

## Infinite Sequences

and Series

OVERVIEW While everyone knows how to add together two numbers, or even several, how to add together infinitely many numbers is not so clear. In this chapter we study such questions, the subject of the theory of infinite series. Infinite series sometimes have a finite sum, as in

$$
\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1
$$

This sum is represented geometrically by the areas of the repeatedly halved unit square shown here. The areas of the small rectangles add together to give the area of the unit square, which they fill. Adding together more and more terms gets us closer and closer to the total.


Other infinite series do not have a finite sum, as with

$$
1+2+3+4+5+\cdots
$$

The sum of the first few terms gets larger and larger as we add more and more terms. Taking enough terms makes these sums larger than any prechosen constant.

With some infinite series, such as the harmonic series

$$
1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots
$$

it is not obvious whether a finite sum exists. It is unclear whether adding more and more terms gets us closer to some sum, or gives sums that grow without bound.

As we develop the theory of infinite sequences and series, an important application gives a method of representing a differentiable function $f(x)$ as an infinite sum of powers of $x$. With this method we can extend our knowledge of how to evaluate, differentiate, and integrate polynomials to a class of functions much more general than polynomials. We also investigate a method of representing a function as an infinite sum of sine and cosine functions. This method will yield a powerful tool to study functions.

## Historical Essay

Sequences and Series

A sequence is a list of numbers

$$
a_{1}, a_{2}, a_{3}, \ldots, a_{n}, \ldots
$$

in a given order. Each of $a_{1}, a_{2}, a_{3}$ and so on represents a number. These are the terms of the sequence. For example the sequence

$$
2,4,6,8,10,12, \ldots, 2 n, \ldots
$$

has first term $a_{1}=2$, second term $a_{2}=4$ and $n$th term $a_{n}=2 n$. The integer $n$ is called the index of $a_{n}$, and indicates where $a_{n}$ occurs in the list. We can think of the sequence

$$
a_{1}, a_{2}, a_{3}, \ldots, a_{n}, \ldots
$$

as a function that sends 1 to $a_{1}, 2$ to $a_{2}, 3$ to $a_{3}$, and in general sends the positive integer $n$ to the $n$th term $a_{n}$. This leads to the formal definition of a sequence.

## DEFINITION Infinite Sequence

An infinite sequence of numbers is a function whose domain is the set of positive integers.

The function associated to the sequence

$$
2,4,6,8,10,12, \ldots, 2 n, \ldots
$$

sends 1 to $a_{1}=2$, 2 to $a_{2}=4$, and so on. The general behavior of this sequence is described by the formula

$$
a_{n}=2 n .
$$

We can equally well make the domain the integers larger than a given number $n_{0}$, and we allow sequences of this type also.

The sequence

$$
12,14,16,18,20,22 \ldots
$$

is described by the formula $a_{n}=10+2 n$. It can also be described by the simpler formula $b_{n}=2 n$, where the index $n$ starts at 6 and increases. To allow such simpler formulas, we let the first index of the sequence be any integer. In the sequence above, $\left\{a_{n}\right\}$ starts with $a_{1}$ while $\left\{b_{n}\right\}$ starts with $b_{6}$. Order is important. The sequence $1,2,3,4 \ldots$ is not the same as the sequence $2,1,3,4 \ldots$.

Sequences can be described by writing rules that specify their terms, such as

$$
\begin{aligned}
a_{n} & =\sqrt{n}, \\
b_{n} & =(-1)^{n+1} \frac{1}{n}, \\
c_{n} & =\frac{n-1}{n}, \\
d_{n} & =(-1)^{n+1}
\end{aligned}
$$

or by listing terms,

$$
\begin{aligned}
\left\{a_{n}\right\} & =\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\} \\
\left\{b_{n}\right\} & =\left\{1,-\frac{1}{2}, \frac{1}{3},-\frac{1}{4}, \ldots,(-1)^{n+1} \frac{1}{n}, \ldots\right\} \\
\left\{c_{n}\right\} & =\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\} \\
\left\{d_{n}\right\} & =\left\{1,-1,1,-1,1,-1, \ldots,(-1)^{n+1}, \ldots\right\}
\end{aligned}
$$

We also sometimes write

$$
\left\{a_{n}\right\}=\{\sqrt{n}\}_{n=1}^{\infty} .
$$

Figure 11.1 shows two ways to represent sequences graphically. The first marks the first few points from $a_{1}, a_{2}, a_{3}, \ldots, a_{n}, \ldots$ on the real axis. The second method shows the graph of the function defining the sequence. The function is defined only on integer inputs, and the graph consists of some points in the $x y$-plane, located at $\left(1, a_{1}\right)$, $\left(2, a_{2}\right), \ldots,\left(n, a_{n}\right), \ldots$


FIGURE 11.1 Sequences can be represented as points on the real line or as points in the plane where the horizontal axis $n$ is the index number of the term and the vertical axis $a_{n}$ is its value.

## Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index $n$ increases.
This happens in the sequence

$$
\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}
$$

whose terms approach 0 as $n$ gets large, and in the sequence

$$
\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, 1-\frac{1}{n}, \ldots\right\}
$$



FIGURE $11.2 a_{n} \rightarrow L$ if $y=L$ is a horizontal asymptote of the sequence of points $\left\{\left(n, a_{n}\right)\right\}$. In this figure, all the $a_{n}$ 's after $a_{N}$ lie within $\epsilon$ of $L$.
whose terms approach 1 . On the other hand, sequences like

$$
\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}
$$

have terms that get larger than any number as $n$ increases, and sequences like

$$
\left\{1,-1,1,-1,1,-1, \ldots,(-1)^{n+1}, \ldots\right\}
$$

bounce back and forth between 1 and -1 , never converging to a single value. The following definition captures the meaning of having a sequence converge to a limiting value. It says that if we go far enough out in the sequence, by taking the index $n$ to be larger then some value $N$, the difference between $a_{n}$ and the limit of the sequence becomes less than any preselected number $\epsilon>0$.

## DEFINITIONS Converges, Diverges, Limit

The sequence $\left\{a_{n}\right\}$ converges to the number $L$ if to every positive number $\boldsymbol{\epsilon}$ there corresponds an integer $N$ such that for all $n$,

$$
n>N \quad \Rightarrow \quad\left|a_{n}-L\right|<\epsilon
$$

If no such number $L$ exists, we say that $\left\{a_{n}\right\}$ diverges.
If $\left\{a_{n}\right\}$ converges to $L$, we write $\lim _{n \rightarrow \infty} a_{n}=L$, or simply $a_{n} \rightarrow L$, and call $L$ the limit of the sequence (Figure 11.2).

The definition is very similar to the definition of the limit of a function $f(x)$ as $x$ tends to $\infty\left(\lim _{x \rightarrow \infty} f(x)\right.$ in Section 2.4). We will exploit this connection to calculate limits of sequences.

## EXAMPLE 1 Applying the Definition

Show that
(a) $\lim _{n \rightarrow \infty} \frac{1}{n}=0$
(b) $\lim _{n \rightarrow \infty} k=k \quad$ (any constant $\left.k\right)$

## Solution

(a) Let $\epsilon>0$ be given. We must show that there exists an integer $N$ such that for all $n$,

$$
n>N \quad \Rightarrow \quad\left|\frac{1}{n}-0\right|<\epsilon
$$

This implication will hold if $(1 / n)<\epsilon$ or $n>1 / \epsilon$. If $N$ is any integer greater than $1 / \epsilon$, the implication will hold for all $n>N$. This proves that $\lim _{n \rightarrow \infty}(1 / n)=0$.
(b) Let $\epsilon>0$ be given. We must show that there exists an integer $N$ such that for all $n$,

$$
n>N \quad \Rightarrow \quad|k-k|<\epsilon
$$

Since $k-k=0$, we can use any positive integer for $N$ and the implication will hold. This proves that $\lim _{n \rightarrow \infty} k=k$ for any constant $k$.

## EXAMPLE 2 A Divergent Sequence

Show that the sequence $\left\{1,-1,1,-1,1,-1, \ldots,(-1)^{n+1}, \ldots\right\}$ diverges.
Solution Suppose the sequence converges to some number $L$. By choosing $\epsilon=1 / 2$ in the definition of the limit, all terms $a_{n}$ of the sequence with index $n$ larger than some $N$ must lie within $\epsilon=1 / 2$ of $L$. Since the number 1 appears repeatedly as every other term of the sequence, we must have that the number 1 lies within the distance $\epsilon=1 / 2$ of $L$. It follows that $|L-1|<1 / 2$, or equivalently, $1 / 2<L<3 / 2$. Likewise, the number -1 appears repeatedly in the sequence with arbitrarily high index. So we must also have that $|L-(-1)|<1 / 2$, or equivalently, $-3 / 2<L<-1 / 2$. But the number $L$ cannot lie in both of the intervals $(1 / 2,3 / 2)$ and $(-3 / 2,-1 / 2)$ because they have no overlap. Therefore, no such limit $L$ exists and so the sequence diverges.

Note that the same argument works for any positive number $\epsilon$ smaller than 1 , not just $1 / 2$.

The sequence $\{\sqrt{n}\}$ also diverges, but for a different reason. As $n$ increases, its terms become larger than any fixed number. We describe the behavior of this sequence by writing

$$
\lim _{n \rightarrow \infty} \sqrt{n}=\infty
$$

In writing infinity as the limit of a sequence, we are not saying that the differences between the terms $a_{n}$ and $\infty$ become small as $n$ increases. Nor are we asserting that there is some number infinity that the sequence approaches. We are merely using a notation that captures the idea that $a_{n}$ eventually gets and stays larger than any fixed number as $n$ gets large.

## DEFINITION Diverges to Infinity

The sequence $\left\{a_{n}\right\}$ diverges to infinity if for every number $M$ there is an integer $N$ such that for all $n$ larger than $N, a_{n}>M$. If this condition holds we write

$$
\lim _{n \rightarrow \infty} a_{n}=\infty \quad \text { or } \quad a_{n} \rightarrow \infty
$$

Similarly if for every number $m$ there is an integer $N$ such that for all $n>N$ we have $a_{n}<m$, then we say $\left\{a_{n}\right\}$ diverges to negative infinity and write

$$
\lim _{n \rightarrow \infty} a_{n}=-\infty \quad \text { or } \quad a_{n} \rightarrow-\infty .
$$

A sequence may diverge without diverging to infinity or negative infinity. We saw this in Example 2, and the sequences $\{1,-2,3,-4,5,-6,7,-8, \ldots\}$ and $\{1,0,2,0,3,0, \ldots\}$ are also examples of such divergence.

## Calculating Limits of Sequences

If we always had to use the formal definition of the limit of a sequence, calculating with $\epsilon$ 's and $N$ 's, then computing limits of sequences would be a formidable task. Fortunately we can derive a few basic examples, and then use these to quickly analyze the limits of many more sequences. We will need to understand how to combine and compare sequences. Since sequences are functions with domain restricted to the positive integers, it is not too surprising that the theorems on limits of functions given in Chapter 2 have versions for sequences.

## THEOREM 1

Let $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ be sequences of real numbers and let $A$ and $B$ be real numbers. The following rules hold if $\lim _{n \rightarrow \infty} a_{n}=A$ and $\lim _{n \rightarrow \infty} b_{n}=B$.

1. Sum Rule:
2. Difference Rule:
3. Product Rule:
4. Constant Multiple Rule:
5. Quotient Rule:

$$
\lim _{n \rightarrow \infty}\left(a_{n}+b_{n}\right)=A+B
$$

$$
\lim _{n \rightarrow \infty}\left(a_{n}-b_{n}\right)=A-B
$$

$$
\lim _{n \rightarrow \infty}\left(a_{n} \cdot b_{n}\right)=A \cdot B
$$

$$
\lim _{n \rightarrow \infty}\left(k \cdot b_{n}\right)=k \cdot B \quad(\text { Any number } k)
$$

$$
\lim _{n \rightarrow \infty} \frac{a_{n}}{b_{n}}=\frac{A}{B} \quad \text { if } B \neq 0
$$

The proof is similar to that of Theorem 1 of Section 2.2, and is omitted.

## EXAMPLE 3 Applying Theorem 1

By combining Theorem 1 with the limits of Example 1, we have:
(a) $\lim _{n \rightarrow \infty}\left(-\frac{1}{n}\right)=-1 \cdot \lim _{n \rightarrow \infty} \frac{1}{n}=-1 \cdot 0=0 \quad$ Constant Multiple Rule and Example 1a
(b) $\lim _{n \rightarrow \infty}\left(\frac{n-1}{n}\right)=\lim _{n \rightarrow \infty}\left(1-\frac{1}{n}\right)=\lim _{n \rightarrow \infty} 1-\lim _{n \rightarrow \infty} \frac{1}{n}=1-0=1 \quad \begin{aligned} & \text { Difference Rule } \\ & \text { and Example 1a }\end{aligned}$
(c) $\lim _{n \rightarrow \infty} \frac{5}{n^{2}}=5 \cdot \lim _{n \rightarrow \infty} \frac{1}{n} \cdot \lim _{n \rightarrow \infty} \frac{1}{n}=5 \cdot 0 \cdot 0=0 \quad$ Product Rule
(d) $\lim _{n \rightarrow \infty} \frac{4-7 n^{6}}{n^{6}+3}=\lim _{n \rightarrow \infty} \frac{\left(4 / n^{6}\right)-7}{1+\left(3 / n^{6}\right)}=\frac{0-7}{1+0}=-7$. Sum and Quotient Rules

Be cautious in applying Theorem 1. It does not say, for example, that each of the sequences $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ have limits if their sum $\left\{a_{n}+b_{n}\right\}$ has a limit. For instance, $\left\{a_{n}\right\}=\{1,2,3, \ldots\}$ and $\left\{b_{n}\right\}=\{-1,-2,-3, \ldots\}$ both diverge, but their sum $\left\{a_{n}+b_{n}\right\}=\{0,0,0, \ldots\}$ clearly converges to 0 .

One consequence of Theorem 1 is that every nonzero multiple of a divergent sequence $\left\{a_{n}\right\}$ diverges. For suppose, to the contrary, that $\left\{c a_{n}\right\}$ converges for some number $c \neq 0$. Then, by taking $k=1 / c$ in the Constant Multiple Rule in Theorem 1, we see that the sequence

$$
\left\{\frac{1}{c} \cdot c a_{n}\right\}=\left\{a_{n}\right\}
$$

converges. Thus, $\left\{c a_{n}\right\}$ cannot converge unless $\left\{a_{n}\right\}$ also converges. If $\left\{a_{n}\right\}$ does not converge, then $\left\{c a_{n}\right\}$ does not converge.

The next theorem is the sequence version of the Sandwich Theorem in Section 2.2. You are asked to prove the theorem in Exercise 95.

## THEOREM 2 The Sandwich Theorem for Sequences

Let $\left\{a_{n}\right\},\left\{b_{n}\right\}$, and $\left\{c_{n}\right\}$ be sequences of real numbers. If $a_{n} \leq b_{n} \leq c_{n}$ holds for all $n$ beyond some index $N$, and if $\lim _{n \rightarrow \infty} a_{n}=\lim _{n \rightarrow \infty} c_{n}=L$, then $\lim _{n \rightarrow \infty} b_{n}=L$ also.


FIGURE 11.3 As $n \rightarrow \infty, 1 / n \rightarrow 0$ and $2^{1 / n} \rightarrow 2^{0}$ (Example 6).

An immediate consequence of Theorem 2 is that, if $\left|b_{n}\right| \leq c_{n}$ and $c_{n} \rightarrow 0$, then $b_{n} \rightarrow 0$ because $-c_{n} \leq b_{n} \leq c_{n}$. We use this fact in the next example.

## EXAMPLE 4 Applying the Sandwich Theorem

Since $1 / n \rightarrow 0$, we know that
(a) $\frac{\cos n}{n} \rightarrow 0 \quad$ because $\quad-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$;
(b) $\frac{1}{2^{n}} \rightarrow 0 \quad$ because $\quad 0 \leq \frac{1}{2^{n}} \leq \frac{1}{n}$;
(c) $(-1)^{n} \frac{1}{n} \rightarrow 0 \quad$ because $\quad-\frac{1}{n} \leq(-1)^{n} \frac{1}{n} \leq \frac{1}{n}$.

The application of Theorems 1 and 2 is broadened by a theorem stating that applying a continuous function to a convergent sequence produces a convergent sequence. We state the theorem without proof (Exercise 96).

## THEOREM 3 The Continuous Function Theorem for Sequences

Let $\left\{a_{n}\right\}$ be a sequence of real numbers. If $a_{n} \rightarrow L$ and if $f$ is a function that is continuous at $L$ and defined at all $a_{n}$, then $f\left(a_{n}\right) \rightarrow f(L)$.

## EXAMPLE 5 Applying Theorem 3

Show that $\sqrt{(n+1) / n} \rightarrow 1$.
Solution We know that $(n+1) / n \rightarrow 1$. Taking $f(x)=\sqrt{x}$ and $L=1$ in Theorem 3 gives $\sqrt{(n+1) / n} \rightarrow \sqrt{1}=1$.

EXAMPLE 6 The Sequence $\left\{2^{1 / n}\right\}$
The sequence $\{1 / n\}$ converges to 0 . By taking $a_{n}=1 / n, f(x)=2^{x}$, and $L=0$ in Theorem 3, we see that $2^{1 / n}=f(1 / n) \rightarrow f(L)=2^{0}=1$. The sequence $\left\{2^{1 / n}\right\}$ converges to 1 (Figure 11.3).

## Using l'Hôpital's Rule

The next theorem enables us to use l'Hôpital's Rule to find the limits of some sequences. It formalizes the connection between $\lim _{n \rightarrow \infty} a_{n}$ and $\lim _{x \rightarrow \infty} f(x)$.

## THEOREM 4

Suppose that $f(x)$ is a function defined for all $x \geq n_{0}$ and that $\left\{a_{n}\right\}$ is a sequence of real numbers such that $a_{n}=f(n)$ for $n \geq n_{0}$. Then

$$
\lim _{x \rightarrow \infty} f(x)=L \quad \Rightarrow \quad \lim _{n \rightarrow \infty} a_{n}=L
$$

Proof Suppose that $\lim _{x \rightarrow \infty} f(x)=L$. Then for each positive number $\epsilon$ there is a number $M$ such that for all $x$,

$$
x>M \quad \Rightarrow \quad|f(x)-L|<\epsilon
$$

Let $N$ be an integer greater than $M$ and greater than or equal to $n_{0}$. Then

$$
n>N \quad \Rightarrow \quad a_{n}=f(n) \quad \text { and } \quad\left|a_{n}-L\right|=|f(n)-L|<\epsilon
$$

## EXAMPLE 7 Applying L'Hôpital's Rule

Show that

$$
\lim _{n \rightarrow \infty} \frac{\ln n}{n}=0
$$

Solution The function $(\ln x) / x$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. Therefore, by Theorem $5, \lim _{n \rightarrow \infty}(\ln n) / n$ will equal $\lim _{x \rightarrow \infty}(\ln x) / x$ if the latter exists. A single application of l'Hôpital's Rule shows that

$$
\lim _{x \rightarrow \infty} \frac{\ln x}{x}=\lim _{x \rightarrow \infty} \frac{1 / x}{1}=\frac{0}{1}=0
$$

We conclude that $\lim _{n \rightarrow \infty}(\ln n) / n=0$.
When we use l'Hôpital's Rule to find the limit of a sequence, we often treat $n$ as a continuous real variable and differentiate directly with respect to $n$. This saves us from having to rewrite the formula for $a_{n}$ as we did in Example 7.

## EXAMPLE 8 Applying L'Hôpital's Rule

Find

$$
\lim _{n \rightarrow \infty} \frac{2^{n}}{5 n}
$$

Solution By l'Hôpital's Rule (differentiating with respect to $n$ ),

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{2^{n}}{5 n} & =\lim _{n \rightarrow \infty} \frac{2^{n} \cdot \ln 2}{5} \\
& =\infty
\end{aligned}
$$

## EXAMPLE 9 Applying L'Hôpital's Rule to Determine Convergence

Does the sequence whose $n$th term is

$$
a_{n}=\left(\frac{n+1}{n-1}\right)^{n}
$$

converge? If so, find $\lim _{n \rightarrow \infty} a_{n}$.
Solution The limit leads to the indeterminate form $1^{\infty}$. We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of $a_{n}$ :

$$
\begin{aligned}
\ln a_{n} & =\ln \left(\frac{n+1}{n-1}\right)^{n} \\
& =n \ln \left(\frac{n+1}{n-1}\right)
\end{aligned}
$$

## Factorial Notation

The notation $n$ ! (" $n$ factorial") means the product $1 \cdot 2 \cdot 3 \cdots n$ of the integers from 1 to $n$. Notice that $(n+1)!=(n+1) \cdot n!$. Thus, $4!=1 \cdot 2 \cdot 3 \cdot 4=24$ and $5!=1 \cdot 2 \cdot 3 \cdot 4 \cdot 5=5 \cdot 4!=120$. We define 0 ! to be 1 . Factorials grow even faster than exponentials, as the table suggests.

| $\boldsymbol{n}$ | $\boldsymbol{e}^{\boldsymbol{n}}$ (rounded) | $\boldsymbol{n}!$ |
| ---: | ---: | ---: |
| 1 | 3 | 1 |
| 5 | 148 | 120 |
| 10 | 22,026 | $3,628,800$ |
| 20 | $4.9 \times 10^{8}$ | $2.4 \times 10^{18}$ |

Then,

$$
\begin{array}{rlrl}
\lim _{n \rightarrow \infty} \ln a_{n} & =\lim _{n \rightarrow \infty} n \ln \left(\frac{n+1}{n-1}\right) & & \infty \cdot 0 \\
& =\lim _{n \rightarrow \infty} \frac{\ln \left(\frac{n+1}{n-1}\right)}{1 / n} & \frac{0}{0} \\
& =\lim _{n \rightarrow \infty} \frac{-2 /\left(n^{2}-1\right)}{-1 / n^{2}} \quad \text { l'Hôpital's Rule } \\
& =\lim _{n \rightarrow \infty} \frac{2 n^{2}}{n^{2}-1}=2
\end{array}
$$

Since $\ln a_{n} \rightarrow 2$ and $f(x)=e^{x}$ is continuous, Theorem 4 tells us that

$$
a_{n}=e^{\ln a_{n}} \rightarrow e^{2}
$$

The sequence $\left\{a_{n}\right\}$ converges to $e^{2}$.

## Commonly Occurring Limits

The next theorem gives some limits that arise frequently.

## THEOREM 5

The following six sequences converge to the limits listed below:

1. $\lim _{n \rightarrow \infty} \frac{\ln n}{n}=0$
2. $\lim _{n \rightarrow \infty} \sqrt[n]{n}=1$
3. $\lim _{n \rightarrow \infty} x^{1 / n}=1 \quad(x>0)$
4. $\quad \lim _{n \rightarrow \infty} x^{n}=0 \quad(|x|<1)$
5. $\lim _{n \rightarrow \infty}\left(1+\frac{x}{n}\right)^{n}=e^{x} \quad(\operatorname{any} x)$
6. $\lim _{n \rightarrow \infty} \frac{x^{n}}{n!}=0 \quad(\operatorname{any} x)$

In Formulas (3) through (6), $x$ remains fixed as $n \rightarrow \infty$.

Proof The first limit was computed in Example 7. The next two can be proved by taking logarithms and applying Theorem 4 (Exercises 93 and 94). The remaining proofs are given in Appendix 3.

## EXAMPLE 10 Applying Theorem 5

(a) $\frac{\ln \left(n^{2}\right)}{n}=\frac{2 \ln n}{n} \rightarrow 2 \cdot 0=0 \quad$ Formula 1
(b) $\sqrt[n]{n^{2}}=n^{2 / n}=\left(n^{1 / n}\right)^{2} \rightarrow(1)^{2}=1 \quad$ Formula 2
(c) $\sqrt[n]{3 n}=3^{1 / n}\left(n^{1 / n}\right) \rightarrow 1 \cdot 1=1 \quad$ Formula 3 with $x=3$ and Formula 2
(d) $\left(-\frac{1}{2}\right)^{n} \rightarrow 0$
Formula 4 with $x=-\frac{1}{2}$
(e) $\left(\frac{n-2}{n}\right)^{n}=\left(1+\frac{-2}{n}\right)^{n} \rightarrow e^{-2} \quad$ Formula 5 with $x=-2$
(f) $\frac{100^{n}}{n!} \longrightarrow 0 \quad$ Formula 6 with $x=100$

## Recursive Definitions

So far, we have calculated each $a_{n}$ directly from the value of $n$. But sequences are often defined recursively by giving

1. The value(s) of the initial term or terms, and
2. A rule, called a recursion formula, for calculating any later term from terms that precede it.

## EXAMPLE 11 Sequences Constructed Recursively

(a) The statements $a_{1}=1$ and $a_{n}=a_{n-1}+1$ define the sequence $1,2,3, \ldots, n, \ldots$ of positive integers. With $a_{1}=1$, we have $a_{2}=a_{1}+1=2, a_{3}=a_{2}+1=3$, and so on.
(b) The statements $a_{1}=1$ and $a_{n}=n \cdot a_{n-1}$ define the sequence $1,2,6,24, \ldots, n!, \ldots$ of factorials. With $a_{1}=1$, we have $a_{2}=2 \cdot a_{1}=2, a_{3}=3 \cdot a_{2}=6, a_{4}=$ $4 \cdot a_{3}=24$, and so on.
(c) The statements $a_{1}=1, a_{2}=1$, and $a_{n+1}=a_{n}+a_{n-1}$ define the sequence $1,1,2,3,5, \ldots$ of Fibonacci numbers. With $a_{1}=1$ and $a_{2}=1$, we have $a_{3}=1+1=2, a_{4}=2+1=3, a_{5}=3+2=5$, and so on.
(d) As we can see by applying Newton's method, the statements $x_{0}=1$ and $x_{n+1}=x_{n}-\left[\left(\sin x_{n}-x_{n}{ }^{2}\right) /\left(\cos x_{n}-2 x_{n}\right)\right]$ define a sequence that converges to a solution of the equation $\sin x-x^{2}=0$.

## Bounded Nondecreasing Sequences

The terms of a general sequence can bounce around, sometimes getting larger, sometimes smaller. An important special kind of sequence is one for which each term is at least as large as its predecessor.

## DEFINITION Nondecreasing Sequence

A sequence $\left\{a_{n}\right\}$ with the property that $a_{n} \leq a_{n+1}$ for all $n$ is called a nondecreasing sequence.

## EXAMPLE 12 Nondecreasing Sequences

(a) The sequence $1,2,3, \ldots, n, \ldots$ of natural numbers
(b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$
(c) The constant sequence $\{3\}$


FIGURE 11.4 If the terms of a nondecreasing sequence have an upper bound $M$, they have a limit $L \leq M$.

There are two kinds of nondecreasing sequences - those whose terms increase beyond any finite bound and those whose terms do not.

## DEFINITIONS Bounded, Upper Bound, Least Upper Bound

A sequence $\left\{a_{n}\right\}$ is bounded from above if there exists a number $M$ such that $a_{n} \leq M$ for all $n$. The number $M$ is an upper bound for $\left\{a_{n}\right\}$. If $M$ is an upper bound for $\left\{a_{n}\right\}$ but no number less than $M$ is an upper bound for $\left\{a_{n}\right\}$, then $M$ is the least upper bound for $\left\{a_{n}\right\}$.

## EXAMPLE 13 Applying the Definition for Boundedness

(a) The sequence $1,2,3, \ldots, n, \ldots$ has no upper bound.
(b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is bounded above by $M=1$.

No number less than 1 is an upper bound for the sequence, so 1 is the least upper bound (Exercise 113).

A nondecreasing sequence that is bounded from above always has a least upper bound. This is the completeness property of the real numbers, discussed in Appendix 4. We will prove that if $L$ is the least upper bound then the sequence converges to $L$.

Suppose we plot the points $\left(1, a_{1}\right),\left(2, a_{2}\right), \ldots,\left(n, a_{n}\right), \ldots$ in the $x y$-plane. If $M$ is an upper bound of the sequence, all these points will lie on or below the line $y=M$ (Figure 11.4). The line $y=L$ is the lowest such line. None of the points $\left(n, a_{n}\right)$ lies above $y=L$, but some do lie above any lower line $y=L-\epsilon$, if $\epsilon$ is a positive number. The sequence converges to $L$ because
(a) $a_{n} \leq L$ for all values of $n$ and
(b) given any $\epsilon>0$, there exists at least one integer $N$ for which $a_{N}>L-\epsilon$.

The fact that $\left\{a_{n}\right\}$ is nondecreasing tells us further that

$$
a_{n} \geq a_{N}>L-\epsilon \quad \text { for all } n \geq N .
$$

Thus, all the numbers $a_{n}$ beyond the $N$ th number lie within $\epsilon$ of $L$. This is precisely the condition for $L$ to be the limit of the sequence $\left\{a_{n}\right\}$.

The facts for nondecreasing sequences are summarized in the following theorem. A similar result holds for nonincreasing sequences (Exercise 107).

## THEOREM 6 The Nondecreasing Sequence Theorem

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

Theorem 6 implies that a nondecreasing sequence converges when it is bounded from above. It diverges to infinity if it is not bounded from above.

## EXERCISES 11.1

## Finding Terms of a Sequence

Each of Exercises 1-6 gives a formula for the nth term $a_{n}$ of a sequence $\left\{a_{n}\right\}$. Find the values of $a_{1}, a_{2}, a_{3}$, and $a_{4}$.

1. $a_{n}=\frac{1-n}{n^{2}}$
2. $a_{n}=\frac{1}{n!}$
3. $a_{n}=\frac{(-1)^{n+1}}{2 n-1}$
4. $a_{n}=2+(-1)^{n}$
5. $a_{n}=\frac{2^{n}}{2^{n+1}}$
6. $a_{n}=\frac{2^{n}-1}{2^{n}}$

Each of Exercises 7-12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.
7. $a_{1}=1, \quad a_{n+1}=a_{n}+\left(1 / 2^{n}\right)$
8. $a_{1}=1, \quad a_{n+1}=a_{n} /(n+1)$
9. $a_{1}=2, \quad a_{n+1}=(-1)^{n+1} a_{n} / 2$
10. $a_{1}=-2, \quad a_{n+1}=n a_{n} /(n+1)$
11. $a_{1}=a_{2}=1, \quad a_{n+2}=a_{n+1}+a_{n}$
12. $a_{1}=2, \quad a_{2}=-1, \quad a_{n+2}=a_{n+1} / a_{n}$

## Finding a Sequence's Formula

In Exercises 13-22, find a formula for the $n$th term of the sequence.
13. The sequence $1,-1,1,-1,1, \ldots \quad 1$ 's with alternating signs
14. The sequence $-1,1,-1,1,-1, \ldots$
15. The sequence $1,-4,9,-16,25, \ldots$
16. The sequence $1,-\frac{1}{4}, \frac{1}{9},-\frac{1}{16}, \frac{1}{25}, \ldots$
17. The sequence $0,3,8,15,24, \ldots$
18. The sequence $-3,-2,-1,0,1, \ldots$
19. The sequence $1,5,9,13,17, \ldots$
20. The sequence $2,6,10,14,18, \ldots$
21. The sequence $1,0,1,0,1, \ldots$
22. The sequence $0,1,1,2,2,3,3,4, \ldots$ 1's with alternating signs
Squares of the positive integers; with alternating signs
Reciprocals of squares of the positive integers, with alternating signs
Squares of the positive integers diminished by 1 Integers beginning with $-3$

Every other odd positive integer
Every other even positive integer
Alternating 1's and 0's
Each positive integer repeated

## Finding Limits

Which of the sequences $\left\{a_{n}\right\}$ in Exercises 23-84 converge, and which diverge? Find the limit of each convergent sequence.

| 23. $a_{n}=2+(0.1)^{n}$ | 24. $a_{n}=\frac{n+(-1)^{n}}{n}$ |
| :---: | :---: |
| 25. $a_{n}=\frac{1-2 n}{1+2 n}$ | 26. $a_{n}=\frac{2 n+1}{1-3 \sqrt{n}}$ |
| 27. $a_{n}=\frac{1-5 n^{4}}{n^{4}+8 n^{3}}$ | 28. $a_{n}=\frac{n+3}{n^{2}+5 n+6}$ |
| 29. $a_{n}=\frac{n^{2}-2 n+1}{n-1}$ | 30. $a_{n}=\frac{1-n^{3}}{70-4 n^{2}}$ |
| 31. $a_{n}=1+(-1)^{n}$ | 32. $a_{n}=(-1)^{n}\left(1-\frac{1}{n}\right)$ |
| 33. $a_{n}=\left(\frac{n+1}{2 n}\right)\left(1-\frac{1}{n}\right)$ | 34. $a_{n}=\left(2-\frac{1}{2^{n}}\right)\left(3+\frac{1}{2^{n}}\right)$ |
| 35. $a_{n}=\frac{(-1)^{n+1}}{2 n-1}$ | 36. $a_{n}=\left(-\frac{1}{2}\right)^{n}$ |
| 37. $a_{n}=\sqrt{\frac{2 n}{n+1}}$ | 38. $a_{n}=\frac{1}{(0.9)^{n}}$ |
| 39. $a_{n}=\sin \left(\frac{\pi}{2}+\frac{1}{n}\right)$ | 40. $a_{n}=n \pi \cos (n \pi)$ |
| 41. $a_{n}=\frac{\sin n}{n}$ | 42. $a_{n}=\frac{\sin ^{2} n}{2^{n}}$ |
| 43. $a_{n}=\frac{n}{2^{n}}$ | 44. $a_{n}=\frac{3^{n}}{n^{3}}$ |
| 45. $a_{n}=\frac{\ln (n+1)}{\sqrt{n}}$ | 46. $a_{n}=\frac{\ln n}{\ln 2 n}$ |
| 47. $a_{n}=8^{1 / n}$ | 48. $a_{n}=(0.03)^{1 / n}$ |
| 49. $a_{n}=\left(1+\frac{7}{n}\right)^{n}$ | 50. $a_{n}=\left(1-\frac{1}{n}\right)^{n}$ |
| 51. $a_{n}=\sqrt[n]{10 n}$ | 52. $a_{n}=\sqrt[n]{n^{2}}$ |
| 53. $a_{n}=\left(\frac{3}{n}\right)^{1 / n}$ | 54. $a_{n}=(n+4)^{1 /(n+4)}$ |
| 55. $a_{n}=\frac{\ln n}{n^{1 / n}}$ | 56. $a_{n}=\ln n-\ln (n+1)$ |
| 57. $a_{n}=\sqrt[n]{4^{n} n}$ | 58. $a_{n}=\sqrt[n]{3^{2 n+1}}$ |
| 59. $a_{n}=\frac{n!}{n^{n}}$ (Hint: Compare | ith $1 / n$. ) |

59. $a_{n}=\frac{n!}{n^{n}} \quad$ (Hint: Compare with $1 / n$.)
60. $a_{n}=\frac{(-4)^{n}}{n!}$
61. $a_{n}=\frac{n!}{10^{6 n}}$
62. $a_{n}=\frac{n!}{2^{n} \cdot 3^{n}}$
63. $a_{n}=\left(\frac{1}{n}\right)^{1 /(\ln n)}$
64. $a_{n}=\ln \left(1+\frac{1}{n}\right)^{n}$
65. $a_{n}=\left(\frac{3 n+1}{3 n-1}\right)^{n}$
66. $a_{n}=\left(\frac{n}{n+1}\right)^{n}$
67. $a_{n}=\left(\frac{x^{n}}{2 n+1}\right)^{1 / n}, \quad x>0$
68. $a_{n}=\left(1-\frac{1}{n^{2}}\right)^{n}$
69. $a_{n}=\frac{3^{n} \cdot 6^{n}}{2^{-n} \cdot n!}$
70. $a_{n}=\frac{(10 / 11)^{n}}{(9 / 10)^{n}+(11 / 12)^{n}}$
71. $a_{n}=\tanh n$
72. $a_{n}=\sinh (\ln n)$
73. $a_{n}=\frac{n^{2}}{2 n-1} \sin \frac{1}{n}$
74. $a_{n}=n\left(1-\cos \frac{1}{n}\right)$
75. $a_{n}=\tan ^{-1} n$
76. $a_{n}=\frac{1}{\sqrt{n}} \tan ^{-1} n$
77. $a_{n}=\left(\frac{1}{3}\right)^{n}+\frac{1}{\sqrt{2^{n}}}$
78. $a_{n}=\sqrt[n]{n^{2}+n}$
79. $a_{n}=\frac{(\ln n)^{200}}{n}$
80. $a_{n}=\frac{(\ln n)^{5}}{\sqrt{n}}$
81. $a_{n}=n-\sqrt{n^{2}-n}$
82. $a_{n}=\frac{1}{\sqrt{n^{2}-1}-\sqrt{n^{2}+n}}$
83. $a_{n}=\frac{1}{n} \int_{1}^{n} \frac{1}{x} d x$
84. $a_{n}=\int_{1}^{n} \frac{1}{x^{p}} d x, \quad p>1$

## Theory and Examples

85. The first term of a sequence is $x_{1}=1$. Each succeeding term is the sum of all those that come before it:

$$
x_{n+1}=x_{1}+x_{2}+\cdots+x_{n}
$$

Write out enough early terms of the sequence to deduce a general formula for $x_{n}$ that holds for $n \geq 2$.
86. A sequence of rational numbers is described as follows:

$$
\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{a}{b}, \frac{a+2 b}{a+b}, \ldots
$$

Here the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let $x_{n}$ and $y_{n}$ be, respectively, the numerator and the denominator of the $n$th fraction $r_{n}=x_{n} / y_{n}$.
a. Verify that $x_{1}^{2}-2 y_{1}^{2}=-1, x_{2}^{2}-2 y_{2}^{2}=+1$ and, more generally, that if $a^{2}-2 b^{2}=-1$ or +1 , then

$$
(a+2 b)^{2}-2(a+b)^{2}=+1 \quad \text { or } \quad-1
$$

respectively.
b. The fractions $r_{n}=x_{n} / y_{n}$ approach a limit as $n$ increases. What is that limit? (Hint: Use part (a) to show that $r_{n}{ }^{2}-2= \pm\left(1 / y_{n}\right)^{2}$ and that $y_{n}$ is not less than $n$.)
87. Newton's method The following sequences come from the recursion formula for Newton's method,

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}
$$

Do the sequences converge? If so, to what value? In each case, begin by identifying the function $f$ that generates the sequence.
a. $x_{0}=1, \quad x_{n+1}=x_{n}-\frac{x_{n}{ }^{2}-2}{2 x_{n}}=\frac{x_{n}}{2}+\frac{1}{x_{n}}$
b. $\quad x_{0}=1, \quad x_{n+1}=x_{n}-\frac{\tan x_{n}-1}{\sec ^{2} x_{n}}$
c. $x_{0}=1, x_{n+1}=x_{n}-1$
88. a. Suppose that $f(x)$ is differentiable for all $x$ in $[0,1]$ and that $f(0)=0$. Define the sequence $\left\{a_{n}\right\}$ by the rule $a_{n}=$ $n f(1 / n)$. Show that $\lim _{n \rightarrow \infty} a_{n}=f^{\prime}(0)$.
Use the result in part (a) to find the limits of the following sequences $\left\{a_{n}\right\}$.
b. $a_{n}=n \tan ^{-1} \frac{1}{n}$
c. $a_{n}=n\left(e^{1 / n}-1\right)$
d. $a_{n}=n \ln \left(1+\frac{2}{n}\right)$
89. Pythagorean triples A triple of positive integers $a, b$, and $c$ is called a Pythagorean triple if $a^{2}+b^{2}=c^{2}$. Let $a$ be an odd positive integer and let

$$
b=\left\lfloor\frac{a^{2}}{2}\right\rfloor \quad \text { and } \quad c=\left\lceil\frac{a^{2}}{2}\right\rceil
$$

be, respectively, the integer floor and ceiling for $a^{2} / 2$.

a. Show that $a^{2}+b^{2}=c^{2}$. (Hint: Let $a=2 n+1$ and express $b$ and $c$ in terms of $n$.)

