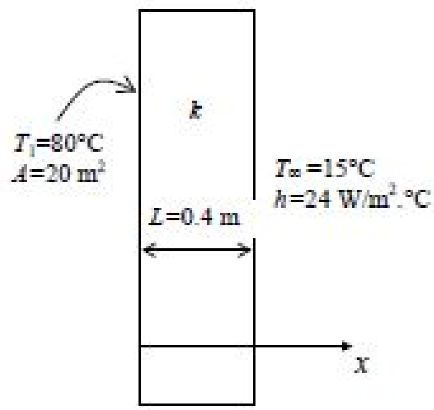
HW sheet 3

Consider a large plane wall of thickness L _ 0.4 m, thermal conductivity k _ 2.3 W/m · °C, and surface area A _ 20 m2. The left side of the wall is maintained at a constant temperature of $T1$ _ 80°C while the right side loses heat by convection to the surrounding air at T _ 15°C with a heat transfer coefficient of h _ 24 W/m2 · °C. Assuming constant thermal conductivity and no heat generation in the wall, (*a*) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the wall, (*b*) obtain a relation for the variation of temperature in the wall by solving the differential equation, and (*c*) evaluate the rate of heat transfer through the wall. *Answer:* (*c*) 6030 W



$$\frac{d^2T}{dx^2} = 0$$

and

$$T(0) = T_1 = 80°\text{C}$$

$$-k\frac{dT(L)}{dx} = h[T(L) - T_\infty]$$

(*b*) Integrating the differential equation twice with respect to x yields

$$\frac{dT}{dx} = C_1$$

$$T(x) = C_1x + C_2$$

where C_1 and C_2 are arbitrary constants. Applying the boundary conditions give

$x = 0$: $\quad T(0) = C_1 \times 0 + C_2 \;\rightarrow\; C_2 = T_1$

$x = L$: $\quad -kC_1 = h[(C_1L + C_2) - T_\infty] \;\rightarrow\; C_1 = -\dfrac{h(C_2 - T_\infty)}{k + hL} \;\rightarrow\; C_1 = -\dfrac{h(T_1 - T_\infty)}{k + hL}$

Substituting C_1 and C_2 into the general solution, the variation of temperature is determined to be

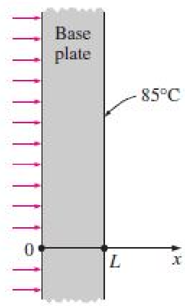
$$\begin{aligned} T(x) &= -\frac{h(T_1 - T_\infty)}{k + hL}x + T_1 \\ &= -\frac{(24\ \text{W/m}^2\cdot°\text{C})(80-15)°\text{C}}{(2.3\ \text{W/m}\cdot°\text{C}) + (24\ \text{W/m}^2\cdot°\text{C})(0.4\ \text{m})}x + 80°\text{C} \\ &= 80 - 131.1x \end{aligned}$$

(*c*) The rate of heat conduction through the wall is

$$\begin{aligned} \dot{Q}_{\text{wall}} &= -kA\frac{dT}{dx} = -kAC_1 = kA\frac{h(T_1 - T_\infty)}{k + hL} \\ &= (2.3\ \text{W/m}\cdot°\text{C})(20\ \text{m}^2)\frac{(24\ \text{W/m}^2\cdot°\text{C})(80-15)°\text{C}}{(2.3\ \text{W/m}\cdot°\text{C}) + (24\ \text{W/m}^2\cdot°\text{C})(0.4\ \text{m})} \\ &= \mathbf{6030\ W} \end{aligned}$$

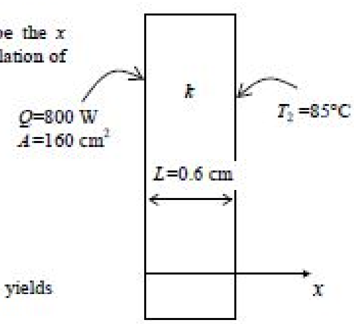
Consider the base plate of a 800-W household iron with a thickness of L _ 0.6 cm, base area of A _ 160 cm2, and thermal conductivity of k _ 20 W/m · °C. The inner surface of the base plate is subjected to uniform heat flux generated by the resistance heaters inside. When steady operating conditions are reached, the outer surface temperature of the plate is measured to be 85°C. Disregarding any heat loss through the upper part of the iron, (*a*) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the plate, (*b*) obtain a relation for the variation of temperature in the base plate by solving the differential equation, and (*c*) evaluate the inner surface temperature.

Answer: (*c*) 100°C



$$\dot{q}_0 = \frac{\dot{Q}_0}{A_{base}} = \frac{800 \text{ W}}{160 \times 10^{-4} \text{ m}^2} = 50{,}000 \text{ W/m}^2$$

Taking the direction normal to the surface of the wall to be the x direction with $x = 0$ at the left surface, the mathematical formulation of this problem can be expressed as



$$\frac{d^2T}{dx^2} = 0$$

and $$-k\frac{dT(0)}{dx} = \dot{q}_0 = 50{,}000 \text{ W/m}^2$$

$$T(L) = T_2 = 85°\text{C}$$

(*b*) Integrating the differential equation twice with respect to x yields

$$\frac{dT}{dx} = C_1$$

$$T(x) = C_1 x + C_2$$

where C_1 and C_2 are arbitrary constants. Applying the boundary conditions give

$x = 0$: $\quad -kC_1 = \dot{q}_0 \rightarrow C_1 = -\frac{\dot{q}_0}{k}$

$x = L$: $\quad T(L) = C_1 L + C_2 = T_2 \rightarrow C_2 = T_2 - C_1 L \rightarrow C_2 = T_2 + \frac{\dot{q}_0 L}{k}$

Substituting C_1 and C_2 into the general solution, the variation of temperature is determined to be

$$T(x) = -\frac{\dot{q}_0}{k}x + T_2 + \frac{\dot{q}_0 L}{k} = \frac{\dot{q}_0 (L - x)}{k} + T_2$$

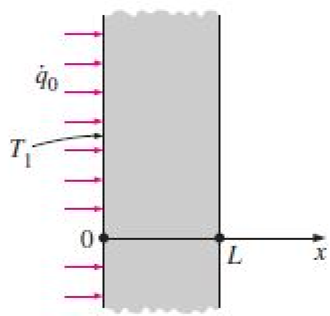
$$= \frac{(50{,}000 \text{ W/m}^2)(0.006 - x)\text{m}}{20 \text{ W/m}\cdot°\text{C}} + 85°\text{C}$$

$$= 2500(0.006 - x) + 85$$

(*c*) The temperature at $x = 0$ (the inner surface of the plate) is

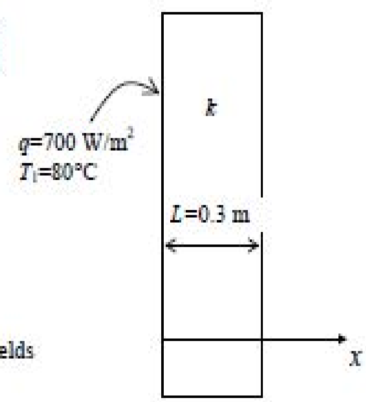
$$T(0) = 2500(0.006 - 0) + 85 = \mathbf{100°C}$$

Consider a large plane wall of thickness L _ 0.3 m, thermal conductivity k _ 2.5 W/m · °C, and surface area A _ 12 m2. The left side of the wall at x _ 0 is subjected to a net heat flux of q· 0 _ 700 W/m2 while the temperature at that surface is measured to be T1 _ 80°C. Assuming constant thermal conductivity and no heat generation in the wall, (*a*) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the wall, (*b*) obtain a relation for the variation of temperature in the wall by solving
the differential equation, and (*c*) evaluate the temperature of the right surface of the wall at x _ L. *Answer:* (*c*) _4°C



Properties The thermal conductivity is given to be k =2.5 W/m·°C.

Analysis (*a*) Taking the direction normal to the surface of the wall to be the x direction with x = 0 at the left surface, the mathematical formulation of this problem can be expressed as



$$\frac{d^2T}{dx^2} = 0$$

and $$-k\frac{dT(0)}{dx} = \dot{q}_0 = 700 \text{ W/m}^2$$

$$T(0) = T_1 = 80°\text{C}$$

(*b*) Integrating the differential equation twice with respect to x yields

$$\frac{dT}{dx} = C_1$$

$$T(x) = C_1x + C_2$$

where C_1 and C_2 are arbitrary constants. Applying the boundary conditions give

Heat flux at x = 0: $$-kC_1 = \dot{q}_0 \rightarrow C_1 = -\frac{\dot{q}_0}{k}$$

Temperature at x = 0: $$T(0) = C_1 \times 0 + C_2 = T_1 \rightarrow C_2 = T_1$$

Substituting C_1 and C_2 into the general solution, the variation of temperature is determined to be

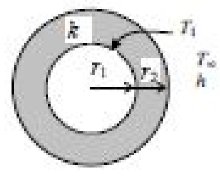
$$T(x) = -\frac{\dot{q}_0}{k}x + T_1 = -\frac{700 \text{ W/m}^2}{2.5 \text{ W/m}\cdot°\text{C}}x + 80°\text{C} = -280x + 80$$

(*c*) The temperature at $x = L$ (the right surface of the wall) is

$$T(L) = -280 \times (0.3 \text{ m}) + 80 = -4°\text{C}$$

Aspherical container of inner radius $r1$ _ 2 m, outer radius $r2$ _ 2.1 m, and thermal conductivity k _ 30 W/m · °C is filled with iced water at 0°C. The container is gaining heat by convection from the surrounding air at T _ 25°C with a heat transfer coefficient of h _ 18 W/m2 · °C. Assuming the inner surface temperature of the container to be 0°C, (*a*) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the container, (*b*) obtain a relation for the variation of temperature in the container by solving the differential equation, and (*c*) evaluate the rate of heat gain to the iced water.

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0$$



and $$T(r_1)=T_1=0^\circ\text{C}$$

$$-k\frac{dT(r_2)}{dr}=h[T(r_2)-T_\infty]$$

(*b*) Integrating the differential equation once with respect to r gives

$$r^2\frac{dT}{dr}=C_1$$

Dividing both sides of the equation above by r to bring it to a readily integrable form and then integrating,

$$\frac{dT}{dr}=\frac{C_1}{r^2}$$

$$T(r)=-\frac{C_1}{r}+C_2$$

where C_1 and C_2 are arbitrary constants. Applying the boundary conditions give

$r=r_1$: $$T(r_1)=-\frac{C_1}{r_1}+C_2=T_1$$

$r=r_2$: $$-k\frac{C_1}{r_2^2}=h\left(-\frac{C_1}{r_2}+C_2-T_\infty\right)$$

Solving for C_1 and C_2 simultaneously gives

$$C_1=\frac{r_2(T_1-T_\infty)}{1-\frac{r_2}{r_1}-\frac{k}{hr_2}} \quad \text{and} \quad C_2=T_1+\frac{C_1}{r_1}=T_1+\frac{T_1-T_\infty}{1-\frac{r_2}{r_1}-\frac{k}{hr_2}}\frac{r_2}{r_1}$$

Substituting C_1 and C_2 into the general solution, the variation of temperature is determined to be

$$T(r)=-\frac{C_1}{r}+T_1+\frac{C_1}{r_1}=C_1\left(\frac{1}{r_1}-\frac{1}{r}\right)+T_1=\frac{T_1-T_\infty}{1-\frac{r_2}{r_1}-\frac{k}{hr_2}}\left(\frac{r_2}{r_1}-\frac{r_2}{r}\right)+T_1$$

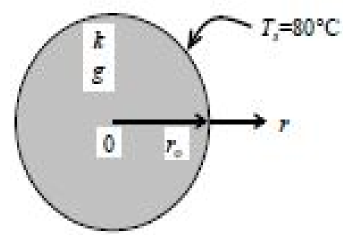
$$=\frac{(0-25)^\circ\text{C}}{1-\frac{2.1}{2}-\frac{30\text{ W/m}\cdot{}^\circ\text{C}}{(18\text{ W/m}^2\cdot{}^\circ\text{C})(2.1\text{m})}}\left(\frac{2.1}{2}-\frac{2.1}{r}\right)+0^\circ\text{C}=29.63(1.05-2.1/r)$$

(*c*) The rate of heat conduction through the wall is

$$\dot{Q}=-kA\frac{dT}{dr}=-k(4\pi r^2)\frac{C_1}{r^2}=-4\pi kC_1=-4\pi k\frac{r_2(T_1-T_\infty)}{1-\frac{r_2}{r_1}-\frac{k}{hr_2}}$$

$$=-4\pi(30\text{ W/m}\cdot{}^\circ\text{C})\frac{(2.1\text{m})(0-25)^\circ\text{C}}{1-\frac{2.1}{2}-\frac{30\text{ W/m}\cdot{}^\circ\text{C}}{(18\text{ W/m}^2\cdot{}^\circ\text{C})(2.1\text{m})}}=\mathbf{23{,}460\text{ W}}$$

Consider a homogeneous spherical piece of radioactive material of radius *r*0 _ 0.04 m that is generating heat at a constant rate of *g*· _ 4 _ 107 W/m3. The heat generated is dissipated to the environment steadily. The outer surface of the
sphere is maintained at a uniform temperature of 80°C and the thermal conductivity of the sphere is *k* _ 15 W/m · °C. Assuming steady one-dimensional heat transfer, (*a*) express the differential equation and the boundary conditions for heat conduction through the sphere, (*b*) obtain a relation for the variation of temperature in the sphere by solving the differential equation, and (*c*) determine the temperature at the center of the sphere.



$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)+\frac{\dot{g}}{k}=0 \quad \text{with } \dot{g}=\text{constant}$$

and $T(r_0)=T_s=80°\text{C}$ (specified surface temperature)

$$\frac{dT(0)}{dr}=0 \quad \text{(thermal symmetry about the mid point)}$$

(*b*) Multiplying both sides of the differential equation by r^2 and rearranging gives

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=-\frac{\dot{g}}{k}r^2$$

Integrating with respect to r gives

$$r^2\frac{dT}{dr}=-\frac{\dot{g}}{k}\frac{r^3}{3}+C_1 \qquad (a)$$

Applying the boundary condition at the mid point,

B.C. at $r=0$: $\quad 0\times\frac{dT(0)}{dr}=-\frac{\dot{g}}{3k}\times 0+C_1 \quad \rightarrow \quad C_1=0$

Dividing both sides of Eq. (*a*) by r^2 to bring it to a readily integrable form and integrating,

$$\frac{dT}{dr}=-\frac{\dot{g}}{3k}r$$

and $T(r)=-\frac{\dot{g}}{6k}r^2+C_2 \qquad (b)$

Applying the other boundary condition at $r=r_0$,

B. C. at $r=r_0$: $\quad T_s=-\frac{\dot{g}}{6k}r_0^2+C_2 \quad \rightarrow \quad C_2=T_s+\frac{\dot{g}}{6k}r_0^2$

Substituting this C_2 relation into Eq. (*b*) and rearranging give

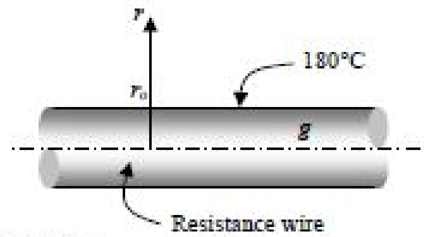
$$T(r)=T_s+\frac{\dot{g}}{6k}(r_0^2-r^2)$$

which is the desired solution for the temperature distribution in the wire as a function of r.

(*c*) The temperature at the center of the sphere (r = 0) is determined by substituting the known quantities to be

$$T(0)=T_s+\frac{\dot{g}}{6k}(r_0^2-0^2)=T_s+\frac{\dot{g}r_0^2}{6k}=80°\text{C}+\frac{(4\times10^7\ \text{W/m}^3)(0.04\ \text{m})^2}{6\times(15\ \text{W/m.°C})}=791°\text{C}$$

A long homogeneous resistance wire of radius *r*0 _ 5 mm is being used to heat the air in a room by the passage of electric current. Heat is generated in the wire uniformly at a rate of *g*· _ 5 _ 107 W/m3 as a result of resistance heating. If the temperature of the outer surface of the wire remains at 180°C, determine the temperature at *r* _ 2 mm after steady operation conditions are reached. Take the thermal conductivity of the wire to be *k* _ 8 W/m · °C. *Answer:* 212.8°C



$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{g}}{k}=0$$

and $T(r_0)=T_s=180°C$ (specified surface temperature)

$$\frac{dT(0)}{dr}=0 \quad \text{(thermal symmetry about the centerline)}$$

Multiplying both sides of the differential equation by r and rearranging gives

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=-\frac{\dot{g}}{k}r$$

Integrating with respect to r gives

$$r\frac{dT}{dr}=-\frac{\dot{g}}{k}\frac{r^2}{2}+C_1 \qquad (a)$$

It is convenient at this point to apply the boundary condition at the center since it is related to the first derivative of the temperature. It yields

B.C. at $r=0$: $\quad 0\times\frac{dT(0)}{dr}=-\frac{\dot{g}}{2k}\times 0+C_1 \quad\rightarrow\quad C_1=0$

Dividing both sides of Eq. (a) by r to bring it to a readily integrable form and integrating,

$$\frac{dT}{dr}=-\frac{\dot{g}}{2k}r$$

and $T(r)=-\frac{\dot{g}}{4k}r^2+C_2 \qquad (b)$

Applying the other boundary condition at $r=r_0$,

B. C. at $r=r_0$: $\quad T_s=-\frac{\dot{g}}{4k}r_0^2+C_2 \quad\rightarrow\quad C_2=T_s+\frac{\dot{g}}{4k}r_0^2$

Substituting this C_2 relation into Eq. (b) and rearranging give

$$T(r)=T_s+\frac{\dot{g}}{4k}(r_0^2-r^2)$$

which is the desired solution for the temperature distribution in the wire as a function of r. The temperature 2 mm from the center line ($r=0.002$ m) is determined by substituting the known quantities to be

$$T(0.002\text{ m})=T_s+\frac{\dot{g}}{4k}(r_0^2-r^2)=180°C+\frac{5\times10^7\text{ W/m}^3}{4\times(8\text{ W/ m.°C})}[(0.005\text{ m})^2-(0.002\text{ m})^2]=\mathbf{212.8°C}$$