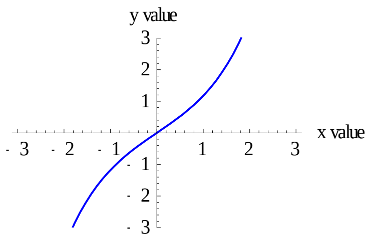
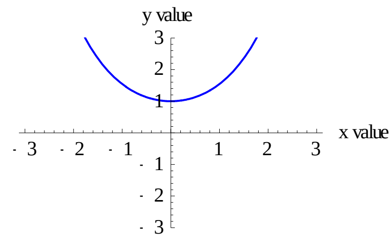
3.4 The hyperbolic functions

The hyperbolic sine and hyperbolic cosine functions define as

1) $\sin h\, x = \frac{e^{x} - e^{-x}}{2}$ 2) $\cos h\, x = \frac{e^{x} + e^{-x}}{2}$





$f(x) = \sin h\, x$

$D_f = R_f = R$

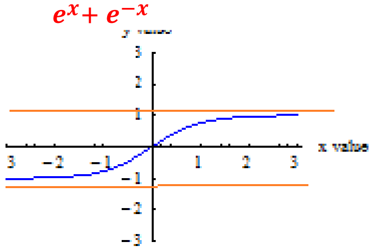
$f(x) = \cos h\, x$

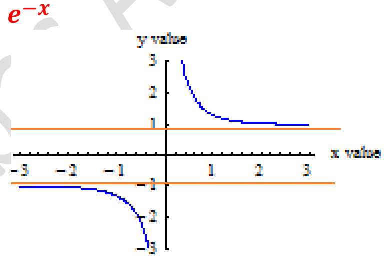
$D_f = R$

$R_f = [1, \infty)$

3) $f(x) = \tan h\, x = \frac{\sin hx}{\cos hx} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

4) $f(x) = \cot hx = \frac{\cos h\, x}{\sin h\, x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$



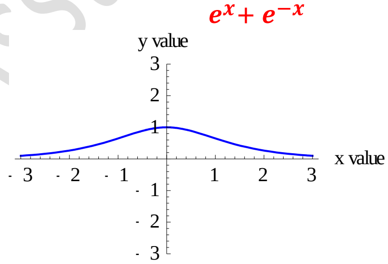


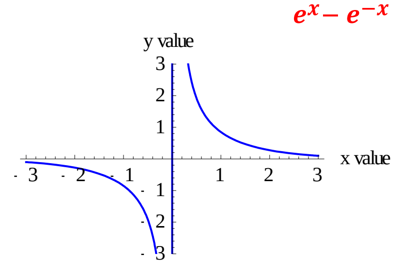
$D_f = R$, $R_f = (-1, 1)$

$D_f = R\backslash\{0\}$, $R_f = R\backslash [-1, 1]$

5) $f(x) = \sec hx = \frac{1}{\cos h\, x} = \frac{2}{e^{x} + e^{-x}}$

6) $f(x) = \csc hx = \frac{1}{\sin h\, x} = \frac{2}{e^{x} - e^{-x}}$





$D_f = R$, $R_f = (0, 1]$

$D_f = R\backslash\{0\}$, $R_f = R\{0\}$

1) $\cos h^2 x - \sin h^2 x = 1$
2) $1 - \tan h^2 x = \sec h^2 x$
3) $\cot h^2 x - 1 = csch^2 x$
4) $\sin h\,(x \pm y) = \sin h\,x \cos hy \pm \cos hx \ \sin hy$
5) $\cos h\,(x \pm y) = \cos h\,x \cos hy \pm \sin hx \ \sin hy$
6) $coshx + \sin hx = e^x$
7) $\cos hx - \sin hx = e^{-x}$
8) $\sin h\, 2x = 2 \sin h\, x \ \cos h\, x$
9) $\cos h\ 2x =$
- $\cos h^2 x + \sin h^2 x$
- $2 \ \sin h^2 x + 1$
- $2 \cos h^2 x - 1$

Derivative

$\frac{d}{dx}(\sin hx) = \cos hx$

$\frac{d}{dx}(\cos hx) = \sin hx$

$\frac{d}{dx}(\tan hx) = \sec h^2 x$

$\frac{d}{dx}(\cot hx) = -\csc h^2 x$

$\frac{d}{dx}(\sec hx) = -sechx \ \tan hx$

$\frac{d}{dx}(cschx) = -\csc hx \ \cot hx$

integrals

1- $\int \cos h\, x dx = \sin h\, x + c$

2- $\int \sin hx\, dx = \cos h\, x + c$

3- $\int \sec h^2 x\, dx = \tan h\, x + c$

4- $\int \csc h^2 x dx = -\cot hx + c$

5- $\int \sec hx \tan hx \ dx = -\sec hx + c$

6- $\int \csc hx \ cothx\, dx = -csch\, x + c$

Ex/ Find $\frac{d}{dx}\left[\int_0^{lnx} \frac{dt}{\sqrt{4+e^t}}\right]$

By the 2^{nd} fundameutal theorem of colculus

$$= \frac{1}{\sqrt{4+e^{lnx}}} \ \frac{1}{x} = \frac{1}{x\sqrt{4+x}}$$

Ex / show that $y = e^{ax} \sinh bx$ satisfies $y'' - 2a\,y' + (a^2 + b^2)y = 0$

$$y' = b\,e^{ax} \cosh\ bx + a\,e^{ax} \sin h\,bx$$

$$y'' = b^2 e^{ax}\ \sinh\ bx + ab\,e^{ax} \cos bx + ab\,e^{ax} \cos h\,bx + a^2 e^{ax} \sin hbx$$

$$y'' = e^{ax} \sinh bx(a^2 + b^2) + 2\,ab\,e^{ax}\ \cos hb\,x$$

$$y'' - 2\,a\,y' + (a^2 + b^2)y = (a^2 + b^2)e^{ax} \sin hb\,x + 2ab\,e^{ax} \cos hbx$$

$$- 2a\,[b\,e^{ax} \cos hx + a\,e^{ax} \sin h\,bx] + (a^2 + b^2)e^{ax} \sin h\,bx) = 0$$

Ex/ evaluate $\int \left[\tan h\,x + \sqrt{\tan h\,x}\ \ \sec h^2 x\right] dx$

$$= \int \frac{\sin h\,x}{\cos h\,x}\,dx + \int (\tan h\,x)^{1/2}\ \sec h^2\,x\ dx$$

$$= \ln|\cosh x| + \frac{(\tan h\,x)^{3/2}}{3/2} + c$$

Ex/ evaluate $\int \frac{\sin h\,2x\,dx}{3+5\cos h\,2x}$

$let \quad u = 3 + 5\cos 2x \quad du = 10 \sin h2\,x\,dx$

$$\int \frac{\sin h\,2x\,dx}{3 + 5\cos h\,2x} = \frac{1}{10}\int \frac{du}{u} = \frac{1}{10}\ln|u| + c$$

$$= \frac{1}{10}\ln|3 + 5\cos 2x| + c$$

Ex / show that $\lim_{x\to\infty} \tan h\,x = 1$

$$= \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x}$$

$$= \lim_{x\to\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = -1$$

3.5 Inverse of Trigonometric and huperbolic functions

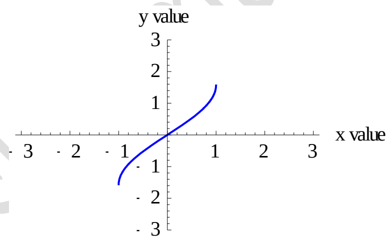
$$if \quad -1 \le x \le 1 \; and \; -\frac{\pi}{2} \le y \le \frac{\pi}{2} \; then$$

$$y = \sin^{-1}x \iff x = \sin y$$

$$\sin^{-1}(\sin y) = y \qquad if \; -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \qquad if \; -1 \le x \le 1$$

1) $f(x) = \sin^{-1}x$

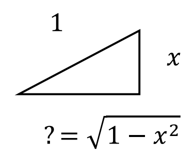


$D_f = [-1,1]$

$R_f = [-\frac{\pi}{2}, \frac{\pi}{2}]$

Ex / simplify $\cos(\sin^{-1}x)$

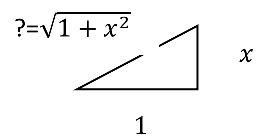
$$Let \; \alpha = \sin^{-1}x \Rightarrow \frac{x}{1} = \sin\alpha$$



$$\therefore \cos(\sin^{-1}x) = \cos\alpha = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

Ex / simplify $\sec^2(\tan^{-1}x)$

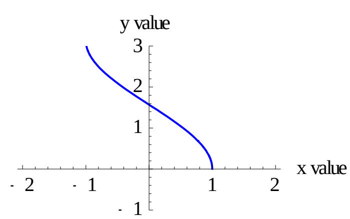
$$let \quad \alpha = \tan^{-1}x \Rightarrow \tan\alpha = \frac{x}{1}$$



$$\therefore \sec(\tan^{-1}x) = \sec\alpha = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$$

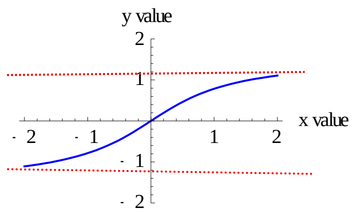
$$\sec^2(\tan^{-1}x) = 1 + x^2$$

2) $f(x) = \cos^{-1}x$



$D_f = [-1, 1]$, $R_f = [0, \pi]$

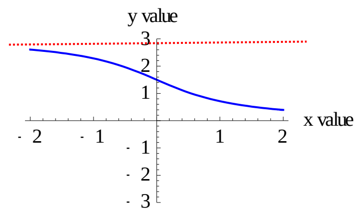
3) $f(x) = \tan^{-1} x$



$D_f = R$

$R_f = (-\frac{\pi}{2}, \frac{\pi}{2})$

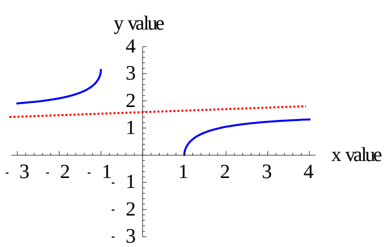
4) $f(x) = \cot^{-1} x$



$D_f = R$

$R_f = (0, \pi)$

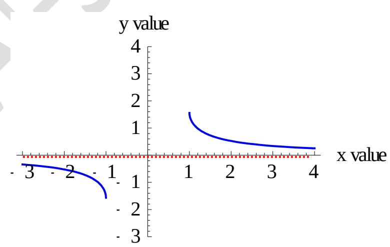
5) $f(x) = \sec^{-1} x$



$D_f = R \backslash (-1, 1)$

$R_f = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

6) $f(x) = \csc^{-1} x$



$D_f = R \backslash (-1, 1)$

$R_f = [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Ex /prove that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Let $y = \cos^{-1}x \quad \Rightarrow \quad x = \cos y$

$$x = \sin(\frac{\pi}{2} - y)$$

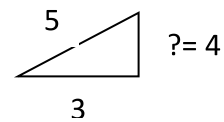
$$\sin^{-1} x = \frac{\pi}{2} - y$$

$$y = \frac{\pi}{2} - \sin^{-1} x \qquad \therefore \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$or \ \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Ex / find the exact value of $\sin\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$

Let $\alpha = \cos^{-1}\left(\frac{3}{5}\right) \leftrightarrow \cos\alpha = \frac{3}{5}$



$$\therefore \sin\left(2\cos^{-1}\left(\frac{3}{5}\right)\right) = \sin(2\alpha)$$

$$= 2\sin\alpha\cos\alpha$$

$$= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$$

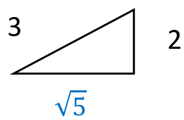
Ex/ find the exact value of $\sin\left[\sin^{-1}\left(\frac{2}{3}\right) + \cos^{-1}\left(\frac{1}{3}\right)\right]$

Let $\alpha = \sin^{-1}\left(\frac{2}{3}\right)$, $\beta = \cos^{-1}\left(\frac{1}{3}\right)$

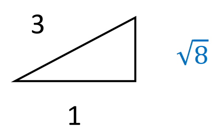
$\Downarrow$ $\Downarrow$

$\sin\alpha = \frac{2}{3}$ $\cos\beta = \frac{1}{3}$

$\sin\alpha = \frac{2}{3}$



$\cos\beta = \frac{1}{3}$



$$\sin[\alpha+\beta] = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{\sqrt{5}}{3}\right)\left(\frac{\sqrt{8}}{3}\right)$$

$$= \frac{2}{9} + \frac{\sqrt{40}}{9} = \frac{2+\sqrt{40}}{9}$$

Ex/ prove that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Let $\alpha = \tan^{-1}x$, $\beta = \tan^{-1}y$

$\Downarrow$ $\Downarrow$

$\tan\alpha = x$ $\tan\beta = y$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$\therefore \alpha+\beta = \tan^{-1}\left(\frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}\right)$$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$