2.4 Length of a curve

1) if $y = f(x)$ and $\frac{dy}{dx}$ is define on $[a, b]$ **then** the **arc length** is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

2) if $x = g(y)$ and $\frac{dx}{dy}$ is define on $[c, d]$ **then** the **arc length** is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Ex / Find the **arc length** of the curve $y = x^{3/2}$ from $(1,1)$ to $(2, 2\sqrt{2})$ in two ways.

Method 1 :

1) $y = x^{3/2}$ from x=1 to x= 2

$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$ is define on $[1,2]$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{9}{4} x}\, dx = \frac{4}{9} \int_1^2 \frac{9}{4} \left(1 + \frac{9}{4} x\right)^{1/2} dx$$

$$= \frac{4}{9} \frac{[1 + \frac{9}{4} x]^{3/2}}{3/2} \Big]_1^2 = \frac{4}{9} \frac{2}{3} \left[\left(1 + \frac{9}{4} x\right)^{3/2}\right]_1^2 = \frac{8}{27} \left[\left(1 + \frac{9}{4}(2)\right)^{3/2} - \left(1 + \frac{9}{4}(1)\right)^{3/2}\right]$$

Method 2:

2) $y = x^{3/2} \Longrightarrow x = y^{2/3}$ from $y = 1$ to $y = 2\sqrt{2}$

$\frac{dx}{dy} = \frac{2}{3} y^{-1/3} = \frac{2}{3} \frac{1}{y^{1/3}}$ is undefine at 0 out $[1, 2\sqrt{2}]$

$$L = \int_1^{2\sqrt{2}} \sqrt{1 + \left(\frac{2}{3} y^{-\frac{1}{3}}\right)^2}\, dy = \int_1^{2\sqrt{2}} \sqrt{1 + \left(\frac{4}{9y^{2/3}}\right)}\, dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{\left(\frac{9y^{2/3}+4}{9y^{2/3}}\right)}\, dy = \frac{1}{3}\int_1^{2\sqrt{2}} y^{-1/3}\sqrt{9\,y^{2/3}+4}\; dy$$

$$= \frac{1}{18}\int_1^{2\sqrt{2}} 6y^{-1/3}\sqrt{9\,y^{2/3}+4}\; dy = \frac{1}{18}\frac{2}{3}\left(9\,y^{2/3}+4\right)^{3/2}\Big]_1^{2\sqrt{2}}$$

3- if $y = f(t)$ and $x = g(t)$ then

$$L = \int_{t1}^{t2}\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\; dt$$

Ex / Find the circumference of acircle of radius a .

equation of circle of radius a is $x^2 + y^2 = a^2$

and the parametric equations are

$x = a\, cost$, $y = a\sin t$

$\Downarrow$ $\Downarrow$

$\frac{dx}{dt} = -a\, sint$ $\frac{dy}{dt} = a\, cost$

$$L = \int_{t1}^{t2}\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\; dt$$

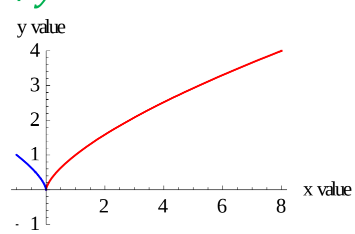
$$L = \int_0^{2\pi}\sqrt{(a\, cost)^2 + (-a\, sint)^2}\;\; dt$$

$$= \int_0^{2\pi}\sqrt{a^2\,(\cos^2 t + \sin^2 t)}\;\; dt = \int_0^{2\pi} a\, dt = a[\,t\,]_0^{2\pi} = 2\pi\, a$$

Ex: Find the arc length of the curve $y = x^{2/3}$ between $x = -1$ and $x = 8$.

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}} \text{ is undefine at } x = 0 \in [-1,8]$$

$$y = x^{2/3} \rightarrow x = \mp y^{3/2}$$



$L_1 : x = -y^{\frac{3}{2}} \rightarrow \frac{dx}{dy} = -\frac{3}{2}\, y^{1/2} \quad , 0 \le y \le 1$

$$L_1 = \int_0^1 \sqrt{1 + \left(\frac{9}{4}\, y\right)}\ dy = \frac{4}{9}\, \frac{(1+\frac{9}{4}y)^{\frac{3}{2}}}{3/2} \Big]_0^1 = 1.74$$

$L_2 : x = y^{3/2} \rightarrow \frac{dx}{dy} = \frac{3}{2}\, y^{1/2} \quad , \quad 0 \le y \le 4$

$$L_2 = \int_0^4 \sqrt{1 + \frac{9}{4}\, y}\ dy = \frac{4}{9}\, \frac{(1+\frac{9}{4}y)^{3/2}}{3/2} \Big]_0^4 = \cdots\cdots$$

$$L = L_1 + L_2$$

H.W.

1. Find the arc length of the curve $x = \frac{1}{3}\,(y^2 + 2)^{3/2}$ from y=0 to y=1

2. Find the arc length of the curve $y = \frac{x^4}{16} + \frac{1}{2x^2}$ from x=2 to x=3

3. Find the arc length of the curve $24xy = y^4 + 48$ from y=2 to y=4

4. Find the arc length of the curve $x = \frac{1}{8}\, y^4 + \frac{1}{4} y^{-2}$ from y=1 to y=4

2.5 Area of the surface of Revolution :

1. if $y = f(x)$ *and* $\frac{dy}{dx}$ is define *on* $[a, b]$ then the area of the Surface geverated by revalving about x - axis is .

$$S = \int_a^b 2\pi\, f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

2. similarly if $x = g(y)$ *and* $\frac{dx}{dy}$ is define on $[c\,, d]$ then the area of the surface generated by revolving about y- axis is

$$S = \int_c^d 2\pi\ g(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

Ex/ Find the ***surface area*** *genereted by revolving*

$y = f(x) = \sqrt{1-x^2}$, $0 \le x \le 1/2$ ***about the x- axis .***

$\frac{dy}{dx} = \frac{1}{2}\,(1-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{1-x2}}$ is undefine at $x = -1$, $x = 1$

$$S = \int_a^b 2\pi\ f(x)\sqrt{1+(\frac{dy}{dx})^2}\ \ dx$$

$$S = \int_0^{1/2} 2\pi\,(\sqrt{1-x^2})\ \sqrt{1+\frac{x^2}{1-x^2}}\ \ dx$$

$$= \int_0^{1/2} 2\pi\sqrt{1-x^2} * \frac{1}{\sqrt{1-x^2}}\ dx = 2\pi\,[x\,]_0^{1/2} = \pi$$

Ex / Find the ***area of the surface*** *geuerated by revolving*

$x = 2\sqrt{1-y}$, $-1 \le y \le 0$ ***about*** $y - axis$

$\frac{dx}{dy} = -\frac{2}{2}\,(1-y)^{-1/2} = \frac{-1}{\sqrt{1-y}}$ is undefine at y= 1 out $-1 \le y \le 0$

$$S = \int_c^d 2\pi\ \ g(y)\sqrt{1+(\frac{dx}{dy})^2}\ \ dy$$

$$\therefore S = \int_{-1}^{0} 2\pi\,(2\sqrt{1-y})\ \sqrt{1+(\frac{-1}{\sqrt{1-y}})^2}\ \ dy$$

$$= 4\pi\int_{-1}^{0}\sqrt{1-y}\ \sqrt{1+\frac{1}{1-y}}\ \ dy$$

$$= 4\,\pi\int_{-1}^{0}\sqrt{1-y}\ \sqrt{\frac{1-y+1}{1-y}}\ \ dy$$

$$= 4\,\pi\int_{-1}^{0}\sqrt{1-y}\ \ \frac{\sqrt{2-y}}{\sqrt{1-y}}\ \ dy = 4\pi\int_{-1}^{0}(2-y)^{1/2}\ \ dy$$

$$= -4\,\pi\ \frac{[2-y]^{3/2}}{3/2}\Big]_{-1}^{0}$$

HW: *Find the **area of the surface** generaled by revolving*

1. $x = \sqrt[3]{y}$, $1 \le y \le 8$ about the x - axis .
2. $y = \sqrt{x} - \frac{1}{3} \times^{\frac{3}{2}}$, $1 \le x \le 3$ about x - axis .
3. $x = y^3$, $0 \le y \le 1$ about y - axis .
4. $x = \sqrt{9 - y^2}$, $-2 \le y \le 2$ about y - axis .
5. $8xy^2 = 2y^6 + 1$, $1 \le y \le 2$ about y - axis .