Chapter One
Fourier Series and Fourier Transform

I. Fourier Series Representation of Periodic Signals

1-Trigonometric Fourier Series:

The trigonometric Fourier series representation of a periodic signal x(t)= x(t + T_0) with fundamental period T_0 is given by

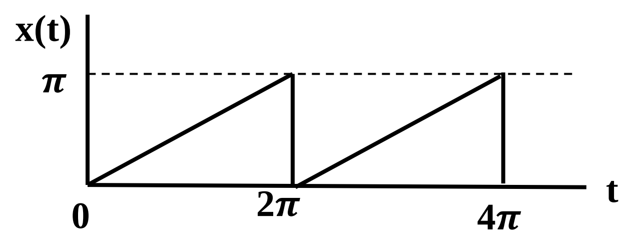
$$\mathbf{x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nw_0 t) + b_n \sin(nw_0 t))}$$

where $\mathbf{w_0} = \frac{2\pi}{T_0}$ is radian frequency,

a_0, a_n and b_n are the Fourier coefficients given by

$$\mathbf{a_0 = \frac{2}{T_0}\int_0^{T_0} x(t)\, dt}\ , \qquad \mathbf{a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\ \cos(nw_0 t)\ dt}\ \mathbf{b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\ \sin(nw_0 t)\ dt}$$

Example 1: Determine the Fourier series to represent the periodic function shown.



Solution:

$x(t) = \frac{t}{2} \qquad 0 < t < 2\pi \qquad , T_0 = 2\pi \qquad w_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$

$$a_0 = \frac{2}{T_0}\int_0^{T_0} x(t)\, dt = \frac{1}{\pi}\int_0^{2\pi} \frac{t}{2}\, dt = \frac{1}{4\pi}\ [t^2]_0^{2\pi} = \pi$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\ \cos(nw_0 t)\ dt = \frac{1}{\pi}\int_0^{2\pi}\frac{t}{2}\ \cos(nt)\ dt = \frac{1}{2\pi}\int_0^{2\pi} t\cos(nt)\ dt$$

Integrating by parts

$$a_n = \frac{1}{2\pi}\left\{\left[\frac{t\sin(nt)}{n}\right]_0^{2\pi} + \left[\frac{\cos(nt)}{n^2}\right]_0^{2\pi}\right\} = 0$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\ \sin(nw_0 t)\ dt = \frac{1}{2\pi}\int_0^{2\pi} t\ \sin(nw_0 t)\ dt = -\frac{1}{n}$$

The general expression for Fourier series is

$$x(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left\{-\frac{1}{n}\sin(nt)\right\} = \frac{\pi}{2} - \left\{\sin t + \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \cdots\right\}$$

Even and Odd Signals:

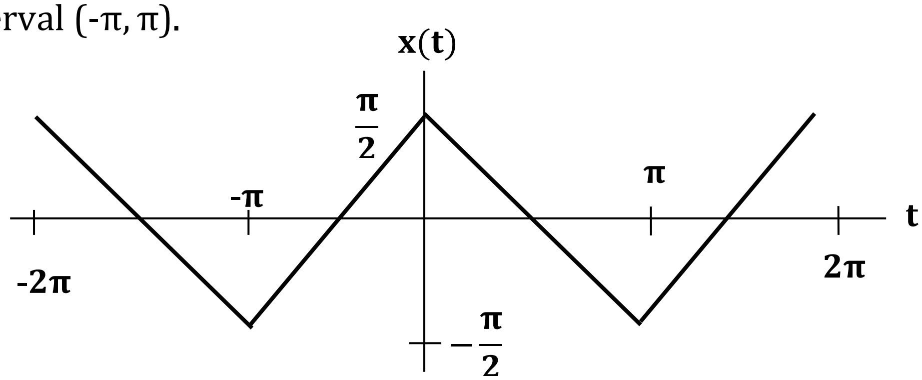
If a periodic signal x(t) is even (x(-t)=x(t)), then b_n= 0 and its Fourier series contains only cosine terms:

$$\mathbf{x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nw_0 t)}$$

If x(t) is odd (x(-t) =-x(t)), then a_n= 0 and its Fourier series contains only sine terms:

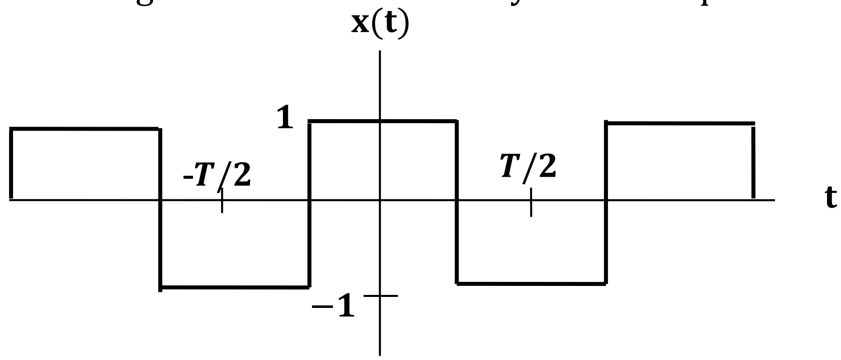
$$\mathbf{x(t) = \sum_{n=1}^{\infty} b_n \sin(nw_0 t)}$$

HW 1: Find the trigonometric Fourier of the triangular waveform shown over the interval (-π, π).



Answer: $x(t) = \frac{4}{\pi}\sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{1}{n^2}\cos nt \qquad -\pi < t < \pi$

HW 2: Find the trigonometric Fourier of symmetric square wave shown.



Answer: $x(t) = \frac{4}{\pi}\left[\cos w_0 t - \frac{1}{3}\cos 3w_0 t + \frac{1}{5}\cos 5w_0 t - \cdots\right]$

2-Complex Exponential Fourier Series Representation:

The complex exponential Fourier series representation of a periodic signal x(t) with fundamental period T_0 is given by

$$\mathbf{x(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{j\, n w_0 t}}$$

where $\boldsymbol{w_0} = \frac{2\pi}{T_0}$ and c_n is complex Fourier coefficients and are given by

$$\mathbf{c_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jnw_0 t}\, dt}$$

The relation between trigonometric and complex exponential Fourier series:

$$\mathbf{a_0/2 = c_0}$$
$$\mathbf{a_n = c_n + c_{-n}} \qquad \mathbf{, n \neq 0}$$
$$\mathbf{b_n = jc_n - jc_{-n}}$$

$$\mathbf{c_n = \frac{a_n - jb_n}{2}}$$
$$\mathbf{c_{-n} = \frac{a_n + jb_n}{2}} \qquad \mathbf{, n \geq 1}$$
$$\mathbf{c_{-n} = c_n^*}$$

Q/ Derive the complex exponential Fourier series from the trigonometric Fourier series?

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nw_0 t) + b_n \sin(nw_0 t))$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{jnw_0 t} + e^{-jnw_0 t}}{2} + b_n \frac{e^{jnw_0 t} - e^{-jnw_0 t}}{2j}\right)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\left\{\frac{a_n - jb_n}{2}\right\} e^{jnw_0 t} + \left\{\frac{a_n + jb_n}{2}\right\} e^{-jnw_0 t}\right)$$

Define $c_n = \frac{a_n - jb_n}{2}$, $c_n^* = \frac{a_n + jb_n}{2} = c_{-n}$ and $c_0 = a_0/2$

$$x(t) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{jnw_0 t} + c_{-n} e^{-jnw_0 t}\right)$$
$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jnw_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jnw_0 t}$$

As n ranges from 1 to ∞ , -n ranges from -1 to -∞

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jnw_0 t} + \sum_{n=-1}^{-\infty} c_n\, e^{jnw_0 t}$$

$$= \sum_{n=-\infty}^{-1} c_n\, e^{jnw_0t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jnw_0t} \quad = \sum_{n=-\infty}^{\infty} c_n\, e^{jnw_0t}$$

Amplitude and Phase Spectra of a Periodic Signal:

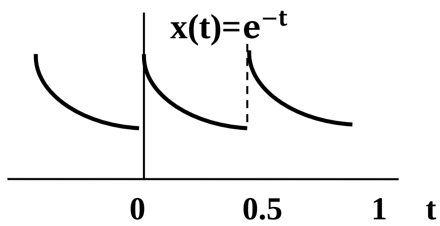
$$\mathbf{c_n = |c_n| e^{j\theta_n}}$$

where $|c_n|$ is the amplitude spectrum of the periodic signal x(t)
θ_n is the phase spectrum of x(t)

$$|\mathbf{c_n}| = |\mathbf{c_{-n}}| = \frac{\sqrt{\mathbf{a_n^2+b_n^2}}}{\mathbf{2}} \quad \text{and} \quad \boldsymbol{\theta_{-n} = -\theta_n}$$

Hence, the amplitude spectrum is an even function of w and the phase spectrum is an odd function of w for a real periodic signal.

Example 2: Find the exponential Fourier series and corresponding frequency spectra for the function x(t) shown.



Solution:

T_0=0.5 , $w_0=\frac{2\pi}{T_0} = 4\pi$

$$c_n = 2\int_0^{0.5} e^{-t}\, e^{-j4\pi nt}\, dt = 2\left[\frac{e^{-(1+j4\pi n)t}}{-(1+j4\pi n)}\right]_0^{0.5} = \frac{2}{1+j4\pi n}\left[1 - e^{-0.5}e^{-j2\pi n}\right]$$

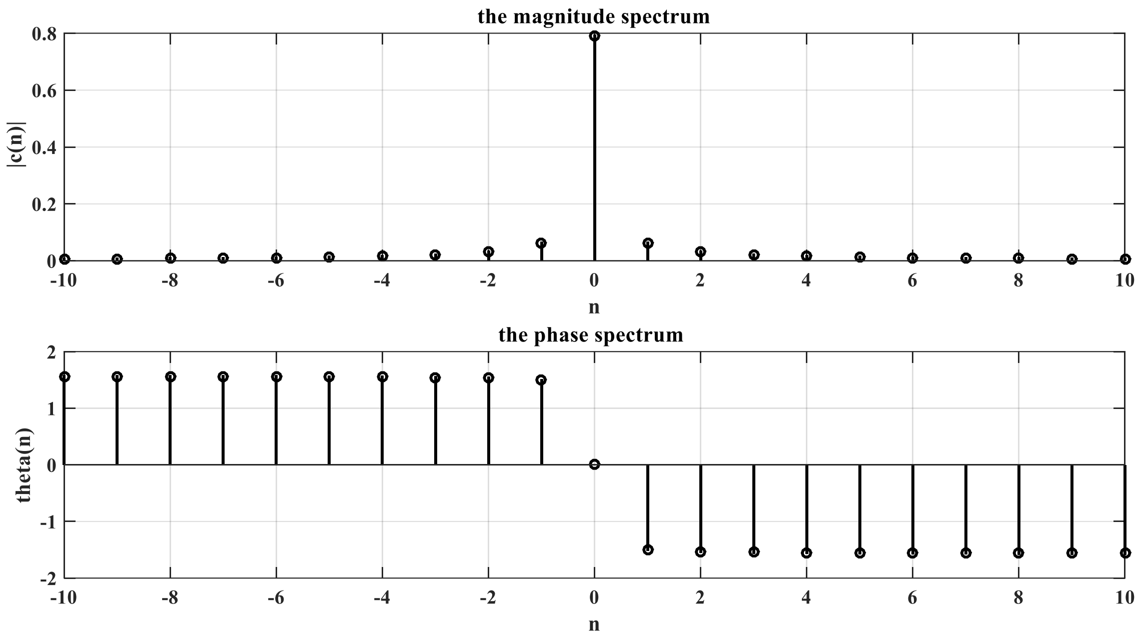
since $e^{-j2\pi n} = 1$

$$c_n = \frac{0.79}{1+j4\pi n}$$

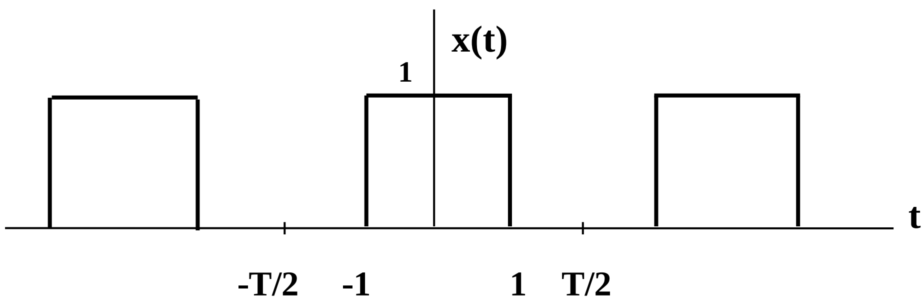
The complex exponential Fourier series, $x(t) = \sum_{n=-\infty}^{\infty} \frac{0.79}{1+j4\pi n} e^{j\,n4\pi t}$

The amplitude spectrum, $|c_n| = \frac{0.79}{\sqrt{1+16\pi^2 n^2}}$, c_0=0.79.

The phase spectrum, $\theta_n = -\tan^{-1}(4\pi n)$



Example 3: Find the complex exponential Fourier series and corresponding frequency spectra for the function shown for T=4,8, and 16.



Solution:

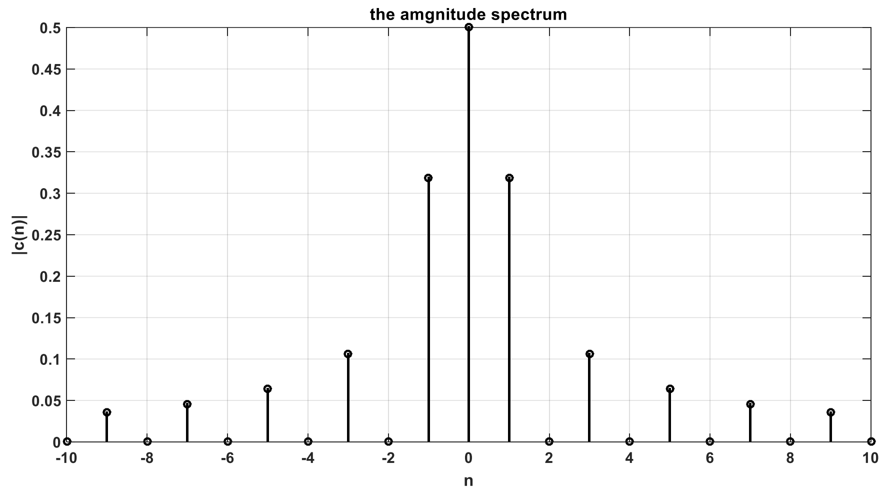
$$c_n = \frac{1}{T}\int_0^T e^{-jw_0 nt}\,dt = \frac{1}{T}\left[\frac{e^{-jnw_0 t}}{-jnw_0 t}\right]_{-1}^{1} = \frac{2}{T}\left[\frac{e^{jnw_o} - e^{-jnw_o}}{2jnw_0}\right] = \frac{2}{T}\,\frac{\sin nw_0}{nw_0}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{2}{T}\,\frac{\sin nw_0}{nw_0}\, e^{j\,nw_0 t}$$

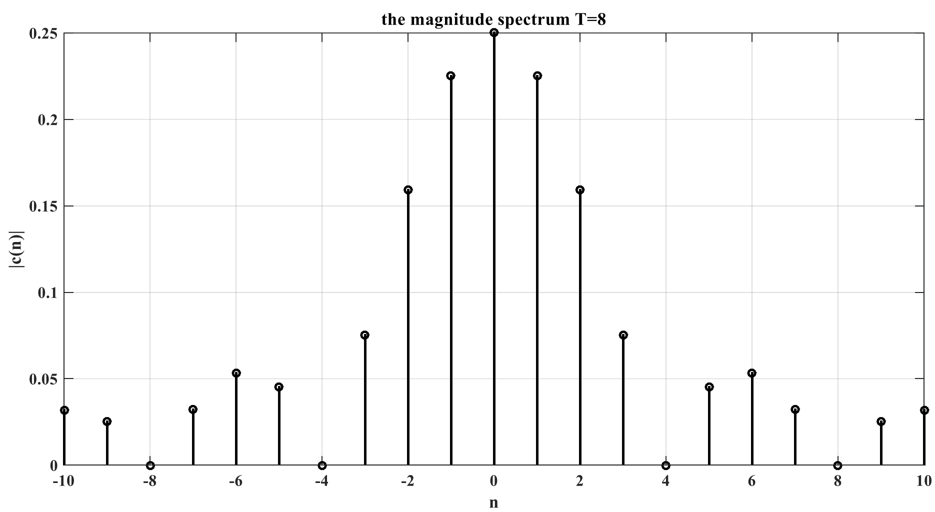
In case T=4: $w_0 = \frac{2\pi}{T} = \frac{\pi}{2}$ $\quad c_n = \frac{1}{2}\,\frac{\sin(\frac{n\pi}{2})}{\frac{n\pi}{2}}$ and $c_0 = \lim_{n\to 0} c_n = \frac{1}{2}$

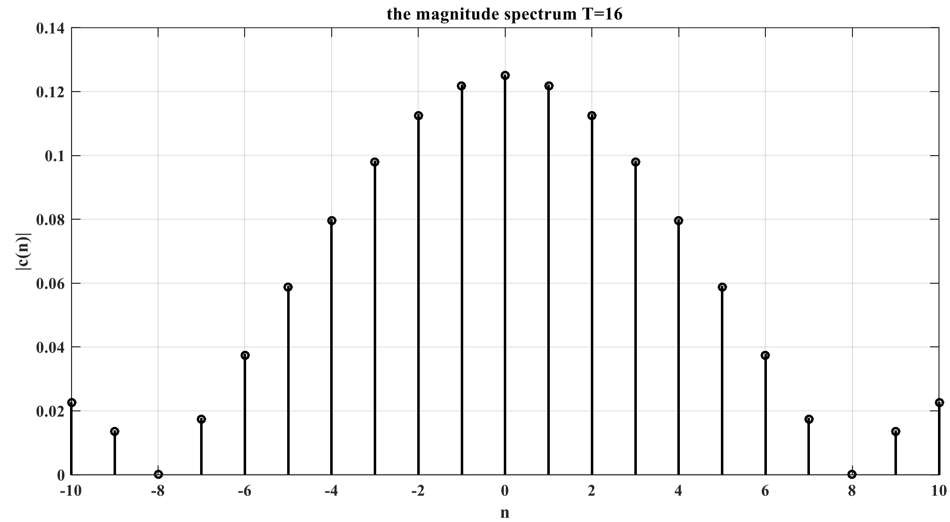
c_n=0 for $\frac{n\pi}{2} = \pm\pi, \pm 2\pi, \pm 3\pi, ...$

n= $\pm 2, \pm 4, \pm 6$, ...



Similarly for T=8 and 16 the magnitude spectrum is shown in respectively





Parseval's Theorem for Power Signal

The average power P of a periodic signal x (t) over any period T is represented by the complex exponential Fourier series as

$$\mathbf{P = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2}$$

Example 4: Determine the average power of x(t)=2 sin (100t).

Solution:

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \frac{1}{T}\int_{-T/2}^{T/2} |2\sin(100t)|^2\, dt = 2$$

$$x(t) = 2\,\frac{e^{j100t} - e^{-j100t}}{2j} = -je^{j100t} + j\, e^{-j100t}$$

Comparing to $x(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{j\, n w_0 t} = c_1 e^{j\,100t} + c_{-1} e^{-j\,100t}$

$c_1=-j$, $c_{-1}=j$ and

$$P = \sum_{n=-\infty}^{\infty} |c_n|^2 = |c_1|^2 + |c_{-1}|^2 = 1 + 1 = 2$$

HW3: find the complex exponential Fourier series and spectral frequency for the functions

a) $x(t) = \begin{cases} 1 & 0 < t < \frac{T_0}{2} \\ -1 & \frac{T_0}{2} < t < T_0 \end{cases}$ where $x(t+T_0)=x(t)$

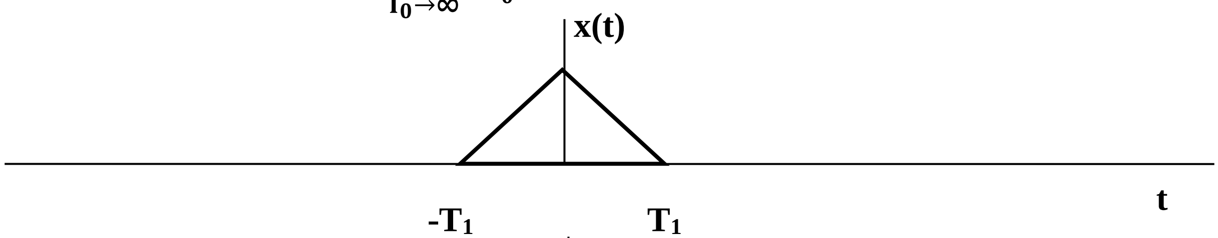
b) $x(t) = \begin{cases} \frac{2}{T}t & 0 < t < \frac{T}{2} \\ -\frac{2}{T}t + 2 & \frac{T}{2} < t < T \end{cases}$ where $x(t+T)=x(t)$

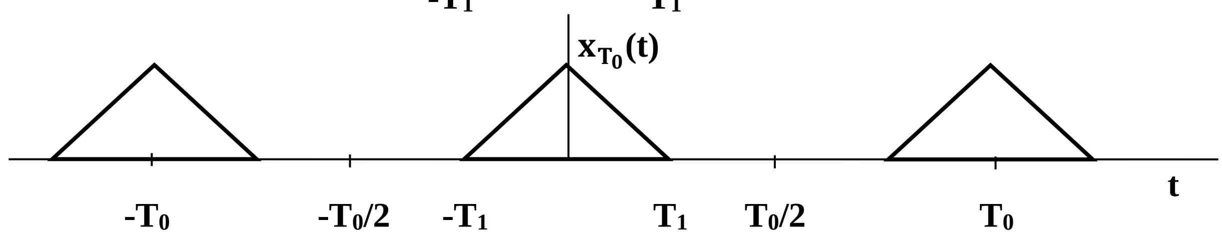
II. Fourier Transform for Non periodic Signal

From Fourier Series to Fourier Transform:

Let x(t) be a *nonperiodic* signal of finite duration, that is $\mathbf{x(t) = 0 \quad |t| > T_1}$

Let $x_{T_0}(t)$ be a *periodic* signal formed by repeating x(t) with fundamental period T_0 in such that $\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$





The complex exponential Fourier series of $x_{T_0}(t)$ is given by

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{j n w_0 t} \qquad w_0 = \frac{2\pi}{T_0}$$

where

$$c_n = \frac{1}{T_0}\int_0^{T_0} x_{T_0}(t)\, e^{-jnw_0 t}\, dt$$

Since $x_{T_0}(t) = x(t)$ for $|t|<T_0/2$ and also since x(t)= 0 outside this interval

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jnw_0 t}\, dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\, e^{-jnw_0 t}\, dt$$

Let us define X(w) as $\mathbf{X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt}$

$$c_n = \frac{1}{T_0} X(nw_0)$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} X(nw_0)\, e^{j\, nw_0 t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(nw_0)\, e^{j\, nw_0 t}\, w_0$$

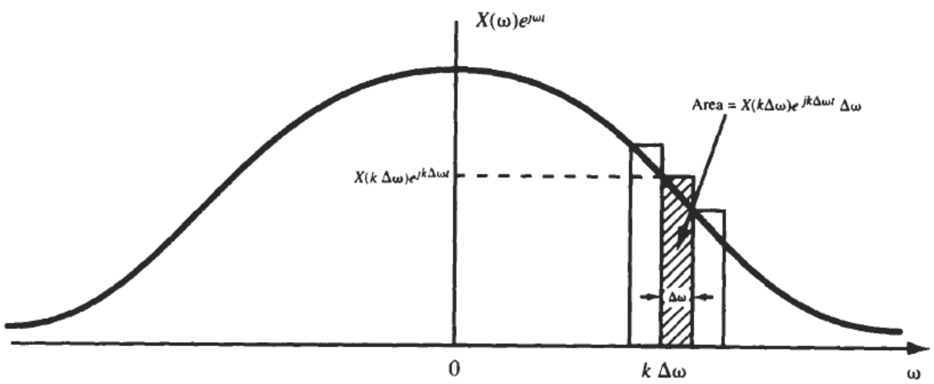
As $T_0 \to \infty$, $w_0=2\pi/T_0$ becomes infinitesimal ($w_0 \to 0$). Thus, let $w_0 = \Delta w$ then

$$x_{T_0}(t)\Big|_{T_0\to\infty} \to \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(nw_0)\, e^{j\, n\, \Delta w\, t}\, \Delta w$$

Therefore

$$x(t) = \lim_{T_0\to\infty} x_{T_0}(t) = \lim_{w_0\to 0} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(nw_0)\, e^{j\, n\, \Delta w\, t}\, \Delta w$$

The sum on the right-hand side of Equation above can be viewed as the area under the function $X(w)e^{jwt}$ as shown



Therefore, the Fourier representation of a nonperiodic x(t) is

$$\mathbf{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w)\, e^{j\,w\,t}\; dw}$$

Fourier Transform Pair:

Define the function X(w) as the Fourier transform of x(t) and x(t) is inverse Fourier transform of X(w). Then

$$\mathbf{X(w) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt \qquad F.\,T.}$$

$$\mathbf{x(t) = \mathcal{F}^{-1}\big(X(w)\big) = \frac{1}{2\pi}\int_{n=-\infty}^{\infty} X(w)\, e^{j\,w\,t}\; dw \qquad I.\,F.\,T.}$$

The pair of a Fourier transform denoted by $\mathbf{x(t) \leftrightarrow X(w)}$

Fourier Spectra:

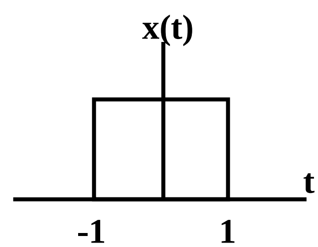
The Fourier transform X(w) is the frequency domain of nonperiodic signal x(t) and is referred to as the spectrum or Fourier spectrum of x(t). In general it is complex and can be expressed as:

$$\boldsymbol{X(w) = |X(w)|e^{j\phi(w)}}$$

where |X(w)| is the magnitude spectrum of x(t)

ϕ(w) is the phase spectrum of x(t)

Example 5: Plot the spectrum for the gate function shown.



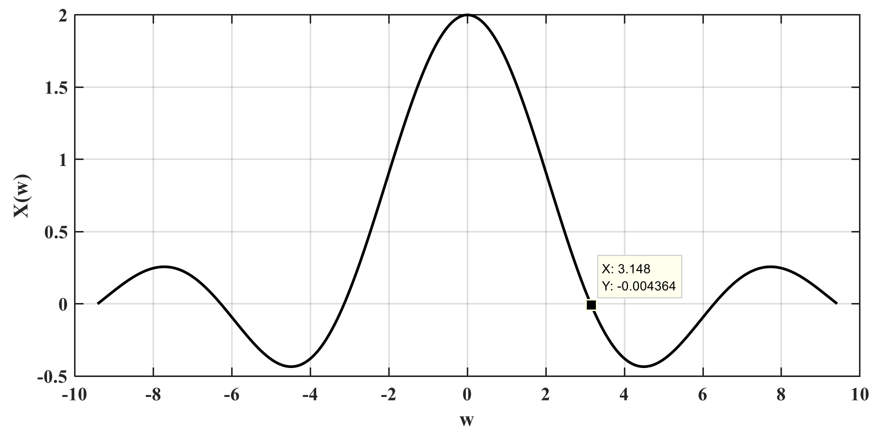
Solution:

$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}\, dt = \int_{-1}^{1} e^{-jwt}\, dt = \left[\frac{e^{-jwt}}{-jw}\right]_{-1}^{1}$$

$$= \frac{e^{-jw} - e^{jw}}{-jw} = \frac{2}{w}\sin w$$

X(0)=$\lim_{w\to 0} \frac{2\sin w}{w} = 2$

X(w)=0 for w=$\pm\pi, \pm 2\pi, \ldots$



Fourier Sine and Cosine Transform

For even function:

$$\mathbf{X(w) = 2\int_0^\infty x(t)\cos wt\,dt} \quad \text{,and} \quad \mathbf{x(t) = \frac{1}{\pi}\int_0^\infty X(w)\cos wt\,dw}$$

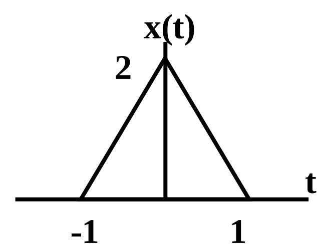
For odd function:

$$\mathbf{X(w) = 2\int_0^\infty x(t)\sin wt\,dt} \quad \text{,and} \quad \mathbf{x(t) = \frac{1}{\pi}\int_0^\infty X(w)\sin wt\,dw}$$

Example 6: Repeat Example 5 using odd or even properties

Solution: $X(w) = 2\int_0^\infty x(t)\cos wt\ dt = 2\int_0^1 \cos wt\,dt =$
$2\left[\frac{\sin wt}{w}\right]_0^1 = 2\ \frac{\sin w}{w}$

HW 4: plot the spectrum for the following function



Answer: $X(w) = 2\left(\frac{\sin\frac{w}{2}}{\frac{w}{2}}\right)^2$

Some Special functions and their transform

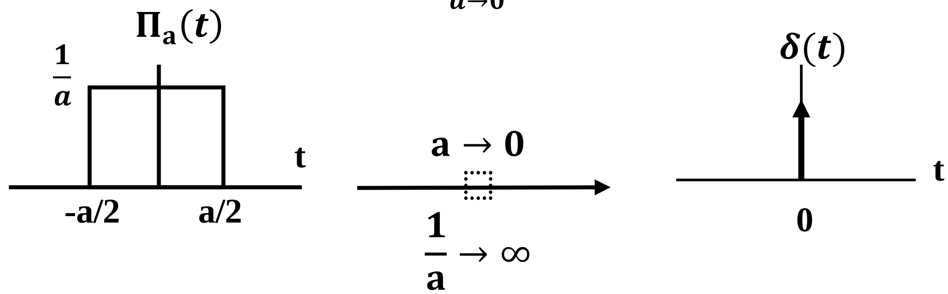
1- The Dirac Delta (unit impulse function) $\delta(t)$

Assume rectangle function (top-hat function) as shown below. It can be denoted by the symbol $\prod_a(t)$ and is defined as:

$$\Pi_a(t) = \begin{cases} 0 & t < -a/2 \\ \frac{1}{a} & -a/2 < t < a/2 \\ 0 & \frac{a}{2} < t \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \Pi_a(t)\,dt = \int_{-a/2}^{a/2} \frac{1}{a}\,dt = 1$$

then dirac delta function $\delta(t)$ is defined as

$$\boldsymbol{\delta(t) = \lim_{a\to 0} \Pi_a(t)}$$



Properties of dirac delta:

- $\int_{-\infty}^{\infty} \delta(t)dt = 1$
- $\int_{-\infty}^{\infty} x(t)\,\delta(t)dt = x(0)$
- $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$
- $\delta(a\,t) = \frac{1}{|a|}\,\delta(t)$ and $\delta(-t) = \delta(t)$
- $\int_{-\infty}^{\infty} f(t)\delta(a(t-t_0))dt = \frac{1}{|a|}\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = \frac{1}{|a|}f(t_0)$

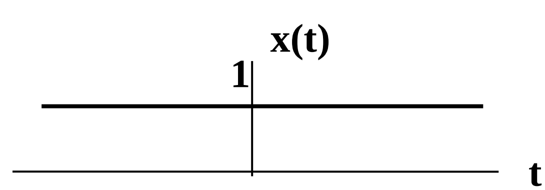
The Fourier transform of dirac delta is given by:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-jwt}\,dt = 1$$

Also, $\mathcal{F}\{\prod_a(t)\} = \int_{-a/2}^{a/2} \frac{1}{a} e^{-jwt}dt = \frac{1}{a}\left[\frac{e^{-jwt}}{-jw}\right]_{-a/2}^{a/2} = \frac{1}{a}\left(\frac{e^{-jwa/2} - e^{jwa/2}}{-jw}\right) = \frac{\sin\frac{wa}{2}}{\frac{wa}{2}}$

Example 7: find the Fourier transform for the following function
Solution:

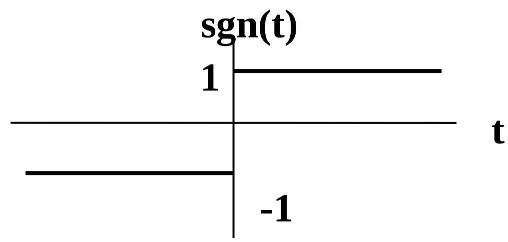
$$\mathcal{F}^{-1}\{\delta(w)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(w)e^{jwt}\,dw = \frac{1}{2\pi}$$



$\mathcal{F}^{-1}\{2\pi\ \delta(\mathrm{w})\} = 1$

$\therefore \mathcal{F}\{1\} = 2\pi\ \delta(\mathrm{w})$

2-*Signum function sgn(t)*

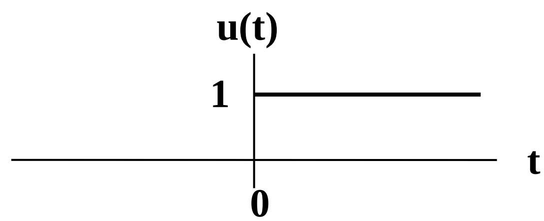


$$\mathrm{sgn(t)}=\begin{cases}1 & t>0\\ -1 & t<0\end{cases}$$

$$\mathcal{F}\{\mathrm{sgn(t)}\} = \int_{-\infty}^{\infty} \mathrm{sgn(t)e^{-jwt}}\,\mathrm{dt} = 2\int_{0}^{\infty} \mathrm{e^{-jwt}}\,\mathrm{dt}$$

$$= \frac{2}{\mathrm{jw}}$$

3-Unit step function u(t)



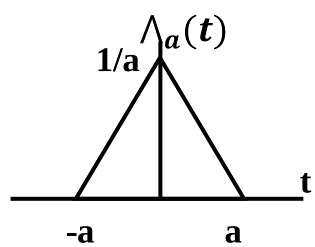
$$\mathrm{u(t)}=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$

$\mathrm{u(t)}=\int_{-\infty}^{t} d(\tau)d\tau$

$\mathrm{u(t)}=\frac{1}{2}+\frac{1}{2}\,\mathrm{sgn(t)}$

$$\mathcal{F}\{\mathrm{u(t)}\} = \mathcal{F}\left\{\frac{1}{2}+\frac{1}{2}\ \mathrm{sgn(t)}\right\} = \frac{1}{2}\mathcal{F}\{1\}+\frac{1}{2}\mathcal{F}\{\ \mathrm{sgn(t)}\} = \pi\delta(\mathrm{w}) + \frac{1}{\mathrm{jw}}$$

4- The triangle function $\Lambda_a(t)$



$$\Lambda_\mathrm{a}(\mathrm{t}) = \begin{cases}\frac{\mathrm{a+t}}{\mathrm{a^2}} & -\mathrm{a}<t<0\\ \frac{\mathrm{a-t}}{\mathrm{a^2}} & 0<t<a\\ 0 & |\mathrm{t}|>a\end{cases}$$

$$\mathcal{F}\{\Lambda_\mathrm{a}(\mathrm{t})\} = \left(\frac{\sin\frac{\mathrm{wa}}{2}}{\frac{\mathrm{wa}}{2}}\right)^2 \quad \text{(see HW 4)}$$

Example 8: $\mathcal{F}\{\mathrm{e^{-at}u(t)}\} = \int_0^\infty \mathrm{e^{-at}\ e^{-jwt}dt} = \left[\frac{\mathrm{e^{-(a+jw)t}}}{\mathrm{-(a+jw)}}\right]_0^\infty = \frac{1}{\mathrm{a+jw}}$

Properties of The Fourier Transform

1-Linearity: $\mathrm{a_1x_1(t)+\ a_2x_2(t) \leftrightarrow a_1X(w) + a_2X_2(w)}$

2-Time shifting: $x(t-t_0) \leftrightarrow e^{-jwt_0}X(w)$

3-Frequecny shifting: $e^{jw_0t}x(t) \leftrightarrow X(w-w_0)$

4-Time scaling: $x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{w}{a}\right)$

5-Time reversal: $x(-t) \leftrightarrow X(-w)$

6-Complex Conjugate: $x^*(t) \leftrightarrow X^*(-w)$

7- Duality (symmetry): $X(t) \leftrightarrow 2\pi\, x(-w)$

8- Differentiation in the Time Domain: $\frac{d\,x(t)}{dt} \leftrightarrow jw\,X(w)$

$$\frac{d^n\,x(t)}{dt^n} \leftrightarrow (jw)^n\,X(w)$$

9- Differentiation in the Frequency Domain: $(-jt)x(t) \leftrightarrow \frac{d\,X(w)}{d}$

10-Integration in the Time Domain: $\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \pi X(0)\delta(w) + \frac{1}{Jw}X(w)$

11-Time convolution: if $x_1(t) \circledast x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$

then $x_1(t) \circledast x_2(t) \leftrightarrow X_1(w)X_2(w)$

12-Multiplication (Frequency convolution): $x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi}X_1(w) \circledast X_2(w)$

13- Parseval's Relation: $\int_{-\infty}^{\infty}|x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(w)|^2\,dw$

Example 9: using properties above to find the Fourier transform of the following functions:

1- $\mathcal{F}\{\delta(t-t_0)\} = e^{-jwt_0}\mathcal{F}\{\delta(t)\} = e^{-jwt_0}$

2- $\mathcal{F}\{f(t)\cos w_0t\} = \mathcal{F}\left\{f(t)\frac{e^{jw_0t}+e^{-jw_0t}}{2}\right\} = \frac{1}{2}\mathcal{F}\{f(t)e^{jw_0t}\} + \frac{1}{2}\mathcal{F}\{f(t)e^{-jw_0t}\}$

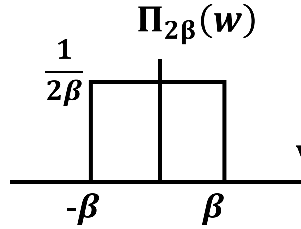
$$= \frac{1}{2}F(w-w_0) + \frac{1}{2}F(w+w_0)$$

3- $\mathcal{F}\{te^{-at}u(t)\} = \frac{1}{-j}\frac{d}{dw}\left[\frac{1}{a+jw}\right] = \frac{1}{-j}\frac{-j}{(a+jw)^2} = \left(\frac{1}{a+jw}\right)^2$

4- $\mathcal{F}\left\{\frac{\sin\beta t}{\beta t}\right\}$

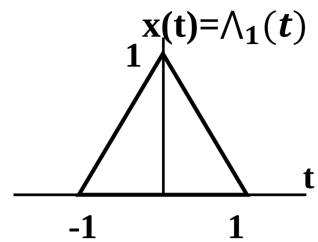
Since $\mathcal{F}\{\prod_a(t)\} = \frac{\sin\frac{wa}{2}}{\frac{wa}{2}}$ use Duality property

$$\mathcal{F}\left\{\frac{\sin\frac{ta}{2}}{\frac{at}{2}}\right\} = 2\pi \prod_a(-t) = 2\pi \prod_a(w)$$



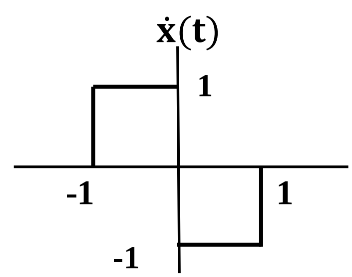
If a=$2\beta \rightarrow \mathcal{F}\left\{\frac{\sin\beta t}{\beta t}\right\} = 2\pi \prod_{2\beta}(w)$

5- $\mathcal{F}\left\{\left(\frac{\sin\beta t}{\beta t}\right)^2\right\}$ **(HW 5)**



Example 10: Use differentiation property to find X(w)

for the function shown. $x(t) = \begin{cases} 1+t & -1<t<0 \\ 1-t & 0<t<1 \\ 0 & |t|>a \end{cases}$



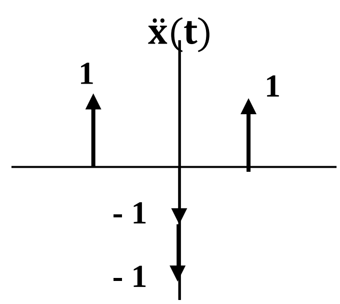
Solution :

$\ddot{x}(t) = \delta(t+1) + \delta(t-1) - 2\delta(t)$

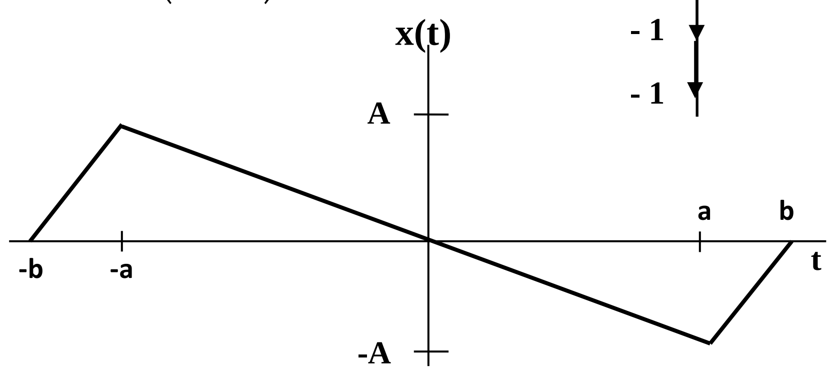
$\mathcal{F}\{\ddot{x}(t)\} = e^{jw} + e^{-jw} - 2 = 2\cos w - 2 = 2(\cos w - 1)$

$\mathcal{F}\{\ddot{x}(t)\} = (jw)^2\, X(w)$

$$X(w) = \frac{\mathcal{F}\{\ddot{x}(t)\}}{-w^2} = \frac{2(1-\cos w)}{w^2} = \left(\frac{\sin\frac{w}{2}}{\frac{w}{2}}\right)^2$$



HW 6:



HW 7: show that $\frac{1}{a+jt} \leftrightarrow 2\pi e^{aw}u(-w)$

Example 11: $\mathcal{F}\{e^{-a|t|}\} = \mathcal{F}\{e^{at}u(-t) + e^{-at}u(t)\} = \frac{1}{a+jw} + \frac{1}{a-jw} = \frac{2a}{a^2+w^2}$

Convolution Theorem

The convolution of two functions ***f(t)*** and ***g(t)*** is defined as:

$$\boldsymbol{h(t) = f(t) \circledast g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau}$$

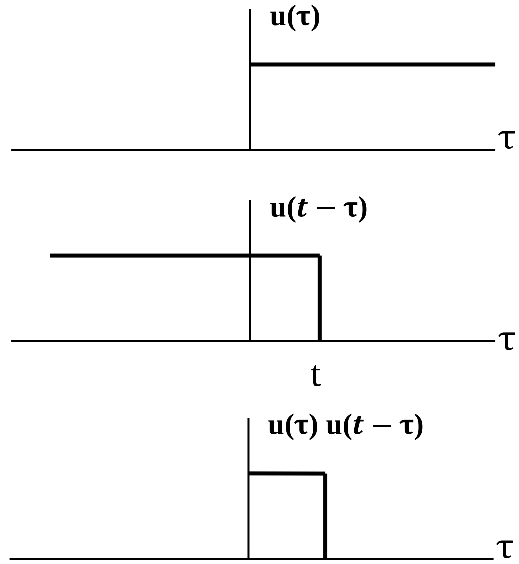
$$\boldsymbol{= \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau}$$

Example 12: find the following convolution

i- $u(t) \circledast \delta(t)$

ii- $u(t) \circledast u(t)$

iii- $e^{-at}u(t) \circledast u(t)$



Solution: i) $h(t)=\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = u(t)$

ii) $h(t)=\int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau$

$$= \int_0^t 1\, d\tau = t$$

iii) $h(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau = \int_0^t e^{-a\tau}d\tau = \left[\frac{e^{-a\tau}}{-a}\right]_0^t = \frac{1-e^{-at}}{a}$

Example 13: Find the inverse Fourier transform for $\mathbf{X(w)} = \frac{\mathbf{1}}{\mathbf{(a+jw)^2}}$ using time convolution property.

Solution: $X(w) = X_1(w)X_2(w) \rightarrow x(t) = \mathcal{F}^{-1}\big(X_1(w)X_2(w)\big) = x_1(t) \circledast x_2(t)$

$$x_1(t) = \mathcal{F}^{-1}\left\{\frac{1}{a+jw}\right\} = e^{-at}\,u(t) \quad , \quad x_2(t) = \mathcal{F}^{-1}\left(\frac{1}{a+jw}\right) = e^{-at}\,u(t)$$

$x(t)= \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-a(t-\tau)}u(t-\tau)d\tau$

$= \int_0^t e^{-a\tau}e^{-a(t-\tau)}d\tau = e^{-at}\int_0^t d\tau = t\,e^{-at} \quad t > 0 = te^{-at}\,u(t)$

HW 8: using time convolution property to find $\mathcal{F}^{-1}\left\{\frac{5}{6+5jw-w^2}\right\}$

Example 14: Find the Fourier transform of $\mathbf{f(t)} = \int_{-\infty}^{t} \boldsymbol{\delta(\tau)d\tau}$ using integration in time domain property.

Solution :

$$\int_{-\infty}^{t} x\,(\tau)d\tau \leftrightarrow \pi X(0)\delta(w) + \frac{1}{Jw}X(w)$$

$X(w)=\mathcal{F}\{\delta(t)\} = 1 \qquad \rightarrow X(0)=1$

$F(w)= \pi X(0)\delta(w) + \frac{1}{Jw}X(w) = \pi\delta(w) + \frac{1}{Jw}$

II. Linear Time Invariant (LTI) System

A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal. Let x (t) and y(t) be the input and output signals, respectively, of a system. Then the system is viewed as a transformation (or mapping) of x(t) into y(t). This transformation is represented by the mathematical notation

$$\mathbf{y(t)=T\{x(t)\}}$$

The system is said to be linear time invariant (LTI) if two conditions are satisfied:

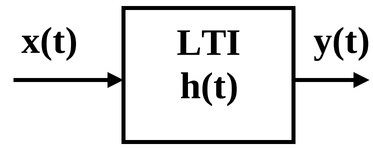
1- $T\{a_1x_1(t)+a_2x_2(t)\}=a_1y_1(t)+a_2y_2(t)$ (linear system)
2- $T\{x(t-t_0)\}=y(t-t_0)$ (time invariant system)

For example y(t)=2x(t) is LTI system

Impulse Response:

The *impulse response* h(t) of a continuous-time LTI system is defined to be the response of the system when the input is δ(t), that is,

$$\mathbf{h(t)=T\{\delta\,(t)\}}$$



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\; d\tau$$

Since the system is linear, the response y(t) of the system to an arbitrary input x(t) can be expressed as:

$$y(t) = T\{x(t)\} = T\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau\right\} = \int_{-\infty}^{\infty} x(\tau)T\{\delta(t-\tau)\}\, d\tau$$

Since the system is time-invariant, $h(t-\tau) = T\{\delta(t-\tau)\}$

$$\mathbf{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = \boldsymbol{x(t) \circledast h(t)}} \quad \textbf{\{convolution integral\}}$$

Frequency Response:

Take Fourier transform for both sides in convolutional integral equation above

$$Y(w) = \mathcal{F}\{y(t)\} = \mathcal{F}\{x(t) \circledast h(t)\} = X(w)H(w)$$

The frequency response of the system:

$$\mathbf{H(w) = \frac{Y(w)}{X(w)} = |H(w)|e^{j\theta_H(w)}}$$

$|H(w)|$=magnitude response of the system.
$\theta_H(w)$=phase response of the system.

The response of complex exponential signal { $x(t)=e^{jw_0t}$ }:

$X(w) = 2\pi\, \delta(w - w_0)$
$Y(w)=X(w)H(w)= 2\pi\, \delta(w - w_0)\, H(w)= 2\pi\, \delta(w - w_0)\, H(w_0)$
$y(t)= \mathcal{F}^{-1}\{2\pi\, \delta(w - w_0)H(w_0)\} = H(w_0)e^{jw_0t}$

$$\mathbf{y(t)= H(w_0)x(t)}$$

where w_0 is the radian frequency of the input signal.

The response for the non periodic signal:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$
$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)H(w)e^{jwt}dw$$

Example 15: find the output response if the frequency response of the system is $H(w)=\frac{1}{jw+2}$ for the following input:

i) $x(t) = e^{-t}u(t)$
ii) $x(t)=e^{j2t}$

Solution :

i) $X(w)=\mathcal{F}\{e^{-t}u(t)\} = \frac{1}{jw+1}$

$Y(w)=X(w)H(w)=\frac{1}{(jw+1)(jw+2)} = \frac{1}{jw+1} - \frac{1}{jw+2}$

$y(t)=\mathcal{F}^{-1}\left\{\frac{1}{jw+1} - \frac{1}{jw+2}\right\} = (e^{-t} - e^{-2t})u(t)$

ii) $y(t)= H(w_0)\, x(t)= \frac{1}{j2+2} e^{j2t}$

IV. Correlation Functions

1- *Correlation of periodic function*

Autocorrelation, $\boldsymbol{\phi}\ (\boldsymbol{\tau}) = \frac{1}{T}\int_{-T/2}^{T/2} \mathbf{x(t)x(t \pm \tau)dt}$

Crosscorrelation, $\boldsymbol{\phi}_{xy}\ (\boldsymbol{\tau}) = \frac{1}{T}\int_{-T/2}^{T/2} \mathbf{x(t)y(t \pm \tau)dt}$

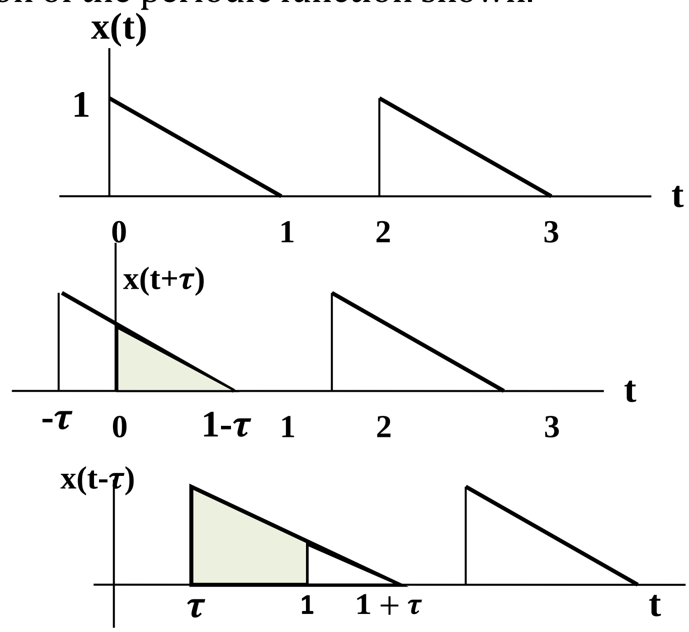
2- *Correlation nonperiodic function*

Autocorrelation, $\boldsymbol{\lambda}\ (\boldsymbol{\tau}) = \int_{-\infty}^{\infty} \mathbf{x(t)x(t \pm \tau)dt}$

Crosscorrelation, $\boldsymbol{\lambda}_{xy}\ (\boldsymbol{\tau}) = \int_{-\infty}^{\infty} \mathbf{x(t)y(t \pm \tau)dt}$

Example 16: Find the autocorrelation function of the periodic function shown.

$x(t)=\begin{cases} 1-t & 0<t<1 \\ 0 & 1<t<2 \end{cases}$



Solution:

$$\phi(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

$$= \frac{1}{2}\int_{0}^{1-\tau} (1-t)\big(1-(t+\tau)\big)dt$$

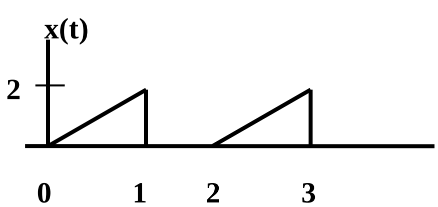
$$= \frac{\tau^3}{12} - \frac{\tau}{4} + \frac{1}{6}$$

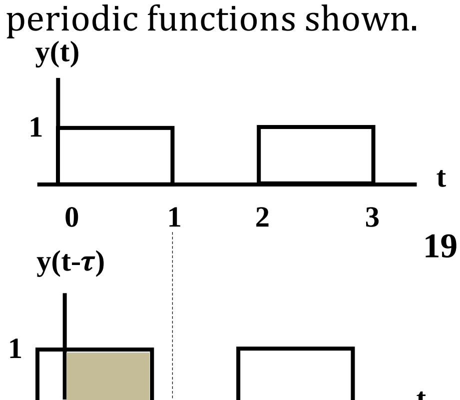
$$\phi(-\tau) = \frac{1}{2}\int_{\tau}^{1} (1-t)\big(1-(t-\tau)\big)dt$$

$$= \frac{\tau^3}{12} - \frac{\tau}{4} + \frac{1}{6}$$

$\therefore \phi(\tau) = \phi(-\tau)$

Example 17: Find the crosscorrelation of the two periodic functions shown.

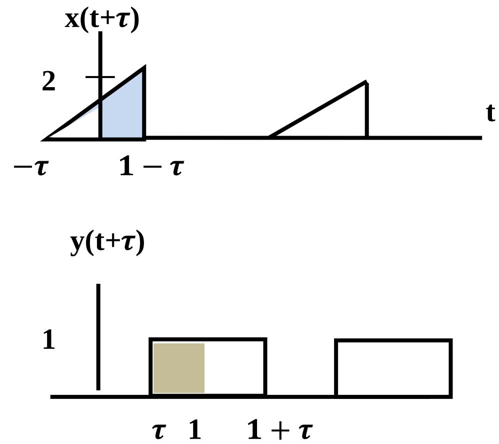




Solution:

i) To find $\boldsymbol{\phi_{xy}(\tau)}$ and $\boldsymbol{\phi_{yx}(-\tau)}$

$$\phi_{xy}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)y(t+\tau)\,dt$$
$$= \frac{1}{2}\int_{0}^{1-\tau} 2t\,dt = \frac{1}{2}(1-\tau)^2$$
$$\phi_{yx}(-\tau) = \frac{1}{T}\int_{-T/2}^{T/2} y(t)x(t-\tau)dt$$
$$= \frac{1}{2}\int_{\tau}^{1+\tau} 2(t-\tau)\ dt = \frac{1}{2}(1-\tau)^2$$



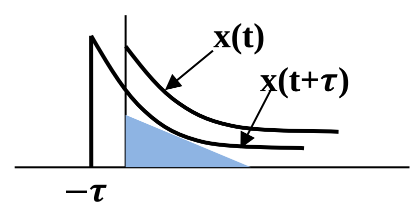
i) To find $\boldsymbol{\phi_{yx}(\tau)}$ and $\boldsymbol{\phi_{xy}(-\tau)}$

$$\phi_{yx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} y(t)x(t+\tau)\,dt$$
$$= \frac{1}{2}\int_{0}^{1-\tau} 2(t+\tau)dt = \frac{1}{2}(1-\tau^2)$$
$$\phi_{xy}(-\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)y(t-\tau)dt$$
$$= \frac{1}{2}\int_{\tau}^{1} 2t\ dt = \frac{1}{2}(1-\tau^2)$$

Example 18: Find the autocorrelation function of the function

$$x(t) = \begin{cases} e^{-at} & t > 0, a > 0 \\ 0 & t < 0 \end{cases}$$

Solution:



$$\lambda(\tau) = \int_{0}^{\infty} e^{-at}e^{-a(t+\tau)}dt = \frac{e^{-a\tau}}{2a}$$

***properties of correlation function**:*

1- Symmetry: $\phi(\tau) = \phi(-\tau)$
2- Average power: $P = \phi(0)$
3- Periodicity : if x(t) =x(t+T) then
$\phi(\tau) = \phi(\tau + T)$
4- Dc value: if f(t)= x(t)+m_1 and g(t)=y(t)+ m_2 then
$\phi_{fg}(\tau) = \phi_{xy}(\tau) + m_1 m_2$
5- Maximum value: $|\phi(\tau)| \leq \phi(0)$
6- Additivity: if z(t)=x(t)+y(t) then
$\phi_z(\tau) = \phi_x(\tau) + \phi_y(\tau) + \phi_{xy}(\tau) + \phi_{yx}(\tau)$

VI. Power Spectrum of periodic signal and Autocorrelation

We can write Autcorrelation in terms of exponential Fourier coefficients as:

$$\boldsymbol{\phi(\tau) = |c_0|^2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |c_n|^2 \, e^{jnw_0 t}}$$

The average power, P= $\boldsymbol{\phi(0) = \sum_{n=-\infty}^{\infty} |c_n|^2}$

PSD= $\boldsymbol{\mathcal{F}\{\phi(\tau)\} = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2 \, \delta(w - nw_0)}$

***Proof*:**

$$\phi(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt \quad = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} c_n \, e^{jnw_0 t} \sum_{m=-\infty}^{\infty} c_m \, e^{jmw_0(t+\tau)} \, dt$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m e^{jnw_0 t} \underbrace{\left(\frac{1}{T}\int_{T/2}^{-T/2} e^{j(m+n)w_0 t} \, dt\right)}_{\substack{=1 \text{ if } m=-n \\ =0 \text{ otherwise}}}$$

$$= \sum_{n=-\infty}^{\infty} c_n c_{-n} e^{jnw_0 t} = \sum_{n=-\infty}^{\infty} |c_n|^2 \, e^{jnw_0 t}$$

Example 19: Find PSD and the average power for the periodic signal using the correlation function. x(t)=A cos (w_0t+θ)

Solution: x(t) = $\frac{A}{2} e^{j\theta} e^{jw_0 t} + \frac{A}{2} e^{-j\theta} e^{-jw_0 t}$

$$c_1 = \frac{A}{2} e^{j\theta} \; \rightarrow |c_1|^2 = \frac{A^2}{4} \quad \text{and} \;\; c_{-1} = \frac{A}{2} e^{-j\theta} \rightarrow |c_{-1}|^2 = \frac{A^2}{4}$$

$$\phi(\tau) = \sum_{n=-\infty}^{\infty} |c_n|^2 e^{jnw_0 t} = \frac{A^2}{4} e^{jw_0 t} + \frac{A^2}{4} e^{-jw_0 t} = \frac{A^2}{2} \cos w_0 t$$

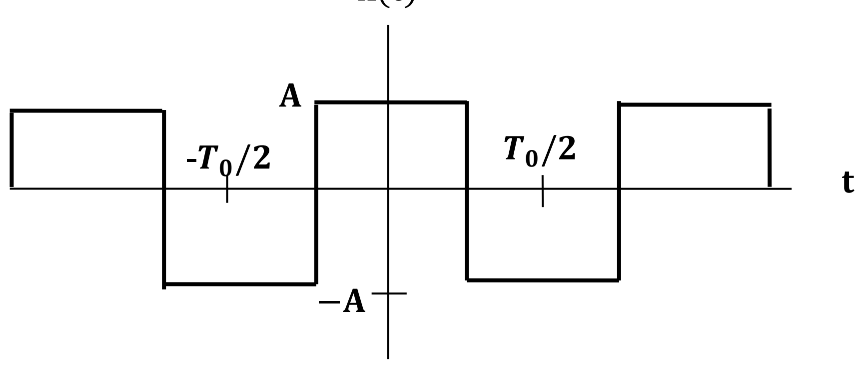
PSD= $\mathcal{F}\{\phi(\tau)\} = \mathcal{F}\left\{\frac{A^2}{2} \cos w_0 t\right\} = \frac{A^2}{4} 2\pi\{\delta(w + w_0) + \delta(w - w_0)\}$

The average power= $\phi(0) = \frac{A^2}{2}$

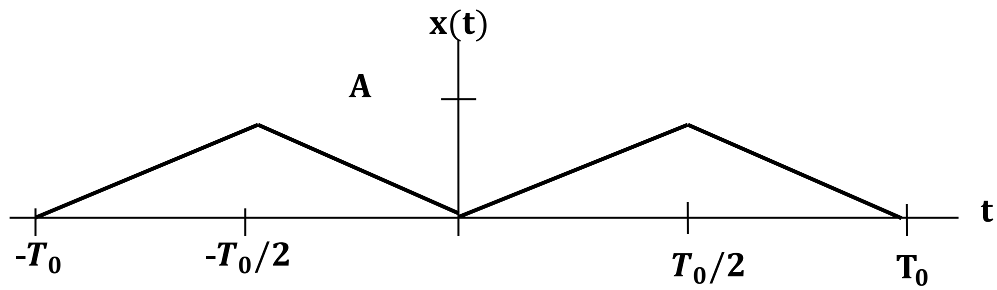
<u>Tutorial Sheet No.1</u>

1) Determine the complex exponential Fourier transform for the following periodic signal shown.

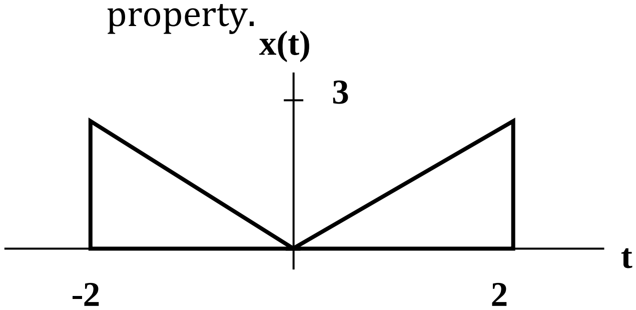
i)

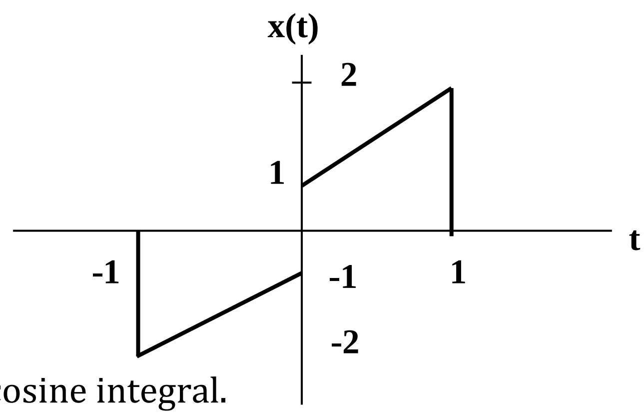


ii)

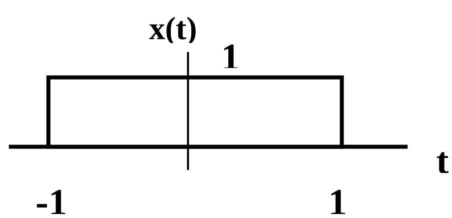


2) Consider the triangular wave x(t) shown in Question (1-ii) using the differentiation technique find the complete Exponential Fourier series of x(t).
3) Find the Fourier transform of the following signals using differentiation property.

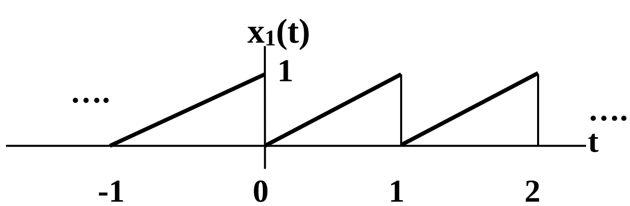


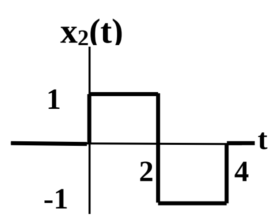


4) Repeat Question (3) using sine and cosine integral.
5) Consider a continuous time LTI system whose step response is given by $s(t)=e^{-t}\,u(t)$. Determine and sketch the output of this system to the input x(t) shown.



6) Find the autocorrelation and the crosscorrelation function for the following functions.

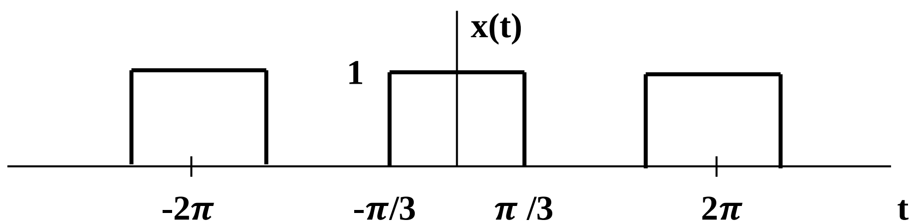




7) Using the Fourier transform to find the impulse response of the system described by: $\acute{\mathbf{y}}(\mathbf{t}) + \mathbf{2y}(\mathbf{t}) = \mathbf{x}(\mathbf{t}) + \acute{\mathbf{x}}(\mathbf{t})$. Find the output y(t) if $x(t)=e^{-t}u(t)$.

8) For the transfer function H(w) is shown below find the power spectral density and the average power of the following signals.

i)



ii)