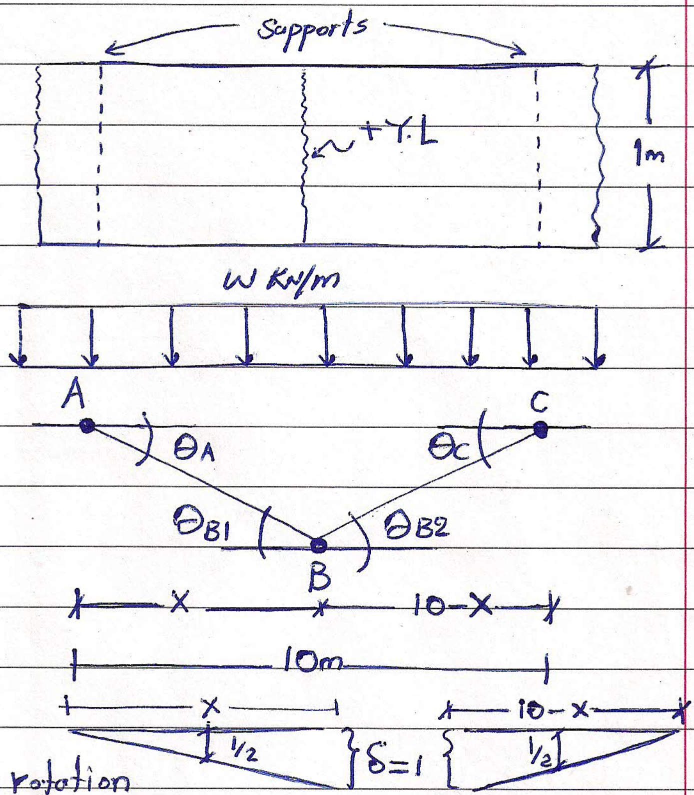
Yield Line Analysis of Slabs
Examples

Example-1-

Determine the load capacity of One-Way Uniformly loaded Continuous slab shown, if $M_{n_A} = 25$ KN.m/m , $M_{n_B} = 25$ KN.m/m and $M_{n_C} = 40$ KN.m/m

Solution

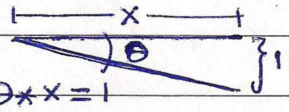


A unit deflection is given at (B).

External Work done = Load x area x (average distance moved for each rigid segment of the slab).

$$W_E = W * x * 1 * \frac{1}{2} + W * (10-x) * 1 * \frac{1}{2}$$

Internal Energy absorbed by rotation along the yield line = Moment * Length * Rotation



$\theta * x = 1$

$\therefore \theta = \frac{1}{x}$

$$W_I = m * 1 * \theta \quad , \quad \theta_A = \theta_{B1} = \frac{1}{x}$$

Also, $\theta_C = \theta_{B2} = \frac{1}{10-x}$

$$\therefore W_I = M_A * 1 * \frac{1}{x} + M_B * 1 * \frac{1}{x} + M_B * 1 * \frac{1}{10-x} + M_C * 1 * \frac{1}{10-x}$$

$$W_E = W_I$$

$$\frac{wx}{2} + \frac{w(10-x)}{2} = \frac{25}{x} + \frac{25}{x} + \frac{25}{10-x} + \frac{40}{10-x}$$

$$\therefore W = \frac{10}{x} + \frac{13}{10-x}$$

To determine minimum $w \Longrightarrow \frac{dw}{dx} = 0 = -\frac{10}{x^2} + \frac{13}{(10-x)^2}$

$$\therefore x = 4.67 m$$

Sub $x = 4.67m \rightarrow w = \frac{10}{4.67} + \frac{13}{10-4.67} = 4.58\ kN/m^2$

Note Answer

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Example ② /

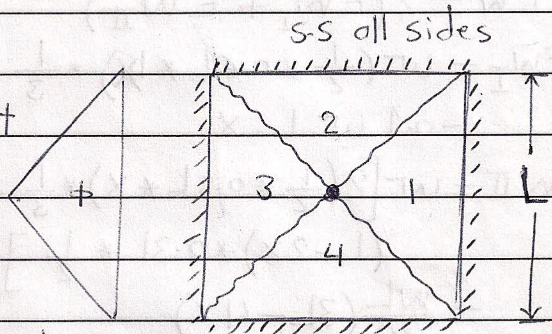
Find collapse load (w) for the square slab of length (L) shown, if the resisting moment in every direction = m kN.m/m (Isotropic slab).

Solution/

$$EW = W * \left(\frac{1}{2} * L * \frac{L}{2}\right) * \frac{1}{3} * 4$$

segment

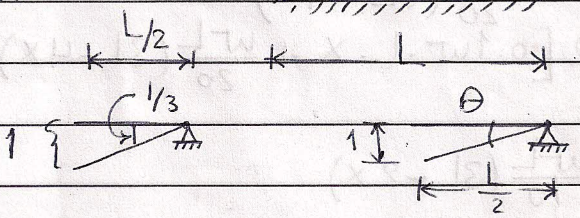
$$= \frac{WL^2}{3}$$



$$IW = m * \theta * l$$

projection of Y.L on R.A

$$= m * \frac{1}{\frac{L}{2}} * L * 4 = 8m$$

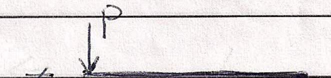


$$EW = IW$$

$$\frac{WL^2}{3} = 8m \rightarrow W = \frac{24m}{L^2}$$ Answer.

Example ③ / For Ex ②, if the slab has the same dimension & boundary conditions, find collapse concentrated force (kN) at center of slab.

Solution/



$$EW = P * 1 = P$$

$$IW = 8m \quad \text{(as before)}$$

$$EW = IW$$

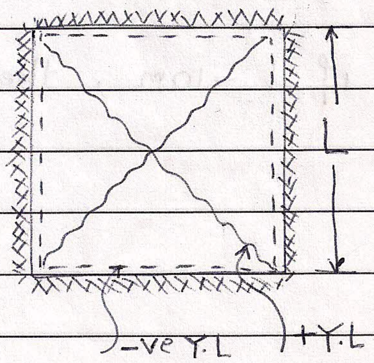
$$P = 8m$$ Answer.

Example ④ / For Ex ②, if the slab is fixed at all sides, find collapse load w, consider $m^+ = m^- = m$

Solution/

$$EW = \frac{WL^2}{3} \quad \text{(as before)}$$

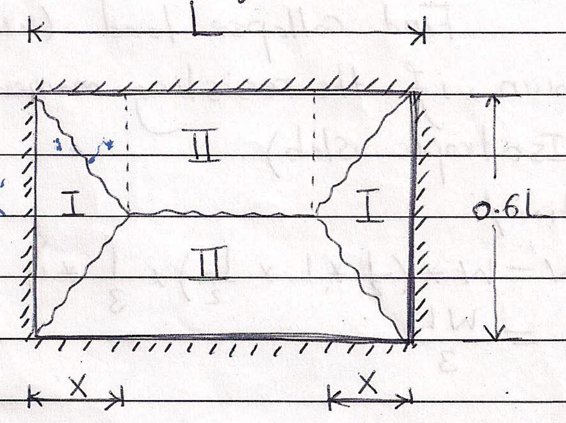
$$IW = m^+ * \frac{1}{L/2} * L * 4 + m^- * \frac{1}{\frac{L}{2}} * L * 4 = 16m$$



$$EW = IW \rightarrow \frac{WL^2}{3} = 16m \rightarrow W = \frac{48m}{L^2}$$

Answer

Example ⑥/ Find collapse load w for the S.S rectangular slab shown, assume ultimate moment of resistance in each direction is "mp".



Solution/

$$EW = 2(EW_I + EW_{II})$$

$$EW_I = w\left(\frac{1}{2} * 0.6L * X\right) * \frac{1}{3}$$

$$= 0.1\, w . L . X$$

$$EW_{II} = w\left[2\left(\frac{1}{2} * 0.3L * X\right) * \frac{1}{3} + (L - 2X) * 0.3L * \frac{1}{2}\right]$$

$$= \frac{wL}{20}(3L - 4X)$$

$$\therefore EW = 2\left[0.1 w . L . X + \frac{wL}{20}(3L - 4X)\right]$$

Note/ the segment (II) is divided to two triangles + one rectangle

$$= \frac{wL}{10}(3L - 2X)$$

$$IW = \left[mp * 0.6L * \frac{1}{X} + mp * 1 * \frac{1}{0.3L}\right] * 2 = \left(\frac{3.6L + 20X}{3X}\right) mp$$

$$EW = IW \Rightarrow \frac{wL}{10}(3L - 2X) = \left(\frac{3.6L + 20X}{3X}\right) mp$$

$$\Rightarrow w = \frac{10}{L}\left[\frac{3.6L + 20X}{3X(3L - 2X)}\right] \Leftrightarrow \text{Rational function } \frac{A}{B}$$

$$\frac{BA' - AB'}{B^2}$$

for minimum load, $\frac{dw}{dX} = 0$

$$= \frac{60X(3L - 2X) - (3.6L + 20X)(9L - 12X)}{[3X(3L - 2X)]^2}$$

$$120X^2 + 43.2LX - 32.4L^2 = 0 \rightarrow X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $L = 10m$, then $X = 3.7m$ and $w = 0.438\, mp$ $\qquad X = 0.37L$

Answer

Example - 7

: For the slab shown;

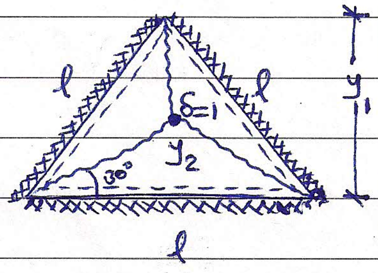
Find $W_{collapse}$ if $m_p^{+} = 20$ kN.m/m , $m_p^{-} = 25$ kN.m/m

$l = 6m$

Solution segments

$\tan 60 = \sqrt{3}$

$\tan 30 = \frac{1}{\sqrt{3}}$



$$W_E = 3\left(\frac{1}{2} * l * \frac{l}{2\sqrt{3}}\right) * w * \frac{1}{3} = \frac{wl^2}{4\sqrt{3}}$$

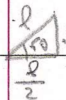
$$W_I = 3\left[\left(m_p^{+} * l * \frac{1}{l/2\sqrt{3}}\right) + \left(m_p^{-} * l * \frac{1}{l/2\sqrt{3}}\right)\right]$$

$$= 3\left[\left(20 \times \cancel{6} * \frac{2\sqrt{3}}{\cancel{6}}\right) + \left(25 * \cancel{6} * \frac{2\sqrt{3}}{\cancel{6}}\right)\right]$$

$$= 3 * 90\sqrt{3} = 90(3)^{1.5}$$

$\tan 60 = y_1 / l/2$

$\therefore y_1 = \frac{\sqrt{3}\, l}{2}$



$\tan 30 = y_2 / l/2$

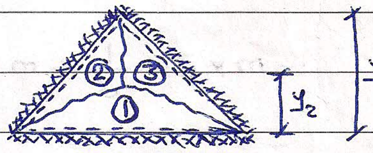
$\therefore y_2 = \frac{l}{2\sqrt{3}}$

$$W_E = W_I \Rightarrow \frac{w * (6)^2}{4\sqrt{3}} = 90(3)^{1.5}$$

$$\therefore w = 90 \text{ kN/m}$$

Note ⟷ للإطلاع

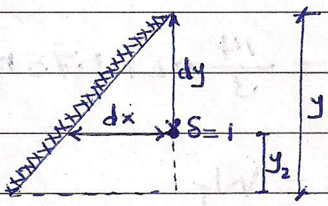
W_I can be calculated by using procedure of page (4) (Rotation about inclined axis)



$$W_{I\,①} = m^{+} * l * \frac{1}{\frac{l}{2\sqrt{3}}} + m^{-} * l * \frac{1}{\frac{l}{2\sqrt{3}}}$$

$$= m^{+} * 2\sqrt{3} + m^{-} * 2\sqrt{3}$$

$$W_{I\,②} = m^{+} * l_x * \frac{1}{dy} + m^{+} * l_y * \frac{1}{dx} + m^{-} * l_x * \frac{1}{dy} + m^{-} * l_y * \frac{1}{dx}$$



$$= m^{+} * \frac{\cancel{l}}{2} * \frac{1}{\frac{\cancel{l}}{\sqrt{3}}} + m^{+} * \frac{\sqrt{3}}{2}\cancel{l} * \frac{1}{\frac{\cancel{l}}{3}}$$

$$+ m^{-} * \frac{\cancel{l}}{2} * \frac{1}{\frac{\cancel{l}}{\sqrt{3}}} + m^{-} * \frac{\sqrt{3}}{2}\cancel{l} * \frac{1}{\frac{\cancel{l}}{3}}$$

$$= m^{+}\left(\frac{\sqrt{3}}{2} + \frac{3}{2}\sqrt{3}\right) + m^{-}\left(\frac{\sqrt{3}}{2} + \frac{3}{2}\sqrt{3}\right)$$

$$= m^{+} * 2\sqrt{3} + m^{-} * 2\sqrt{3}$$

$y_1 = \frac{\sqrt{3}}{2} l$ $\quad y_2 = \frac{l}{2\sqrt{3}}$

$dy = y_1 - y_2 = \frac{l}{\sqrt{3}}$

$\frac{dx}{dy} = \frac{l/2}{y_1} \Rightarrow dx = \frac{l}{3}$

$$W_{I\,③} = W_{I\,②}$$

$$W_{I\,Total} = W_{I\,①} + W_{I\,②} + W_{I\,③} = \left[m^+ * 2\sqrt{3} + m^- * 2\sqrt{3}\right] * 2 + m^+ * 2\sqrt{3} + m^- * 2\sqrt{3}$$

$$= m^+ * 6\sqrt{3} + m^- * 6\sqrt{3}$$

$$= 20 * 6\sqrt{3} + 25 * 6\sqrt{3} = 270\sqrt{3}$$

$$= 90 \times 3 * \sqrt{3} = 90 * (3)^{1.5}$$

As before

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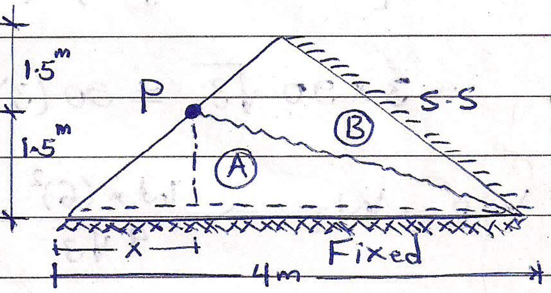
Example ⑧

Find Collapse Force (P) for the isotropic triangular slab shown

Assume $m_p^- = m_p^+ = m_p$

Solution



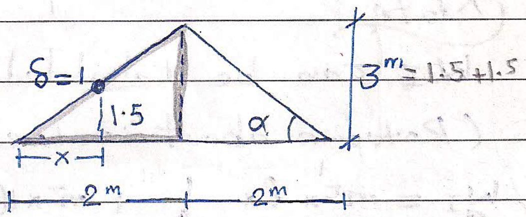
Segment (B) Rotate about inclined axis

$$W_E = P * 1 = P$$

$$W_{I\,(A)} = m^+ * 3 * \frac{1}{1.5} + m^- * 4 * \frac{1}{1.5}$$

$$= \frac{14}{3} m$$

$$\frac{x}{1.5} = \frac{2}{3} \Rightarrow x = 1m$$

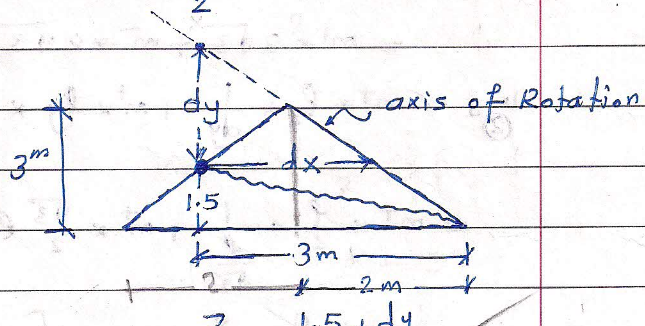


$$W_{I\,(B)} = m^+ * l_x * \frac{1}{d_y} + m^+ * l_y * \frac{1}{d_x}$$

$$= m * 3 * \frac{1}{3} + m * 1.5 * \frac{1}{2}$$

$$\alpha = \tan^{-1}\frac{3}{2} \approx 56.31^\circ$$

$$= 1.75\,m$$



$$W_{I\,Total} = \frac{14}{3} m + 1.75 m = 6.417 m$$

$$W_E = W_I$$

$$\frac{3}{2} = \frac{1.5 + d_y}{3}$$

$$d_y = \frac{9}{2} - 1.5 = 3m$$

$$P = 6.417\, m_p$$

Answer

$$\frac{3}{d_x} = \frac{d_y + 1.5}{3}$$

$$d_x = 2m$$

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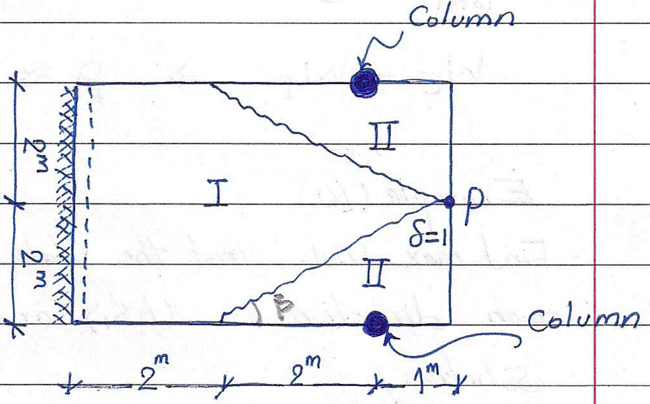
Example ⑨

Find $P_{collapse}$ $\quad m_p^+ = m_p^- = m_p$

Solution

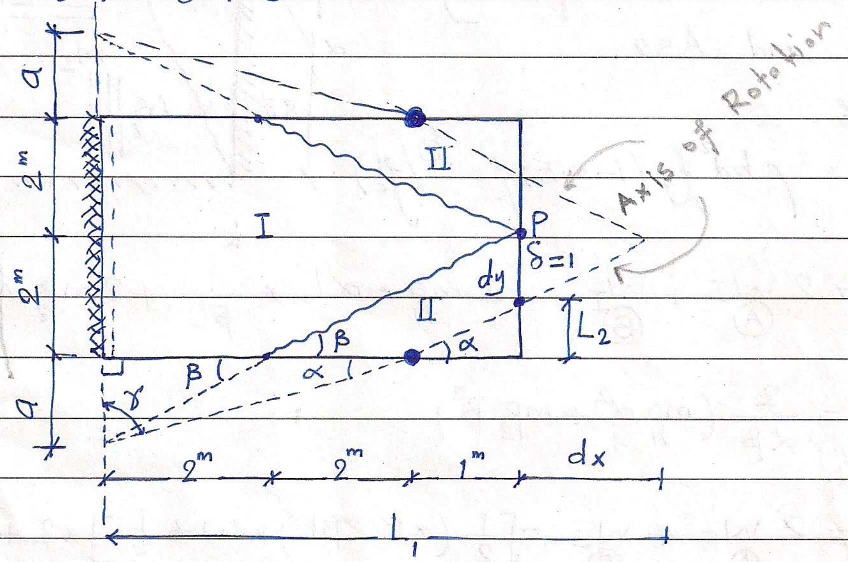


$$\beta = \tan^{-1}\frac{2}{3} \approx 33.7^\circ$$

$$\tan\beta = \frac{a}{2} \Rightarrow \boxed{a = \frac{4}{3}}$$

$$\alpha = \tan^{-1}\frac{a}{4} = \tan^{-1}\frac{4/3}{4} \approx 18.43^\circ$$

$$\gamma = 90 - \alpha = 90 - 18.43^\circ \approx 71.57^\circ$$



$$\tan\gamma = \frac{L_1}{2+a} \Rightarrow L_1 = (2+a) * \tan\gamma$$

$$= (2 + \tfrac{4}{3}) * \tan 71.57 = 10\,m$$

$$\therefore dx = 10 - (2+2+1) = 5m$$

$$L_2 = \tan\alpha * 1 = \tan(18.43) * 1 = 0.33\,m$$

$$\therefore dy = 2 - L_2 = 2 - 0.33 = 1.67\,m$$

$$W_E = P * 1 = P$$

$$W_{I\,(\mathrm{I})} = m_p^+ * 4 * \frac{1}{5} + m_p^- * 4 * \frac{1}{5} = \frac{8}{5} m_p$$

$$W_{I\,(\mathrm{II})} = m_p^+ * l_x * \frac{1}{dy} + m_p^+ * l_y * \frac{1}{dx}$$

$$= m_p * 3 * \frac{1}{1.67} + m_p * 2 * \frac{1}{5} = 2.2\,m_p$$

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Two segments

$$W_{I\,\text{Total}} = W_{I\,(I)} + 2W_{I\,(II)} = \left(\frac{8}{5} + 2\times 2.2\right)m_p \approx 6\,m_p$$

$$W_E = W_I \Rightarrow P \approx 6\,m_p$$

Answer

Example (10)

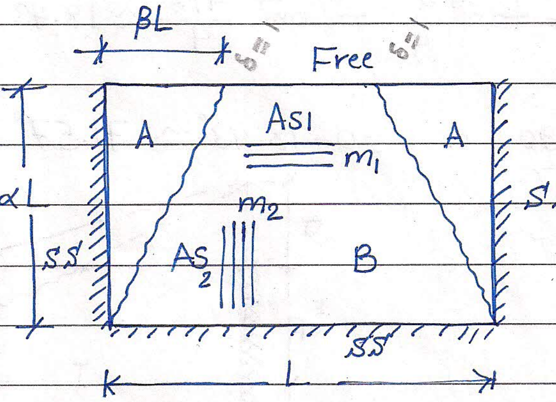
Find max. UDL that the slab can sustain if its reinforcement in each direction is (AS_1) and (AS_2) as shown.

Solution/

Let mp_1 and mp_2 the moment of resistance corresponding to AS_1 and AS_2

Note

$$mp = \rho b d^2 f_y \left(1 - 0.59\rho\, f_y/f'_c\right)$$



$$W_I = 2W_{I\,(A)} + W_{I\,(B)} = 2\,mp_1 * \alpha L * \frac{1}{\beta L} + 2\,mp_2 * \beta L * \frac{1}{\alpha L}$$

$$= \frac{2}{\alpha\beta}\left(mp_1\alpha^2 + mp_2\beta^2\right)$$

$$W_E = 2W_{E\,(A)} + W_{E\,(B)} = \left[\frac{1}{2}(\alpha L\cdot\beta L)*w*\frac{1}{3}\right]*2 + 2\Big[\underbrace{\overset{\frac{1}{2}*\frac{1}{3}}{\frac{1}{6}}\,w\,\alpha\beta L^2}_{\text{مثلث}}$$

$$+ \underbrace{\alpha L\left(\frac{L}{2} - \beta L\right)w * \frac{1}{2}}_{\text{مستطيل}}\Big]$$

$$= \frac{1}{6}\alpha(3-2\beta)L^2 * w$$

$$W_E = W_I \Rightarrow w = \frac{\frac{2}{\alpha\beta}\left(mp_1\alpha^2 + mp_2\beta^2\right)}{\frac{\alpha}{6}(3-2\beta)L^2} = \frac{12\left(mp_1\alpha^2 + mp_2\beta^2\right)}{\alpha^2L^2(3\beta-2\beta^2)}$$

It varied according to β.

Let $mp_2 = \mu\, mp_1$ أي أن النسبة بين العزمين $= \mu$

$$\therefore w = \frac{12\,mp_1\left(\alpha^2 + \mu\beta^2\right)}{\alpha^2L^2\left(3\beta - 2\beta^2\right)}$$