Derivatives and Integrals of inverse trigonometric functions .

Ex/ Show that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

Let $y = \sin^{-1} x \Longrightarrow \sin y = x$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

1) $\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

2) $\frac{d}{dx}[\cos^{-1} u] = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$

3) $\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2} \frac{du}{dx}$

4) $\frac{d}{dx}[\cot^{-1} u] = \frac{-1}{1+u^2} \frac{du}{dx}$

5) $\frac{d}{dx}[\sec^{-1} u] = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$

6) $\frac{d}{dx}[\csc^{-1} u] = \frac{-1}{u\sqrt{u^2-1}} \frac{du}{dx}$

1) $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c$

2) $\int \frac{du}{1+u^2} = \tan^{-1} u + c$

3) $\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + c$

Ex/ Find dy/dx **of**

1) $y = \tan^{-1}(xe^{2x})$

$$\frac{dy}{dx} = \frac{1}{1+(x\,e^{2x})^2} \cdot (2x\,e^{2x} + e^{2x}.1) = \frac{e^{2x}(2x+1)}{1+(x\,e^{2x})^2}$$

2) $y = \cos^{-1}(\sin x)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1+(\sin x)^2}} . \cos x = \frac{-\cos x}{\sqrt{1+(\sin x)^2}}$$

3) $y = \ln(\cos^{-1} x)$

$$\frac{dy}{dx} = \frac{1}{\cos^{-1} x} , \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-x^2}\,\cos^{-1} x}$$

4) $x^3 + x\tan^{-1} y = e^y$

$$3x^2 + x.\frac{1}{1+y^2}\frac{dy}{dx} + \tan^{-1} y\ .1 = e^y.\frac{dy}{dx}$$

$$3x^2 + \tan^{-1} y = \left(e^y - \frac{x}{1+y^2}\right)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 + \tan^{-1} y}{e^y - \frac{x}{1+y^2}}$$

ex: $\int \frac{e^x\,dx}{4+e^{2x}}$

$$= \int \frac{e^x\,dx}{4\,(1+\frac{e^{2x}}{4})} = \int \frac{e^x dx}{4[1+(\frac{e^x}{2})^2]}$$

Let $u = \frac{e^x}{2} \Rightarrow du = \frac{1}{2}e^x\,dx$

$$= \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1} u + c = \frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right) + c$$

Ex/ $\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{9\,(1-\frac{4x^2}{9})}} = \int \frac{dx}{3\sqrt{1-(\frac{2x}{3})^2}}$

Let $u = \frac{2x}{3} \Rightarrow du = \frac{2}{3}dx$

$$= \frac{1}{2}\int \frac{\frac{2}{3}dx}{\sqrt{1-(\frac{2x}{3})^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1} u + c$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$$

Ex/ $\int \frac{t\,dt}{1+t^4} = \int \frac{t\,\,dt}{1+(t^2)^2}$

Let $x = t^2 \Rightarrow dx = 2t\,\,dt$ or $\frac{1}{2}dx = t\,\,dt$

$$= \frac{1}{2}\int \frac{dx}{1+x^2}$$

$$= \frac{1}{2}\tan^{-1} x + c = \frac{1}{2}\tan^{-1}(t^2) + c$$

Ex/ Find the area of the region enclosed by graphs of

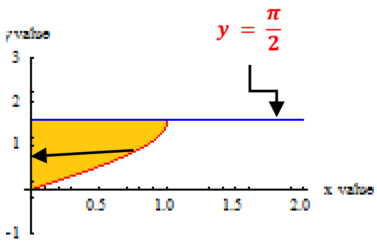
$$y = \sin^{-1} x \;,\quad x = 0 \quad \text{and} \quad y = \frac{\pi}{2}.$$

$$A = \int_c^d [\, f(y) - g(y) \,] dy$$

$$= \int_0^{\pi/2} [\, \sin y - 0 \,]\, dy$$

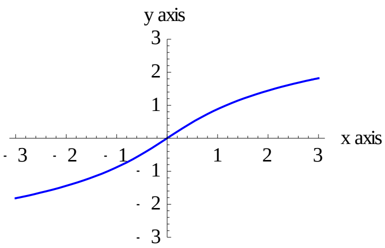
$$= -[\, \cos y \,]_0^{\pi/2}$$

$$= -[\cos(\tfrac{\pi}{2}) - \cos(0)\,] = 1$$



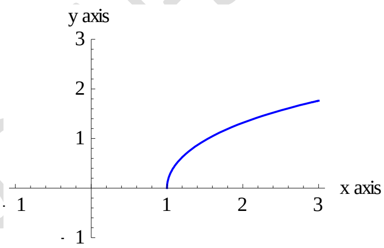
Inverse of Hyperbolic functions

1) $f(x) = \sinh^{-1} x$



$D_f = R_f = R$

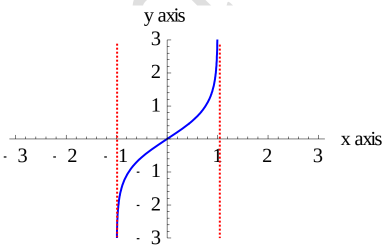
2) $f(x) = \cosh^{-1} x$



$D_f = \{x : x \geq 1\}$

$R_f = \{y : y \geq 0\}$

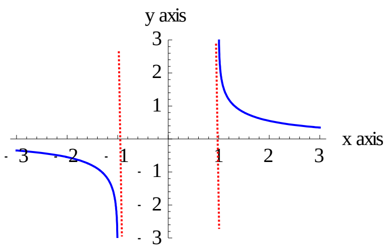
3) $f(x) = \tanh^{-1} x$



$D_f = (-1, 1)$

$R_f = R$

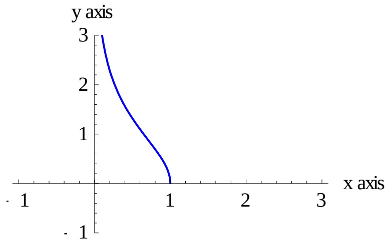
4) $f(x) = \coth^{-1} x$

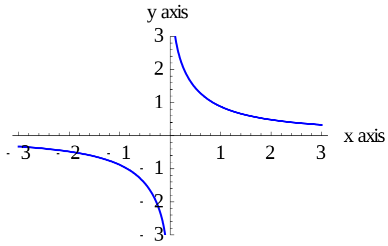


$D_f = R \setminus [-1, 1]$

$R_f = R \setminus \{0\}$

5) $f(x) = sech^{-1} x$ $\qquad$ $f(x) = csch^{-1} x$





$D_f = (0, 1]$ $\qquad$ $D_f = R_f = R \setminus \{0\}$

$R_f = \{y : y \geq 0\}$

Ex/ prove that $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

Let $y = \sinh^{-1} x \implies \sinh y = x$

$$\frac{e^y - e^{-y}}{2} = x$$

$$[e^y - e^{-y} = 2x] * e^y$$

$$e^{2y} - 1 = 2x e^y$$

$$1e^{2y} - 2x e^y - 1 = 0$$

$$a = 1 \ , \ b = -2x \ , \ c = -1$$

$$\therefore \ e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$e^y = x \pm \sqrt{x^2+1}$ $\quad$ *since* $e^y > 0$ always neglect minus sign

$$e^y = x + \sqrt{x^2+1}$$

$$y = \ln\left(x + \sqrt{x^2+1}\right) \rightarrow$$

$$\therefore \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

Similarly we can prove that $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$

Ex/ prove that $\tanh^{-1}x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$

Let $y = \tanh^{-1}x \implies \tanh y = x$

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{x}{1}$$

$$[\,e^y - e^{-y} = x\,e^y + x\,e^{-y}] * e^y$$

$$e^{2y} - 1 = x\,e^{2y} + x$$

$$e^{2y} - xe^{2y} = 1 + x$$

$$e^{2y}(1-x) = 1 + x$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \ln\frac{1+x}{1-x}$$

$$y = \frac{1}{2} ln \frac{1+x}{1-x}$$

$$\therefore \tanh^{-1}x = \frac{1}{2} \ln \frac{1+x}{1-x}$$

Similarly we can prove that

$$\coth^{-1}x = \frac{1}{2} \ln \frac{x+1}{x-1}$$

Ex/prove that $\text{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x2}}{x}\right)$

Let $y = \text{sech}^{-1}x \Rightarrow \text{sech}\, y = x$

$$\frac{1}{\cosh y} = x$$

$$\frac{2}{e^y + e^{-y}} = x$$

$$[\,xe^y + xe^{-y} = 2] * e^y$$

$$xe^{2y} + x = 2e^y$$

$$xe^{2y} - 2e^y + x = 0$$

$$a = x\ , \quad b = -2\ , \quad c = x$$

$$e^y = \frac{2 \pm \sqrt{4-4x^2}}{2x} = \frac{2 \pm 2\sqrt{1-x^2}}{2x}$$

$$e^y = \frac{1 \pm \sqrt{1-x^2}}{x} \quad \text{since} \quad e^y > 0$$

$$e^y = \frac{1+\sqrt{1-x^2}}{x} \quad \rightarrow y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

$$\therefore \text{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

Similarly we can prove that $\text{csch}^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$

Derivatives

$$\frac{d}{dx}(\sin h^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}(\cos h^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\tan h^{-1} x) = \frac{1}{1-x^2} \quad |x| < 1$$
$$\frac{d}{dx}(\cot h^{-1} x) = \frac{1}{1-x^2} \quad |x| > 1$$
$$\frac{d}{dx}(\sec h^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\csc h^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$$

Integrals

$$\int \frac{dx}{\sqrt{1+x^2}} = \sin h^{-1} x + c$$
$$\int \frac{dx}{\sqrt{x^2-1}} = \cos h^{-1} x + c$$
$$\int \frac{dx}{1-x^2} = \begin{cases} \tan h^{-1} x + c\ , & |x| < 1 \\ \cot h^{-1} x + c\ , & |x| > 1 \end{cases}$$
$$\int \frac{dx}{x\sqrt{1-x^2}} = -\sec h^{-1} x + c$$
$$\int \frac{dx}{x\sqrt{1+x^2}} = -\csc h^{-1} x + c$$

Ex/ Find dy/dx of

(1) $y = 2^x + \sin h^{-1}(x^2)$

$$\frac{dy}{dx} = 2^x . \ln 2 + \frac{1}{\sqrt{1+x^4}} . (2x)$$

(2) $y = (1 + x \csc h^{-1} x)^{10}$

$$\frac{dy}{dx} = 10\,(1 + x \csc h^{-1} x)^9 . [\,x . \frac{-1}{|x|\sqrt{1+x^2}} + \csc h^{-1} x . 1]$$

Ex/ evaluate $\int \frac{\sin\theta\, d\theta}{\sqrt{1+\cos^2\theta}}$

Let $x = \cos\theta \rightarrow dx = -\sin\theta\, d\theta$

$$= \int \frac{-dx}{\sqrt{1+x^2}} = -\sin h^{-1} x + c$$
$$= -\sin h^{-1}(\cos\theta) + c$$

Ex/ Find $\lim\limits_{x \to \infty} (\cosh^{-1} x - \ln x)$ $(\infty - \infty)$

since $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

$$= \lim_{x \to \infty} [\ln(x + \sqrt{x^2 - 1}) - \ln x]$$

$$= \lim_{x \to \infty} \left(\ln \frac{x + \sqrt{x^2 - 1}}{x}\right)$$

$$= \ln\left[\lim_{x \to \infty} \frac{x + \sqrt{x^2 - 1}}{x}\right] = \ln\left[\lim_{x \to \infty} \frac{1 + \sqrt{1 - \frac{1}{x^2}}}{1}\right] = \ln 2$$

Ex/ show that if $y = \tan^{-1} x$ ***then*** $y'' = -2 \sin y \cos^3 y$

$$y = \tan^{-1} x \Longrightarrow \tan y = x$$

$$\sec^2 y \; y' = 1$$

Or $y' = \frac{1}{\sec^2 y} = \cos^2 y$

Or $y' = [\cos y]^2$

$$y'' = 2[\cos y](-\sin y)\, y'$$

$$y'' = 2[\cos y](-\sin y)(\cos^2 y)$$

$$\therefore\; y'' = -2 \sin y \cos^3 y$$

Ex/ evaluate $\int \frac{dx}{\sqrt{1 - e^{2x}}}$

$$= \int \frac{e^x\, dx}{e^x \sqrt{1 - e^{2x}}}$$ *Let* $u = e^x \rightarrow du = e^x\, dx$

$$= \int \frac{du}{u\sqrt{1 - u^2}} = -\,\mathrm{sech}^{-1} u + c = -\,\mathrm{sech}^{-1} e^x + c$$