3.1 Inverse function

if $f : A \to B$ then $f^{-1} : B \to A$

or if $y = f(x)$ then $x = f^{-1}(y)$

$D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$

Ex / let $f(x) = \frac{x}{x-2}$ find $f^{-1}(x)$

$$y = \frac{x}{y-2} \qquad x = yx - 2y \Rightarrow x - yx = -2y$$

$$x(1-y) = -2y \Rightarrow x = \frac{-2y}{1-y}$$

Replace x by y to obtain $\Rightarrow y = \frac{-2x}{1-x}$

$$y = \frac{-2x}{1-x} = f^{-1}(x)$$

Note: if $f(x)$ is increase or decrease always.then f has an inverse.

Ex / show that $f(x) = x^5 + 7x^3 + 4x + 1$ has an inverse.

$f'(x) = 5x^4 + 21x^2 + 4 > 0$ always increase thus

$f(x)$ has an inverse .

Note : $f(x) o f^{-1}(x) = f^{-1}(x) o f(x) = x$

Ex/ determine whether $f(x) = \sqrt[3]{x-2}$ is the inverse of $g(x) = x^3 + 2$

$fog(x) = f(g(x))$

$= f(x^3 + 2)$

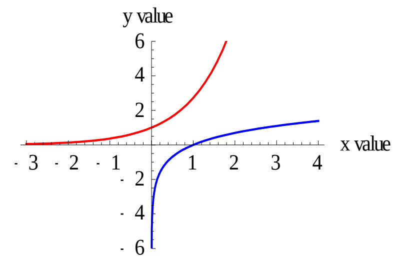
$= \sqrt[3]{(x^3+2) - 2}$

$= \sqrt[3]{x^3}$

$= x \qquad \therefore \; fog(x) = x$

$\therefore \; g(x) = f^{-1}(x)$

3.2 Logarithms :



Def : $Log_a\, x = y \iff x = a^{y}$

Properties:

1. $Log_a\,(x.y) = Log_a\, x + Log_a\, y$
2. $Log_a\left(\frac{x}{y}\right) = Log_a\, x - Log_a\, y$
3. $Log_a\, x^n = n\, Log_a\, x$
4. $Log_a\, a = 1$, $Log_a\, 1 = 0$

Ex/ Solve for *x*

$$Log_5\,(5^{2x}) = 8$$
$$2\,x\, Log_5 5 = 8$$
$$2x\,(1) = 8 \implies x = 4$$

Ex/ Solve for *x* $\quad Log_{10}\, x^2 + Log_{10}\, x = 30$

$$2\, \mathrm{Log}_{10}\, \mathrm{x} + \mathrm{Log}_{10}\, \mathrm{x} = 30$$
$$3\, Log_{10}\, x = 30 \implies Log_{10}\, x = 10 \implies x = 10^{10}$$

Note : if $y = Log_a\, u$ then $\frac{dy}{dx} = \frac{1}{u}\, Log_a e\, \frac{du}{dx}$ where $e \simeq 2.7$

Ex / Find dy/dx if $y = Log_3(2x^2 + 5x + 1)$

$$\frac{dy}{dx} = \frac{1}{2\,x^2 + 5x + 1}\, Log_3\, e\, .(4x + 5)$$
$$= \frac{4x+5}{2x^2 + 5x + 1}\, Log_3 e$$

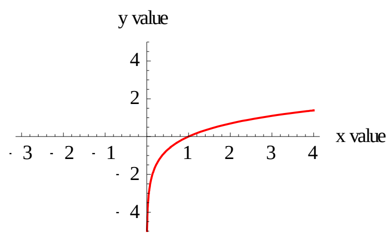
The Natural Logarithm

If $a=e$ *then* $Log_a x = Log_e x = \ln x$

$$y = f(x) = \ln x$$

$D_f = \{x : x > 0\}$

$R_f = R$



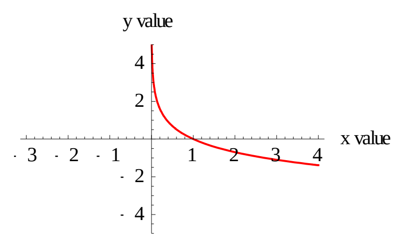
Ex / Find domein of $f(x) = \log_3 (x-2)$

$$x - 2 > 0$$
$$x > 2$$
$$\therefore D_f = \{x : x > 2\}$$

Properties

1. $\ln(x.y) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
3. $\ln x^n = n \ln x$
4. $\ln 1 = 0$, $\ln e = 1$

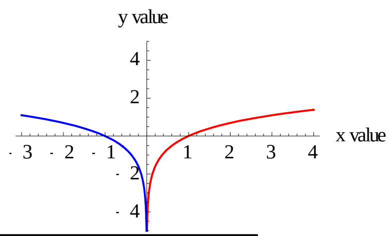
ex/ *draw the groph of* $y = \ln\left(\frac{1}{x}\right)$



$$\therefore \ln\left(\frac{1}{x}\right) = \ln 1 - \ln x$$
$$= 0 - \ln x$$
$$= -\ln x$$
$$\therefore \ln\left(\frac{1}{x}\right) = -\ln x$$

Ex/ draw the graph of $y = \ln|x|$

$$y = \ln|x| = \begin{cases} \ln x & x \geq 0 \\ \ln(-x) & x < 0 \end{cases}$$



if $y = \ln u$ then $\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$

Ex/ Let $y = \ln(\sin x)$ Find dy/dx

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x$$

Ex/ Find $\frac{dy}{dx}$ if $y = \ln\left(\frac{x^2 \sin^3 x}{\cos x^4 \sqrt{1+x}}\right)$

$$y = \ln\left(\frac{x^2 \sin^3 x}{\cos x^4 \sqrt{1+x}}\right)$$

$$y = \ln(x^2 \sin^3 x) - \ln(\cos x^4 \sqrt{1+x})$$

$$y = \ln x^2 + \ln \sin^3 x - \left[\ln \cos x^4 + \ln(1+x)^{1/2}\right]$$

$$y = 2\ln x + 3\ln \sin x - \ln \cos x^4 - \frac{1}{2}\ln(1+x)$$

$$\frac{dy}{dx} = 2.\frac{1}{x} + 3.\frac{1}{\sin x} \cos x - \frac{1}{\cos x^4} .(-\sin x^4 (4x^3) - \frac{1}{2}.\frac{1}{1+x}$$

$$\frac{dy}{dx} = \frac{2}{x} + 3\cot x + 4x^3 \cot x^2 - \frac{1}{2(1+x)}$$

Ex/ Let $y = x^{\sin x}$ Find $\frac{dy}{dx}$

$$y = x^{\sin x} \implies \ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x . \frac{1}{x} + \ln x . \cos x$$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \ln x \cos x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \ln x \cos x\right]$$

Ex/ Find $\frac{dy}{dx}$ ***if*** $y = \frac{x^3\sqrt{x-5}}{\sin^2 x\ \cos^3 x}$

$$\ln y = \ln\left(\frac{x^3\ \sqrt{x-5}}{\sin^2 x\cos^3 x}\right)$$

$$\ln y = \ln\,[x^3\ \sqrt{x-5}] - \ln[\sin^2 x\cos^3 x]$$

$$\ln y = \ln\ x^3 + \ln(x-5)^{1/2} - [\ln\sin^2 x + \ln\cos^3 x]$$

$$\ln y = 3\ln x + \frac{1}{2}\ln(x-5) - 2\ln\sin x - 3\ln\cos\ x$$

$$\frac{1}{y}\frac{dy}{dx} = 3\frac{1}{x} + \frac{1}{2}\ \frac{1}{x-5} - 2\frac{1}{\sin x}\cos x - 3\frac{1}{\cos x}(-\sin x)$$

$$\frac{dy}{dx} = y\,[\frac{3}{x} + \frac{1}{2(x-5)} - 2\cot x + 3\tan x]$$

$$\frac{dy}{dx} = \frac{x^3\sqrt{x-5}}{\sin^2\ \ con^3 x}\ \left[\frac{3}{x} + \frac{1}{2(x-5)} - 2\cot x + 3\tan x\right]$$

$$\boxed{\int\frac{du}{u} = \ln|u| + c}$$

Ex/ find $\int\frac{x^2\ dx}{x^3-4} = \frac{1}{3}\int\frac{3x^2\ dx}{x^3-4} = \frac{1}{3}\ln\ |x^3-4| + c$

Ex /find $\int\left(\frac{1}{\sqrt{x}(1+\sqrt{x})}\right)\ dx$

$$Let\quad u = 1+\sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}}\ dx \implies 2\ du = \frac{dx}{\sqrt{x}}$$

$$= \int\frac{2\ du}{u} = 2\int\frac{du}{u} = 2\ln|u| + c$$

$$= 2\ln\left|1+\sqrt{x}\right| + c$$

Ex / $\int \frac{\cos(\ln x)}{x}\,dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$= \int \cos u\; du = \sin u + c$

$= \sin(\ln x) + c$

Ex / $\int \sec x\,dx = \int \sec x * \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx = \ln|\sec x + \tan x| + c$

Ex / $\int \frac{\ln x}{x}\,dx$ let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

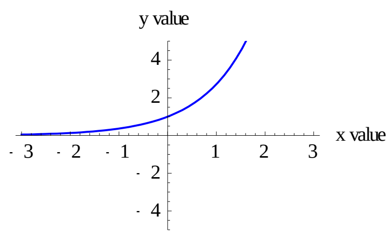
$= \int u\,du = \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$

Ex / $\int \frac{dx}{x \ln x}$ let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$= \int \frac{du}{u} = \ln|u| + c = \ln|\ln x| + c$

3.3 The exponential function

$y = f(x) = e^x$



$D_f = R$

$R_f = \{y : y > 0\}$

$\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to -\infty} e^x = 0$

Properties:

1. $e^x . e^y = e^{x+y}$
2. $\frac{e^x}{e^y} = e^{x-y}$
3. $(e^x)^n = e^{nx}$

Ex / Sketch the groph of $\boldsymbol{x = 2 + ln(y-1)}$

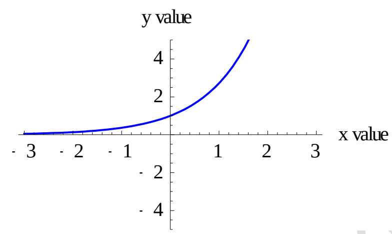
$$x = 2 + \ln(y-1)$$

$$x - 2 = \ln(y-1)$$

$$e^{x-2} = y - 1$$

$$1 + e^{x-2} = y$$

$$\boldsymbol{y = 1 + e^{x-2}}$$



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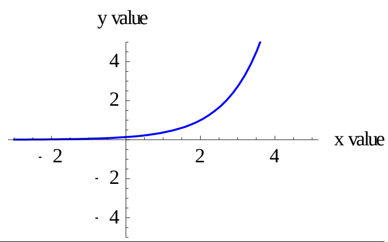

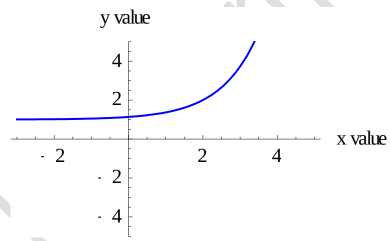

$$if\ \boldsymbol{y = e^{u}}\ then\ \boldsymbol{\frac{dy}{dx} = e^{u} \cdot \frac{du}{dx}}$$

$\boldsymbol{ex}$ $/Find\ \frac{dy}{dx}\ \ if\ \ y = e^{x\tan x}$

$$\frac{dy}{dx} = e^{x\tan x}.[\,x.sec^{2}x + \tan x\ .1]$$

$$\boldsymbol{\int e^{u}du = e^{u} + c}$$

ex / $\int \frac{e^{\tan x}}{\cos^{2}}\boldsymbol{dx} = \int e^{\tan x}\ \sec^{2}x\,dx = e^{\tan x} + c$

$\boldsymbol{ex/}$ $\int \boldsymbol{e^{2lnx}\,dx} = \int e^{lnx^{2}}\,dx = \int x^{2}\,dx = \frac{x^{3}}{3} + c$

Ex / $\int \boldsymbol{e^{x+e^{x}}\,dx} = \int \boldsymbol{e^{x}.e^{e^{x}}\,dx} = \boldsymbol{e^{e^{x}} + c}$

The function $y = f(x) = a^x \quad , \quad a > 0$

1) $a^x . a^y = a^{x+y}$

2) $\frac{a^x}{a^y} = a^{x-y}$

3) $(a^x)^n = a^{nx}$

if $y = a^u$ then $\frac{dy}{dx} = a^u . \frac{du}{dx} . \ln a$

Ex/ find $\frac{dy}{dx}$ if $y = 2^{sinx}$

$$\frac{dy}{dx} = 2^{sinx} . \cos x . \ln 2$$

Ex/ Find $\frac{dy}{dx}$ if $y = \pi^{x^2} + e^x . x^e$

$$\frac{dy}{dx} = \pi^{x^2} . (2x) . \ln \pi + [\, e^x . e x^{e-1} + x^e . e^x]$$

$$\int a^u \, du = \frac{1}{\ln a} a^u + c$$

Ex/ $\int 2^{\sin x} \cos x \, dx = \frac{1}{ln2} 2^{\sin x} + c$

Ex/ Find $\lim_{x \to \infty} \frac{2+e^x}{1+3e^x}$ $\left(\frac{\infty}{\infty}\right)$ note : $e^{\infty} \to \infty$, $e^{-\infty} \to 0$

$$= \lim_{x \to \infty} \frac{2+e^x}{1+3e^x} * \frac{e^{-x}}{e^{-x}}$$

$$= \lim_{x \to \infty} \frac{2e^{-x}+1}{e^{-x}+3} = \frac{0+1}{0+3} = \frac{1}{3}$$