Gaussian Elimination and Gauss-Jordan Elimination

Definition of Matrix

If m and n are positive integers, then an $m \times n$ **matrix** is a rectangular array

$$\underbrace{\left.\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}\right\} m \text{ rows}}_{n \text{ columns}}$$

in which each **entry,** a_{ij}, of the matrix is a number. An $m \times n$ matrix (read "m by n") has m **rows** (horizontal lines) and n **columns** (vertical lines).

The entry a_{ij} is located in the ith row and the jth column. The index i is called the **row subscript** because it identifies the row in which the entry lies, and the index j is called the **column subscript** because it identifies the column in which the entry lies.

A matrix with m rows and n columns (an m n matrix) is said to be of **size** m n. If m n, the matrix is called **square** of **order** n. For a square matrix, the entries a_{11}, a_{22}, a_{33}, . . . are called the **main diagonal** entries.

Example 1

Examples of Matrices

Each matrix has the indicated size.

(a) Size: 1×1

$$[2]$$

(b) Size: 2×2

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) Size: 1×4

$$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$$

(d) Size: 3×2

$$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$$

One very common use of matrices is to represent systems of linear equations. The matrix derived from the coefficients and constant terms of a system of linear equations is called the **augmented matrix** of the system. The matrix containing only the coefficients of the system is called the **coefficient matrix** of the system. Here is an example.

System	*Augmented Matrix*	*Coefficient Matrix*
$$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \qquad - 4z &= 6 \end{aligned}$$	$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}$$	$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by aligning the variables in the equations vertically.

Given System	*Align Variables*	*Augmented Matrix*
$$\begin{aligned} x_1 + 3x_2 &= 9 \\ -x_2 + 4x_3 &= -2 \\ x_1 - 5x_3 &= 0 \end{aligned}$$	$$\begin{aligned} x_1 + 3x_2 \qquad\qquad &= 9 \\ -x_2 + 4x_3 &= -2 \\ x_1 \qquad\quad - 5x_3 &= 0 \end{aligned}$$	$$\begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & -5 & 0 \end{bmatrix}$$

Elementary Row Operations

In the previous section you studied three operations that can be used on a system of linear equations to produce equivalent systems.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology these three operations correspond to **elementary row operations.**

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Example 2: Elementary Row Operations

(a) Interchange the first and second rows.

Original Matrix	*New Row-Equivalent Matrix*	*Notation*
$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$	$$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$	$R_1 \leftrightarrow R_2$

(b) Multiply the first row by $\frac{1}{2}$ to produce a new first row.

Original Matrix	*New Row-Equivalent Matrix*	*Notation*
$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$	$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$	$\left(\frac{1}{2}\right)R_1 \rightarrow R_1$

(c) Add -2 times the first row to the third row to produce a new third row.

Original Matrix	*New Row-Equivalent Matrix*	*Notation*
$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$

Notice in Example 2(c) that adding 2 times row 1 to row 3 does not change row 1

<u>Example 3: Using Elementary Operations to Solve a System</u>

Linear System

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \quad\;\; &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Associated Augmented Matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

Add the first equation to the second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Add the first row to the second row to produce a new second row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} \qquad R_2 + R_1 \rightarrow R_2$$

Add -2 times the first equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Add -2 times the first row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \qquad R_3 + (-2)R_1 \rightarrow R_3$$

Add the second equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Add the second row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \qquad R_3 + R_2 \rightarrow R_3$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiply the third row by $\frac{1}{2}$ to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \left(\tfrac{1}{2}\right)R_3 \rightarrow R_3$$

Now you can use back-substitution to find the solution. The solution is $x = 1$, $y = 1$, and $z = 2$

The last matrix in Example 3 is said to be in **row-echelon form.** The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix. To be in row-echelon form, a matrix must have the properties listed below.

Definition of Row-Echelon Form of a Matrix

A matrix in **row-echelon form** has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is in reduced row-echelon form if every column that has a leading 1 has zeros in every position above and below its leading 1

Example 4: Row-Echelon Form

The matrices below are in row-echelon form.

$$\text{(a)} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(c)} \begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrices shown in parts (b) and (d) are in *reduced* row-echelon form. The matrices listed below are not in row-echelon form.

$$\text{(e)} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \qquad \text{(f)} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

It can be shown that every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 4 you could change the matrix in part (e) to row-echelon form by multiplying the second row in the matrix by 1/2 .
The method of using Gaussian elimination with back-substitution to solve a system is as follows.

Gaussian Elimination with Back Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Gaussian elimination with back-substitution works well as an algorithmic method for solving systems of linear equations. For this algorithm, the order in which the elementary row operations are performed is important. Move from *left to right by columns*, changing all entries directly below the leading 1's to zeros.

Example 5: Gaussian Elimination with Back-Substitution

Solve the system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

SOLUTION The augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}.$$

Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

← The first two rows are interchanged. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

← Adding -2 times the first row to the third row produces a new third row. $R_3 + (-2)R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix}$$

Adding -1 times the first row to the fourth row produces a new fourth row. $R_4 + (-1)R_1 \to R_4$

Now that the first column is in the desired form, you should change the second column as shown below.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$

Adding 6 times the second row to the fourth row produces a new fourth row. $R_4 + (6)R_2 \to R_4$

To write the third column in proper form, multiply the third row by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$

Multiplying the third row by $\frac{1}{3}$ produces a new third row. $\left(\frac{1}{3}\right)R_3 \to R_3$

Similarly, to write the fourth column in proper form, you should multiply the fourth row by $-\frac{1}{13}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Multiplying the fourth row by $-\frac{1}{13}$ produces a new fourth row. $\left(-\frac{1}{13}\right)R_4 \to R_4$

The matrix is now in row-echelon form, and the corresponding system of linear equations is as shown below.

$$\begin{aligned} x_1 + 2x_2 - x_3 \qquad &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

Using back-substitution, you can determine that the solution is

$$x_1 = -1, \qquad x_2 = 2, \qquad x_3 = 1, \qquad x_4 = 3.$$

When solving a system of linear equations, remember that it is possible for the system to have no solution. If during the elimination process you obtain a row with all zeros except for the last entry, it is unnecessary to continue the elimination process. You can simply conclude that the system is inconsistent and has no solution.

Example 6: A system with No Solution

Solve the system.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 \qquad + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

SOLUTION The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}.$$

Apply Gaussian elimination to the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_2 + (-1)R_1 \to R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_3 + (-2)R_1 \to R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_4 + (-3)R_1 \to R_4$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_3 + R_2 \to R_3$$

Note that the third row of this matrix consists of all zeros except for the last entry. This means that the original system of linear equations is inconsistent. You can see why this is true by converting back to a system of linear equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 2 \\ 0 &= -2 \\ 5x_2 - 7x_3 &= -11 \end{aligned}$$

Because the third "equation" is a false statement, the system has no solution.

Gauss-Jordan Elimination

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** after Carl Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in the next example.

Example 7: Gauss-Jordan Elimination

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION In Example 3, Gaussian elimination was used to obtain the row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now, rather than using back-substitution, apply elementary row operations until you obtain a matrix in reduced row-echelon form. To do this, you must produce zeros above each of the leading 1's, as follows.

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (2)R_2 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_2 + (-3)R_3 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (-9)R_3 \rightarrow R_1$$

Now, converting back to a system of linear equations, you have

$$\begin{aligned} x \qquad\qquad &= 1 \\ y \qquad &= -1 \\ z &= 2. \end{aligned}$$

The Gaussian and Gauss-Jordan elimination procedures employ an algorithmic approach easily adapted to computer use. These elimination procedures, however, make no effort to avoid fractional coefficients. For instance, if the system in Example 7 had been listed as

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ x - 2y + 3z &= 9 \\ -x + 3y \qquad &= -4 \end{aligned}$$

both procedures would have required multiplying the first row by $1/2$, which would have introduced fractions in the first row. For hand computations, fractions can sometimes be avoided by judiciously choosing the order in which elementary row operations are applied.

No matter which order you use, the reduced row-echelon form will be the same.

The next example demonstrates how Gauss-Jordan elimination can be used to solve a system with an infinite number of solutions.

Example 8: A System with an Infinite Number of Solutions

Solve the system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \qquad &= 1 \end{aligned}$$

SOLUTION The augmented matrix of the system of linear equations is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}.$$

Using a graphing utility, a computer software program, or Gauss-Jordan elimination, you can verify that the reduced row-echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}.$$

The corresponding system of equations is

$$\begin{aligned} x_1 \quad\; + 5x_3 &= \;\;\,2 \\ x_2 - 3x_3 &= -1. \end{aligned}$$

Now, using the parameter t to represent the *nonleading* variable x_3, you have

$$x_1 = 2 - 5t, \qquad x_2 = -1 + 3t, \qquad x_3 = t, \qquad \text{where } t \text{ is any real number.}$$

Note that in Example 8 an arbitrary parameter was assigned to the nonleading variable x3. You subsequently solved for the leading variables x1 and x2 as functions of t.

<u>Homogeneous System of Linear Equations</u>

As the final topic of this section, you will look at systems of linear equations in which each of the constant terms is zero. We call such systems **homogeneous.** For example, a homogeneous system of *m* equations in *n* variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdot\cdot\cdot + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdot\cdot\cdot + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdot\cdot\cdot + a_{3n}x_n &= 0 \\ &\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdot\cdot\cdot + a_{mn}x_n &= 0. \end{aligned}$$

It is easy to see that a homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations must be satisfied. Such a solution is called **trivial** (or **obvious**). For instance, a homogeneous system of three equations in the three variables x_1, x_2, and x_3 must have $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ as a trivial solution.

Example 9: Solving Homogeneous System of Linear Equations

Solve the system of linear equations.

$$x_1 - x_2 + 3x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$

SOLUTION Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

yields the matrix shown below.

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \qquad R_2 + (-2)R_1 \to R_2$$

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad \left(\tfrac{1}{3}\right)R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad R_1 + R_2 \to R_1$$

The system of equations corresponding to this matrix is

$$x_1 \qquad + 2x_3 = 0$$
$$x_2 - \quad x_3 = 0.$$

Using the parameter $t = x_3$, the solution set is

$$x_1 = -2t, \qquad x_2 = t, \qquad x_3 = t, \quad t \text{ is any real number.}$$

This system of equations has an infinite number of solutions, one of which is the trivial solution (given by $t = 0$).

Example 9 illustrates an important point about homogeneous systems of linear equations. You began with two equations in three variables and discovered that the system has an infinite number of solutions. In general, a homogeneous system with fewer equations than variables has an infinite number of solutions.

The Number of Solutions of a Homogeneous System

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have an infinite number of solutions.