2.1 Integration

if $\frac{d}{dx}[f(x)] = F(x)$ **then** $\int F(x)\,dx = f(x) + c$

Properties of integration:

1. $\int c\, f(x)\, dx = c \int f(x)\, dx$
2. $\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
3. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1$

$$\int \cos u\, du = \sin u + c$$
$$\int \sin u\, du = -\cos u + c$$
$$\int \sec^2 u\, du = \tan u + c$$
$$\int \csc^2 u\, du = -\cot u + c$$
$$\int \sec u \tan u\, du = \sec u + c$$
$$\int \csc u \cot u\, du = -\csc u + c$$

Ex / evaluate $\int \frac{t^2 - 2t^4}{t^4}\, dt = \int \left(\frac{1}{t^2} - 2\right) dt$

$= \int (t^{-2} - 2)\, dt$

$= \frac{t^{-1}}{-1} - 2t + c = -\frac{1}{t} - 2t + c$

Ex / **evaluate** $\int \frac{\cos x}{\sin^2 x}\, dx = \int \frac{1}{\sin x} \frac{\cos x}{\sin x}\, dx$

$= \int \csc x \cot x\, dx = -\csc x + c$

Ex / evaluate $\int x^3 \sqrt{x}\, dx = \int x^3 x^{1/2}\, dX$

$= \int x^{7/2}\, dx = \frac{x^{9/2}}{9/2} + c = \frac{2}{9} x^{9/2} + c$

Ex / evaluate $\int (2+ y^2)^2\, dy = \int (4 + 4y^2 + y^4)\, dx$

$$= 4y + \frac{4y^3}{3} + \frac{y^5}{5} + c$$

Ex / evaluate $\int sec\, x\,(\sec x + \tan x)\, dx$

$$= \int (sec^2 x + sec\, x\ tan\, x)\, dx = tan\, x + sec\, x + c$$

Ex / evaluate $\int \frac{\cos^3\theta - 5}{\cos^2\theta}\, d\theta = \int (\cos\theta - 5\ \sec^2\theta)\, d\theta$

$$= \sin\theta - 5\ \tan\theta + c$$

Ex/ $\int \tan^2 x\, dx = \int (sec^2 x - 1)\, dx = \tan x - x + c$

Integration by substitution

Ex / $\int x^2 \sqrt{x-1}\, dx$

Let $u = x-1 \Rightarrow x = u + 1 \Rightarrow dx = du$

$$\int (u+1)^2 \sqrt{u}\quad du = \int (u^2 + 2u + 1) u^{1/2}\ du$$

$$= \int (u^{5/2} + 2\, u^{3/2} + u^{1/2})\ du$$

$$= \frac{u^{7/2}}{7/2} + 2\ \frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} + c$$

$$= \frac{2}{7}\,(x-1)^{7/2} + \frac{4}{5}\,(x-1)^{\frac{5}{2}} = \frac{2}{3}\,(x-1)^{3/2} + c$$

The Definite Integral:

If $\frac{d}{dx}\,[F(x)] = f(x)$ Then $\int_a^b f(x)\, dx = F(b) - F(a)$

Ex / evaluate $\int_0^3 (x^3 - 4x + 1) dx$

$$= \frac{x^4}{4} - 4\frac{x^2}{2} + x\ \Big]_0^3 = \left(\frac{81}{4} - 18 + 3\right) - (0) = \frac{21}{4}$$

Properties

1) $\int_a^a f(x)\ dx = o$
2) $\int_b^a f(x)\ dx = -\int_a^b f(x)\,dx$
3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ if $a < c < b$

Ex / $\int_1^1 x^2\,dx = o$

Ex / $\int_4^0 x\ dx = -\int_0^4 x\,dx = -\left.\frac{x^2}{2}\right]_o^4 = -8$

2.2 Application of the definite integral:

a. Area under a curve:

Ex / *Find the area under the curve* $y = \cos x$ *over the interval* $[\,0, \frac{\pi}{2}\,]$

since $\cos x \geq 0$ *for* $0 \leq x \leq \pi/2$

$$\therefore A = \int_0^{\pi/2} \cos x\,dx = \sin x\Big]_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1$$

Ex: Show that $\int_o^{\pi/2} \sin^n x\,dx = \int_o^{\pi/2} \cos^n x\,dx$

$$\int_o^{\pi/2} \sin^n x\,dx = \int_o^{\pi/2} \cos\left(\frac{\pi}{2} - x\right)^n dx$$

$$let\quad u = \frac{\pi}{2} - x \Rightarrow du = -dx$$

x	u
0	$\pi/2$
$\pi/2$	0

$$= -\int_{\pi/2}^{o} \cos^n u\,du = \int_o^{\pi/2} \cos^n u\,du$$

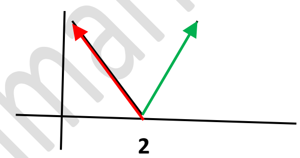
Ex / Evaluate $\int_0^6 f(x)\,dx\,,\quad f(x) = \begin{cases} x^2, & x < 2 \\ 3x-2, & x \geq 2 \end{cases}$

$$\int_0^6 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^6 (3x-2)\,dx$$

$$= \frac{x^3}{3}\Big]_0^2 + \left(\frac{3x^2}{2} - 2x\right)\Big]_2^6 = \frac{128}{3}$$

Ex / $\int_1^5 |x-2|\,dx = \int_1^2 -(x-2)\,dx + \int_2^5 (x-2)\,dx$

$$= \left[-\frac{x^2}{2} + 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^5 = 5$$



The Second Fundamental Theorem of Integral

if $F(x) = \int_a^x f(t)\,dt$ **then** $F'(x) = f(x)$

Ex / Find $F'(x)$ **if** $F(x) = \int_0^x \frac{\sin t}{t}\,dx$

By the 2nd fundamental theorem of integral. $F'(x) = \frac{\sin x}{x}$

Ex / Find $\frac{d}{dx}\left[\int_0^x \frac{dt}{1+\sqrt{t}}\right]$

by the 2nd fundamental theorem of integral

$$\frac{d}{dx}\left[\int_0^x \frac{dt}{1+\sqrt{t}}\right] = \frac{1}{1+\sqrt{x}}$$

1- **if** $F(x) = \int_a^{g(x)} f(t)dt$ **then** $F'(x) = f(g(x))\acute{g}(x)$

2- **if** $F(x) = \int_{h(x)}^{g(x)} f(t)dt$ **then** $F'(x) = f(g(x))g'(x) - f(h(x))h'(x)$

Ex / $\frac{d}{dx}\left[\int_3^{sinx} \frac{1}{1+t^2} dt\right] = \frac{1}{1+(\sin x)^2} \,.\, (\cos x) = \frac{\cos x}{1+ sin^2\ x}$

Ex / $\frac{d}{dx}\left[\int_{x^2}^{x^3} sin^2\ t\ dt\right]$ = $sin^2\ (x^3).(3x^2) - sin^2\ (x^2).(2x)$

= $3x^2\ sin^2\ (x^3) - 2x\sin^2(x^2)$

Ex / evaluate $\int \frac{x^{1/3}\ dx}{x^{8/3}+2x^{4/3}+1}$

= $\int \frac{x^{1/3}\ dx}{(x^{4/3}\ + 1)^2}$

= $\int (x^{4/3} + 1)^{-2} \left(x^{1/3} dx\right)$

Let u = $x^{4/3} + 1 \implies du = \frac{4}{3} x^{1/3}\ dx$

$\frac{3}{4}\ du = x^{1/3}\ dx$

$= \frac{3}{4} \int u^{-2}\ du = \frac{3}{4}\ \frac{u^{-1}}{-1} + c = -\frac{3}{4}\,(x^{\frac{4}{3}} + 1)^{-1} + c$