Area between two curves:

a. If $f(x) \geq g(x)$ for $a \leq x \leq b$ then $A = \int_a^b [f(x) - g(x)]\,dx$

b. If $f(y) \geq g(y)$ for $c \leq y \leq d$ then $A = \int_c^d [(f(y) - g(y)]\,dy$

Ex / Find the area of the region enclosed between the curves
$y = x^2$ ***and*** $y = x + 6$

$$x^2 = x + 6$$

$$x^2 - x - 6 = 0$$

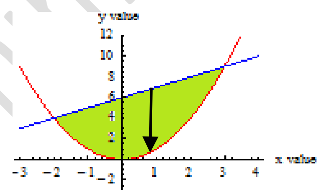
$$(x + 2)(x - 3) = 0$$

$$\Downarrow \qquad \Downarrow$$

$$x = -2 \qquad x = 3$$

$$\Downarrow$$

Check $x = 0$



$y_1 = 0$ (*below*) $\quad y_2 = 6$ (*above*)

A= $\int_{-2}^{3} (y_2 - y_1)\,dx = \int_{-2}^{3} [(x + 6) - x^2]\,dx$

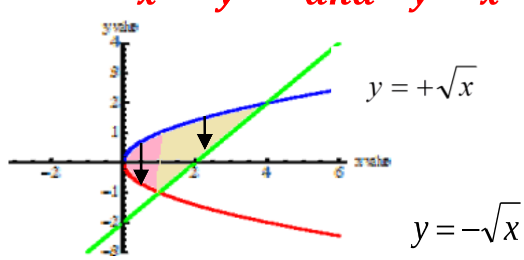
$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \left(\frac{(3)^2}{2} + 6(3) - \frac{(3)^3}{3}\right) - \left(\frac{(-2)^2}{2} + 6(-2) - \frac{(-2)^3}{3}\right)$$

$$= \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6}$$

Ex / Find the area of the region enclosed between the curves

$x = y^2$ ***and*** $y = x - 2$



Method (1) $y_1 = \sqrt{x}$, $y_2 = -\sqrt{x}$, $y_3 = x - 2$

$$\sqrt{x} = -\sqrt{x} \qquad\qquad \pm\sqrt{x} = x - 2$$

$$\sqrt{x} + \sqrt{x} = 0 \qquad\qquad x = (x-2)^2$$

$$2\sqrt{x} = 0 \qquad\qquad x^2 - 5x + 4 = 0$$

$$\sqrt{x} = 0 \qquad\qquad (x-1)(x-4) = 0$$

$$\Downarrow \qquad \Downarrow$$

$$x = 0 \qquad\qquad x = 1 \qquad x = 4$$

$$A = A_1 + A_2$$

$$= \int_0^1 [\sqrt{x} - (-\sqrt{x})]dx + \int_1^4 [\sqrt{x} - (x-2)]dx$$

$$= \int_0^1 2\sqrt{x}\; dx + \int_1^4 [\sqrt{x} - x + 2]dx$$

$$= 2\int_0^1 x^{1/2}\,dx + \int_1^4 [x^{1/2} - x + 2]dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^1 + \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right]_1^4 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$$

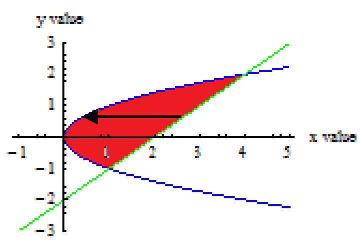
Method (2)

$$y^2 = y + 2 \qquad y^2 - y - 2 = 0$$

$$(y+1)(y-2) = 0$$

$$\Downarrow \qquad \Downarrow$$

Y=-1 *y=2*



$$A = \int_{-1}^{2} [(y+2) - y^2]\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{(2)^2}{2} - 2(2) - \frac{(2)^3}{3}\right] - \left[\frac{(-1)^2}{2} - 2(-1) - \frac{(-1)^3}{3}\right] = \frac{9}{2}$$

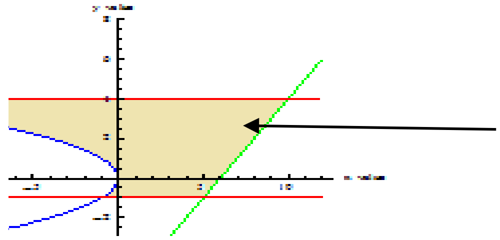
Ex / find the area of the region enclosed between the curves

$$y^2 = -x \ , \ y = x - 6 \ , \ y = -1 \ , \ y = 4$$

$y^2 = -x$, $y = x - 6$

$\Downarrow$ $\Downarrow$

$x = -y^2$ $x = y+6$



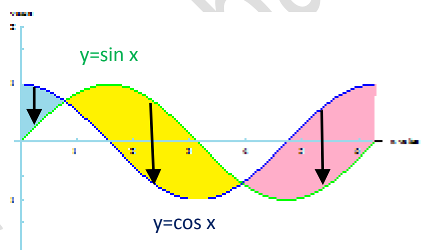
$$A = \int_{-1}^{4} (x_1 - x_2)\,dy$$

$$= \int_{-1}^{4} [(y+6) - (-y^2)]\,dy$$

$$= \int_{-1}^{4} [y + 6 + y^2]\,dy$$

$$= \left[\frac{y^2}{2} + 6y + \frac{y^3}{3}\right]_{-1}^{4} = \frac{160}{3}$$

Ex / *Find the area bounded by y = sin x , y = cosx , from x = 0 to x = 2π Sketch the region)*

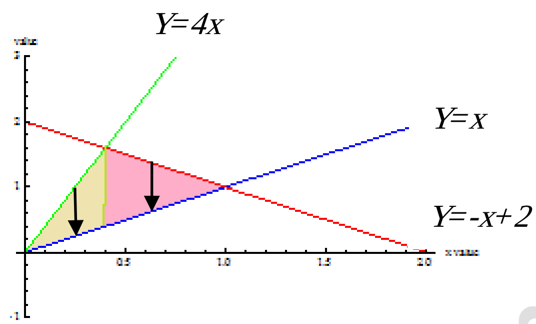


$A = A_1 + A_2 + A_3$

$$= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{5\frac{\pi}{4}} (\sin x - \cos x)\,dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\,dx$$

$$= \sin x + \cos x\,]_0^{\pi/4} - \cos x - \sin x\,]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + \sin x + \cos x\,]_{\frac{5\pi}{4}}^{2\pi} = 4\sqrt{2}$$

Ex / Find the area of the region enclosed between the curves
$y = x$, $y = 4x$, $y = -x + 2$, *sketch the region* .



$y_1 = y_2$

$4x = -x + 2$

$5x = 2 \xrightarrow{yields} x = \frac{2}{5}$

$y_1 = y_3$

$4x = x \xrightarrow{yields} x = 0$

$y_2 = y_3$

$-x + 2 = x$

$2x = 2 \xrightarrow{yields} x = 1$

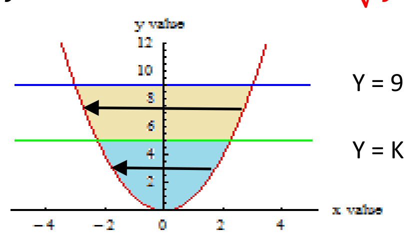
$A = A_1 + A_2$

$$= \int_0^{2/5} (4x - x)dx + \int_{2/5}^{1} [(-x + 2) - x]\, dx$$

$$= \left[4\frac{x^2}{2} - \frac{x^2}{2}\right]_0^{2/5} + \left[-\frac{x^2}{2} + 2x - \frac{x^2}{2}\right]_{2/5}^{1}$$

$$= \left[3\frac{x^2}{2}\right]_0^{2/5} + \left[-x^2 + 2x\right]_{2/5}^{1} = \frac{3}{5}$$

Ex: Find k that divide the area between $y = x^2$ and $y = 9$ into two equal parts . $y = x^2 \rightarrow x = \pm\sqrt{y}$



to be equal parts $A_1 = A_2$

$$\int_k^9 [\sqrt{y} - (-\sqrt{y})]\, dy = \int_0^k [\sqrt{y} - (-\sqrt{y})]dy$$

$$2\int_k^9 \sqrt{y}\, dy = 2\int_0^k \sqrt{y}\, dy$$

$$\int_k^9 y^{1/2}\, dy = \int_0^k y^{1/2}\, dy$$

$$\frac{y^{3/2}}{3/2}\Big]_k^9 = \frac{y^{3/2}}{3/2}\Big]_0^k$$

$$\frac{2}{3}\left[y^{3/2}\right]_k^9 = \frac{2}{3}\left[y^{3/2}\right]_0^k$$

$$9^{3/2} - k^{3/2} = k^{3/2} - 0$$

$$27 = 2\,k^{3/2}$$

$$27/2 = k^{3/2} \rightarrow \quad (27/2)^{2/3} = k$$

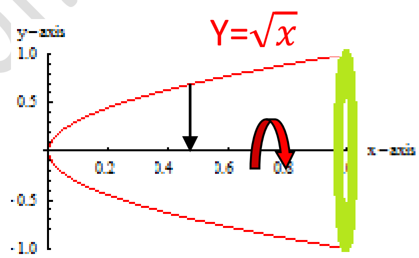
2.3 -Volumes

a. Volumes by Disks

$$V = \int_a^b \pi\,(f(x))^2\,dx$$

Ex/ Find the volume of the solid obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1,4]$ is revolved about the $x-axis$.

$$V = \int_a^b \pi\,(f(x))^2\,dx$$



$$V = \int_1^4 \pi\,[\sqrt{x}]^2 dx = \pi\left[\frac{x^2}{2}\right]_1^4 = 8\,\pi - \frac{\pi}{2} = \frac{15\pi}{2}$$

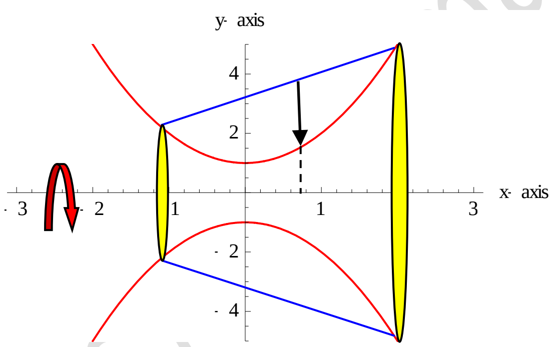
b. Volumes by washers:

$$V = \int_0^2 \pi \left[(f(x))^2 - (g(x))^2\right] dx$$

Ex: find the volume generated by revolving the area enclosed by

$y = x^2 + 1 \quad , \quad y = x + 3$ *about the* $x - axis$

$$y_1 = y_2$$
$$x^2 + 1 = x + 3$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$\Downarrow \qquad \Downarrow$$
$$x = -1 \qquad x = 2$$



By washer method

$$V = \int_a^b \pi\ [[f(x)]^2 - [g(x)]^2]\ dx$$

$$= \int_{-1}^{2} \pi\ [(x+3)^2 - (x^2+1)^2]\ dx$$

$$= \pi \int_{-1}^{2} [x^2 + 6x + 9 - x^4 - 2x^2 - 1]\ dx$$

$$= \pi \int_{-1}^{2} [-x^4 - x^2 + 6x + 8]\ dx$$

$$= \pi \left[-\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x \right]_{-1}^{2}$$

$$= \frac{27\pi}{5}$$

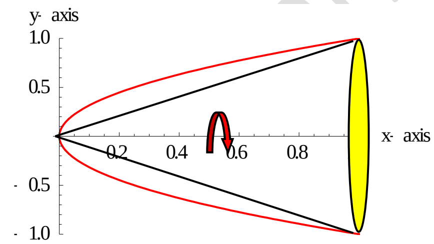
Ex/ Find the volume of solid generated by revolving the region enclosed by $x = y^2$ and $x = y$

1- about the x-axis

2- about the Line $y = -1$.

1- about the $x - axis$ $\qquad x = y^2 \Rightarrow y = \pm\sqrt{x}$

$$y_1 = y_2$$
$$\sqrt{x} = x$$
$$x = x^2$$
$$x^2 - x = 0$$
$$x\,(x - 1) = 0$$
$$\Downarrow \qquad \Downarrow$$
$$x = 0 \qquad x = 1$$



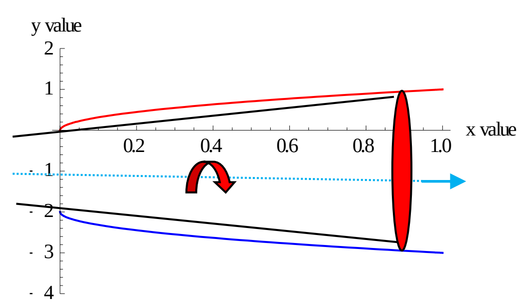
By washer method

$$V = \int_a^b \pi\left[[f(x)]^2 - [g(x)]^2\right]dx$$

$$= \int_0^1 \pi\left((\sqrt{x})^2 - (x)^2\right)dx$$

$$= \pi\int_0^1 (x - x^2)dx = \pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \pi\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{\pi}{6}$$

2- about the line $\quad y = -1$



$$V = \int_a^b \pi\,[f(x) - (-1)]^2 - [g(x) - (-1)]^2 dx$$

$$= \int_0^1 \pi\left[(\sqrt{x} + 1)^2 - (x + 1)^2\right]dx$$

$$= \pi\int_0^1 \left[x + 2\sqrt{x} + 1 - x^2 - 2x - 1\right]dx$$

$$= \pi\int_0^1 \left[-x + 2\sqrt{x} - x^2\right]dx$$

$$= \pi\left[\frac{-x^2}{2} + 4\frac{x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right]_0^1 = \frac{\pi}{2}$$

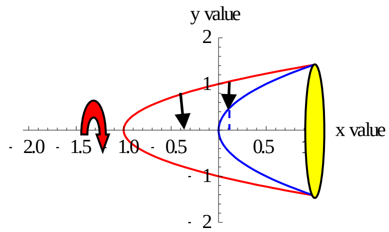
Ex / Find the volume of the solid generated when the region enclosed by $y = \sqrt{x+1}$, $y_2 = \sqrt{2x}$ and $y_2 = 0$ is revolved about the $x - axis$.

$$y_1 = y_2$$

$$\sqrt{x+1} = \sqrt{2x}$$

$$x + 1 = 2x$$

$$x = 1$$



$$y_1 = y_3$$

$$\sqrt{x+1} = 0$$

$$x = -1$$

$$y_2 = y_3$$

$$\sqrt{2x} = 0$$

$$x = 0$$

$$V = v_1 + v_2$$

$$= \int_{-1}^{0} \pi \, [\sqrt{x+1}]^2 \, dx + \int_{0}^{1} \pi \left([\sqrt{x+1}]^2 - [\sqrt{2x}]^2 \right) dx$$

$$= \pi \int_{-1}^{0} (x+1) \, dx + \pi \int_{0}^{1} [x + 1 - 2x] \, dx$$

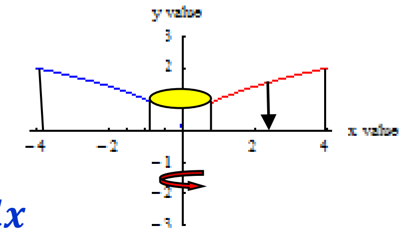
$$= \pi \int_{-1}^{0} (x+1) dx + \pi \int_{0}^{1} (1 - x) \, dx$$

$$= \pi \left[\frac{x^2}{2} + x \right]_{-1}^{0} + \pi \left[x - \frac{x^2}{2} \right]_{0}^{1} =$$

c. Volume by Cylindrical shells:

$$V = \int_a^b 2\pi \; x \; f(x) \; dx$$

Ex / Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$ over [1,4] is revolved about the y-axis.



$$V = \int_a^b 2\pi \; x \; f(x)\, dx$$

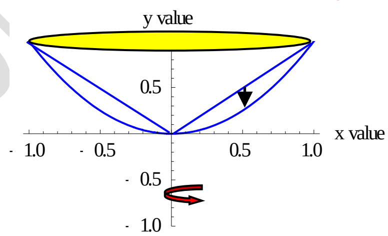
$$= \int_1^4 2\pi\, x \sqrt{x}\; dx$$

$$= 2\pi \int_1^4 x^{3/2}\; dx = 2\pi \left[\frac{x^{5/2}}{5/2}\right]_1^4$$

$$= 2\pi . \frac{2}{5}\left[x^{5/2}\right]_1^4 = \frac{4\pi}{5}[32-1] = \frac{124\pi}{5}$$

Ex / *use **cylindrical shell** method to find the volume of the solid generated when the region in the first quadrant enclosed by*

$y = x$ ***and*** $y = x^2$ *is revolved about* $y - axis$



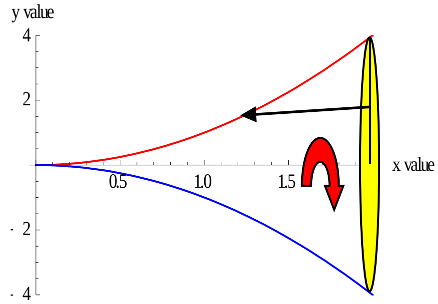
$$y_1 = y_2$$
$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$\Downarrow \quad \Downarrow$$
$$x = 0 \quad x = 1$$

$$V = \int_a^b 2\pi\, x\, [f(x) - g(x)]\, dx$$

$$= \int_0^1 2\pi\, x\, [x - x^2]dx = 2\pi \int_0^1 (x^2 - x^3)dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{\pi}{6}$$

Ex** / use **cylindrical shell** to find the volume of the solid generated when the region under* $y = x^2$ *over the internal* $[0,2]$ *is revolved about* $x-$ ***axis



x	y
0	0
2	4

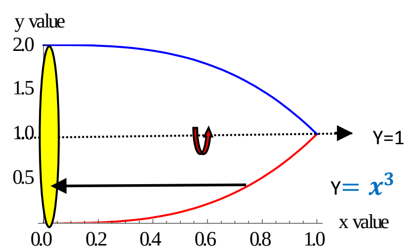
$$y = x^2 \rightarrow x = \sqrt{y}$$

$$V = \int_a^b 2\,\pi\, y\,[f(y) - g(y)]\, dx$$

$$= \int_0^4 2\,\pi\, y\,[2 - \sqrt{y}]\, dy$$

$$= 2\,\pi\, \int_0^4 [2\,y - y^{3/2}]\, dy = 2\,\pi \left[y^2 - \frac{y^{5/2}}{5/2} \right]_0^4 = \frac{32}{5}\,\pi$$

***Ex** / use **cylndrical shells** method to find the volume of the solid ge nerated when the region is enclosed by* $y = x^3$ $y = 1$, $x = 0$ *is revolved* ***1.about the line** y = 1 ,* ***2. about the line** y = -1*

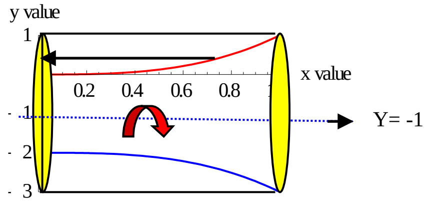


1. $V = \int_a^b 2\,\pi\,[1 - y]\,[f(y) - g(y)]\, dy$

$$= 2\,\pi\, \int_0^1 [1 - y]\,[\sqrt[3]{y} - 0]\, dy$$

$$= 2\,\pi\, \int_0^1 [y^{1/3} - y^{4/3}]\, dy = 2\,\pi \left[\frac{y^{4/3}}{4/3} - \frac{y^{7/3}}{7/3} \right]_0^1 = 2\,\pi \left[\frac{3}{4} - \frac{3}{7}\right] = \frac{9\,\pi}{14}$$

Note : about y = -1



2. $V = \int_a^b 2\pi\,[y-(-1)]\,[f(y) - g(y)]\;dy$

$= 2\pi \int_0^1 [y+1]\;[\sqrt[3]{y} - 0]\,dy$

$= 2\pi \int_0^1 [y^{4/3} + y^{1/3}]\,dy = 2\pi \left[\frac{y^{\frac{7}{3}}}{\frac{7}{3}} + \frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_0^1 = 2\pi\left[\frac{3}{7} + \frac{3}{4}\right] = \frac{33\pi}{14}$