}$$

$$\frac{dy}{dx} = \left(\frac{x-1}{x+1}\right)(14x^6 - 2x) + \frac{2(2x^7 - x^2)}{(x+1)^2}$$

Ex: Find $\frac{d}{d\lambda}[\frac{\lambda\lambda_o + \lambda^6}{2-\lambda_o}]$

$$= \frac{1}{2-\lambda_o}\frac{d}{d\lambda}[\lambda\lambda_o + \lambda^6]$$

$$= \frac{1}{2-\lambda_o}[\lambda_o + 6\lambda 5] = \frac{[\lambda_o + 6\lambda 5]}{2-\lambda_o}$$

Ex: Find $\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{d^2y}{dx^2}(1)$ **where** $y = 6x^5 - 4x^2$

$$\frac{dy}{dx} = 30x^4 - 8x$$
$$\frac{d^2y}{dx^2} = 120x^3 - 8$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 112$$

Ex: **Find an equation for the line that is tangent to $y = (1-x)(1+x)$ at the point where x=2.**

$y = (1-x)(1+x) = 1 - x^2$

$\acute{y} = -2x \quad \rightarrow \quad slope = m = \acute{y}(2) = -4$

$y(2) = 1 - (2)^2 = -3 \quad \rightarrow \quad (x_1, y_1) = (2, -3)$

The equation of the line is $(y - y_1) = m(x - x_1)$

$$y - (-3) = (-4)(x - 2)$$

$$y = -4x + 5$$

Derivatives of Trigonometric Functions:

1. $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$
2. $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$
3. $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$
4. $\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$
5. $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$
6. $\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$

Ex: Find $\acute{f}(x)$ if $f(x) = \frac{\sin x \ \sec x}{1+x\tan x}$

$$\acute{f}(x) = \frac{(1+x\tan x)\,(\sin x\,.\ \sec x \tan x + \sec x\,.\cos x) - (\sin x \sec x)(x.sec^2 x + \tan x\,.1)}{(1+x\tan x)^2}$$

The Chain rule:

1. If y = $f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Or $\frac{dy}{dx} = f'(g(x))g'(x)$

2. If y = f (t) and $x = g(t)$ then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Ex: ***Let*** $y = u^{10}$ **and** $u = x^3 + 7x + 1$, ***Find*** $\frac{dy}{dx}$ ***in two ways.***

1. $\frac{dy}{du} = 10\,u^9$, $\frac{du}{dx} = 3x^2 + 7$ $\rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = (10\,u^9).(3x^2 + 7)$$

$$\frac{dy}{dx} = [10\,(x^3 + 7x + 1)^9].(3x^2 + 7)$$

2. $y = (x^3 + 7x + 1)^{10}$ $\rightarrow \frac{dy}{dx} = [10\,(x^3 + 7x + 1)^9].(3x^2 + 7)$

: ***Let*** $y = t^2$ ***and*** $x = 5\,t + 2$ **Find** $\frac{dy}{dx}$

$\frac{dy}{dt} = 2t$, $\frac{dx}{dt} = 5$ $\rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{5} = \frac{2}{5}\left(\frac{x-2}{5}\right)$

Implicit Differentiation:

Ex: Find $\frac{dy}{dx}$ ***if*** $5\,y^2 + \sin y = x^2$

$$10\,y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x$$

$$(10\,y + \cos y)\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{(10\,y + \cos y)}$$

Ex: find $\frac{dw}{d\lambda}$ **if** $a^2w^2 + b^2\lambda^2 = 1$ **where a , b are constants.**

$$2a^2\,w\,\frac{dw}{d\lambda} + 2\,b^2\,\lambda = 0 \rightarrow \frac{dw}{d\lambda} = \frac{-b^2\,\lambda}{a^2\,w}$$