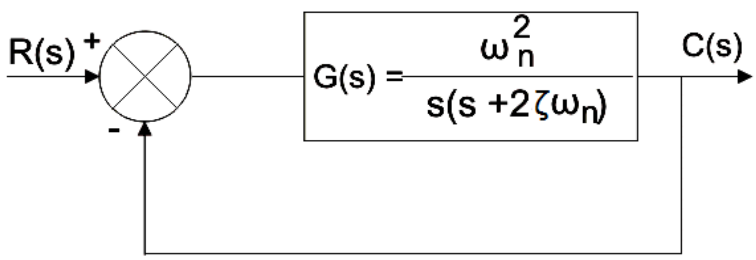
Time Response of Second-Order Control System

The order of a control system is determined by the power of s in the denominator of its transfer function. If the power of s in the denominator of transfer function of a control system is 2, then the system is said to be second-order control system. The general expression of transfer function of a second order control system is given as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Here, ζ and ωn are damping ratio and natural frequency of the system respectively and we will learn about these two terms in detail later on. Therefore, the output of the system is given as

$$C(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Characteristics equation of time response of second-order control system:

The general equation of transfer function of second order control system is given as:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

If the denominator of the expression is zero,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\Rightarrow s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 - \zeta^2\omega_n^2 + \omega_n^2 = 0$$

$$\Rightarrow (s + \zeta\omega_n)^2 - \zeta^2\omega_n^2 + \omega_n^2 = 0$$

$$\Rightarrow (s + \zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2 = 0$$

$$\Rightarrow s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \; or \; -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

$$Putting, \; \omega_d = \omega_n\sqrt{1-\zeta^2}$$

And , α = ζωn

Then , s = α ± jωd

The dynamic behavior of the second-order system can then be described in terms of two parameters ζ and ω_n. If $0 < \zeta < 1$, the closed-loop poles are complex conjugates and lie in the left-half s plane. The system is then called underdamped, and the transient response is oscillatory. If $\zeta = 1$, the system is called critically damped. Overdamped systems correspond to $\zeta > 1$. The transient response of critically damped and overdamped systems do not oscillate. If $\zeta = 0$, the transient response does not die out.

We shall now solve for the response of the system shown in Figure above to a unit-step input. We shall consider three different cases: the underdamped $(0 < \zeta < 1)$, critically damped $(\zeta = 1)$, and overdamped $(\zeta > 1)$ cases.

If we consider a unit step function as the input of the system,

(1) Underdamped case $(0 < \zeta < 1)$: In this case, $C(s)/R(s)$ can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The frequency ω_d is called the *damped natural frequency.* For a unit-step input, $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

The inverse Laplace transform

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Hence the inverse Laplace transform

$$\mathcal{L}^{-1}[C(s)] = c(t)$$

$$= 1 - e^{-\zeta\omega_n t}\left(\cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin\left(\omega_d t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}\right), \qquad \text{for } t \geq 0$$

The error of the signal of the response is given by

$$e(t) = r(t) - c(t)$$

$$= e^{-\zeta\omega_n t}\left(\cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t\right) \qquad \text{for } t \geq 0$$

From the above expression it is clear that the error of the signal is of oscillation type with exponentially decaying magnitude when $\zeta < 1$. The frequency of the oscillation is ωd and the time constant of exponential decay is $1/\zeta\omega n$. Where, ωd, is referred as damped frequency of the oscillation, and ωn is natural frequency of the oscillation. The term ζ affects that damping a lot and hence this term is called damping ratio.

(2) Critically damped case ($\zeta = 1$): If the two poles of $C(s)/R(s)$ are nearly equal, the system may be approximated by a critically damped one.

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2 s}$$

The inverse Laplace transform

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \qquad \text{for } t \geq 0$$

(3) Overdamped case ($\zeta > 1$): In this case, the two poles of $C(s)/R(s)$ are negative real and unequal. For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

The inverse Laplace transform

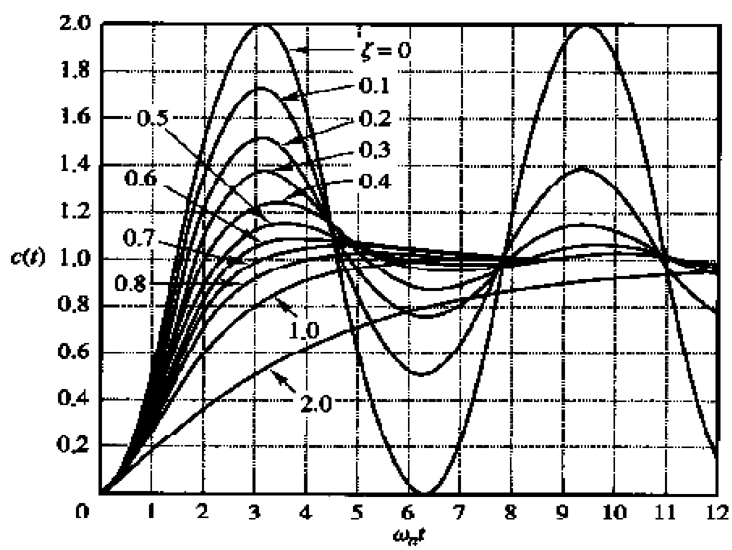
$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta + \sqrt{\zeta^2 - 1})} e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t}$$

$$- \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta - \sqrt{\zeta^2 - 1})} e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

$$= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \qquad \text{for } t \geq 0$$

where $s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$. Thus, the response $c(t)$ includes two decaying exponential terms.

A family of curves $c(t)$ with various values of ζ is shown in Figure



A second-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\zeta > 1$:

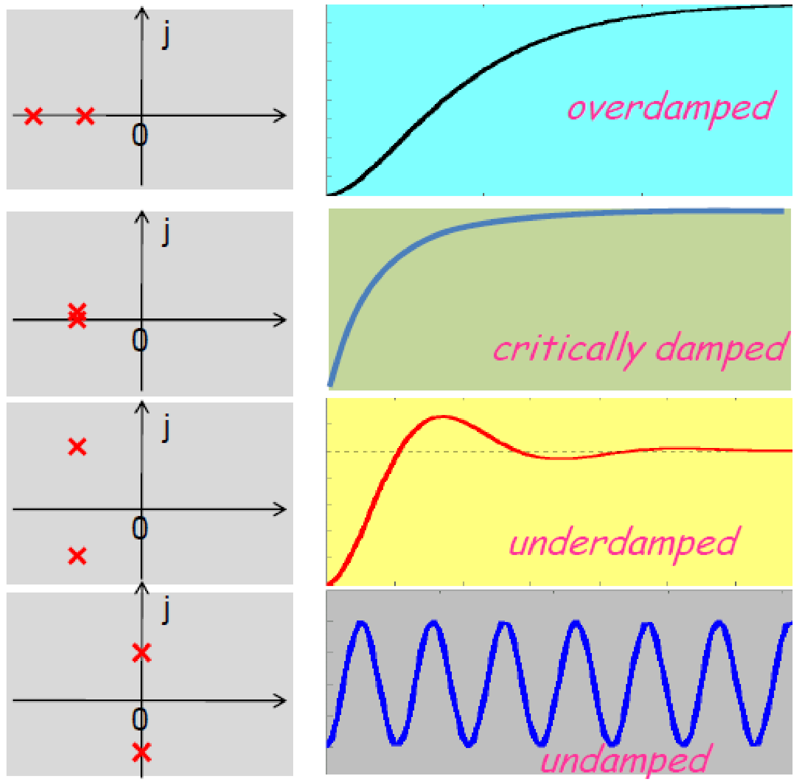
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$\zeta = 1$: $s_{1,2} = -\omega_n$

$0 < \zeta < 1$:

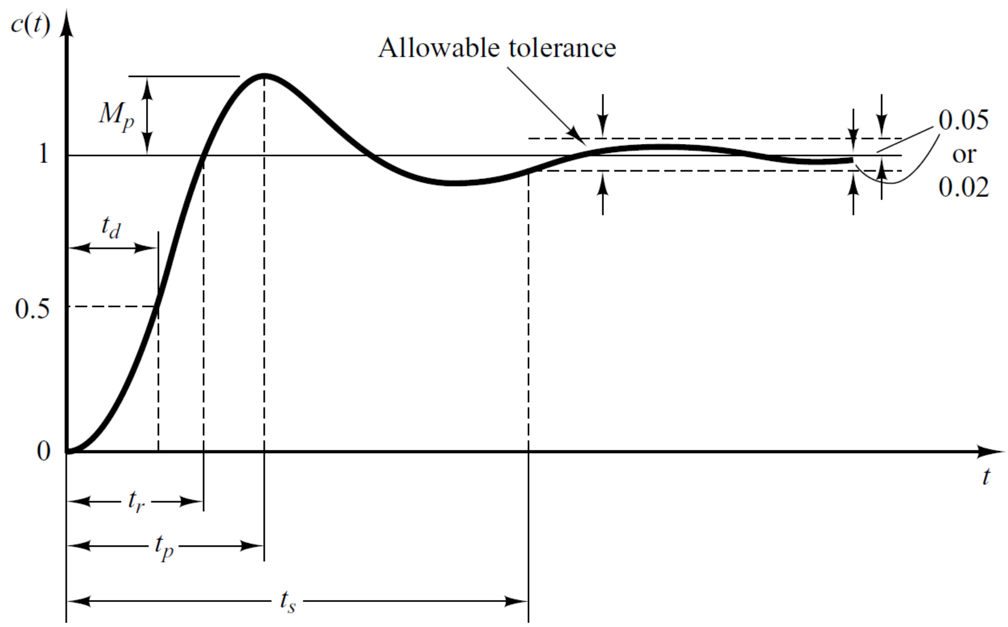
$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$\zeta = 0$: $s_{1,2} = \pm j\omega_n$



Definitions of transient-response specifications.

The performance of the control system can be expressed in the term of transient response to a unit step input function because it is easy to generate. Let us consider a second-order control system in which a unit step input signal is given and it is also considered that the system is initially at rest. That is all initial conditions of the system are zero. The time response characteristics of the system at underdamped condition is drawn below.



There are number of common terms in transient response characteristics and which are:

1. **Delay time (td)** is the time required to reach at 50% of its final value by a time response signal during its first cycle of oscillation.

2. **Rise time (tr)** is the time required to reach at final value by a under damped time response signal during its first cycle of oscillation. If the signal is over damped, then rise time is counted as the time required by the response to rise from 10% to 90% of its final value.

In order to derive the expression for the rise time we have to equate the expression for c(t) = 1. From the above we have:

$$c(t) = 1 = 1 - \frac{e^{-\zeta\omega t}\sin\left[\omega\sqrt{1-\zeta^2}t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]}{\sqrt{1-\zeta^2}}$$

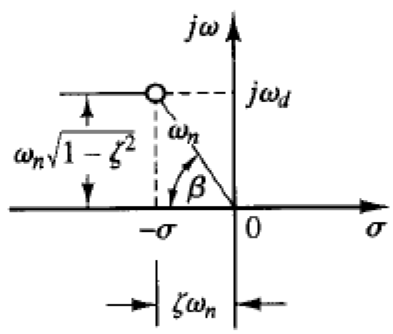
On solving above equation we have expression for rise time equal to

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega\sqrt{1-\zeta^2}}$$

Thus, the rise time t_r is

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d}$$

where β is defined in Figure



3. **Peak time (tp)** is simply the time required by response to reach its first peak i.e. the peak of first cycle of oscillation, or first overshoot.

On differentiating the expression of c(t) we can obtain the expression for peak time. dc(t)/ dt = 0 we have expression for peak time,

$$t_p = \frac{\pi}{\omega\sqrt{1-\zeta^2}}$$

$$t_p = \frac{\pi}{\omega_d}$$

4. **Maximum overshoot (Mp)** is straight way difference between the magnitude of the highest peak of time response and magnitude of its steady

state. Maximum overshoot is expressed in term of percentage of steady-state value of the response. As the first peak of response is normally maximum in magnitude, maximum overshoot is simply normalized difference between first peak and steady-state value of a response.

$$Maximum\ \%\ Overshoot = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

Now it is clear from the figure that the maximum overshoot will occur at peak time tp hence on putting the value of peak time we will get maximum overshoot as

$$\%\ MP = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

5. **Settling time (ts)** is the time required for a response to become steady. It is defined as the time required by the response to reach and steady within specified range of 2% to 5% of its final value. Settling time is given by the expression:

$$t_s = \frac{4}{\omega\zeta}$$

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad \text{(2\% criterion)}$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad \text{(5\% criterion)}$$

6. **Steady-state error ($e$ss)** is the difference between actual output and desired output at the infinite range of time.

$$e_{ss} = \lim_{t\to\infty} [r(t) - c(t)]$$

Deriving an Expression of Rise Time:

The expression of underdamped second-order control system with unit step input function,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right\}$$

Where

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Again, as per definition, the magnitude of output signal at Rice times is 1. That is c(t) = 1, hence

$$1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t_r + \phi\right\}$$

$$\Rightarrow \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t_r + \phi\right\} = 0$$

$$\Rightarrow \sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t_r + \phi\right\} = 0$$

$$\Rightarrow \left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t_r + \phi\right\} = \pi$$

$$\Rightarrow t_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$$

Deriving an Expression of Peak Time:

As per definition at the peak time, the response curve reaches to its maximum value. Hence at that point,

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = 0$$

$$Now,\ c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right\}$$

$$\therefore \frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cdot \omega_n\sqrt{1-\zeta^2} \cos\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right\}$$

$$-\frac{(-\zeta\omega_n)\,e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right\}$$

$$Putting,\ \frac{\mathrm{d}c(t)}{\mathrm{d}t} = 0$$

$$\therefore \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[-\omega_n\sqrt{1-\zeta^2} \cos\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right\} + \zeta\omega_n \sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right\}\right]$$

$$= 0$$

$$\omega_n\sqrt{1-\zeta^2} \cos\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right\} = \zeta\omega_n \sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right\}$$

$$\Rightarrow \tan\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right] = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\phi$$

$$\therefore \left(\omega_n\sqrt{1-\zeta^2}\right)t = n\pi$$

$$Where,\ n = 1, 2, 3\ldots$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

The maximum overshoot occurs at n = 1.

<u>Deriving an Expression of Maximum Overshoot:</u>

If we put the expression of peak time in the expression of output response c(t), we get,

$$c(t)_{max} = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right) t_p + \phi\right]$$

$$\Rightarrow c(t)_{max} = 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right) \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} + \phi\right]$$

$$\Rightarrow c(t)_{max} = 1 - \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\pi + \phi) = 1 - \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} (-\sin\phi)$$

$$= 1 + \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\phi = 1 + \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$M_p = c(t)_{max} - 1 = \left(1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right) - 1$$

$$\Rightarrow M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Deriving an Expression of Settling Time

It is already defined that settling time of a response is that time after which the response reaches to its steady-state condition with value above nearly 98% of its final value. It is also observed that this duration is approximately 4 times of time constant of a signal. At the time constant of a second-order control system is $1/\zeta\ \omega n$, the expiration of settling time can be given as

$$t_s = \frac{4}{\zeta\omega_n}$$

Now we will see the effect of different values of ζ on the response. We have three types of systems on the basis of different values of ζ.

1. **Under damped system**: A system is said to be under damped system when the value of **ζ is less than one**. In this case **roots** are **complex** in nature and the real parts are always negative. System is asymptotically **stable**. **Rise time is lesser than the other system** with the **presence of finite overshoot**.

2. **Critically damped system**: A system is said to be critically damped system when the value of **ζ is one**. In this case **roots** are **real** in nature and the real parts are always repetitive in nature. System is asymptotically stable. **Rise time is less in this system** and there is **no presence of finite overshoot**.

3. **Over damped system**: A system is said to be over damped system when the value of **ζ is greater than one**. In this case **roots** are **real** and **distinct** in nature and the real parts are always negative. System is asymptotically **stable**. **Rise time is greater than the other system** and there is **no presence of finite overshoot**.

4. **Sustained Oscillations**: A system is said to be sustain damped system when the value of **zeta is zero**. **No damping occurs in this case**.

$\zeta > 1:$

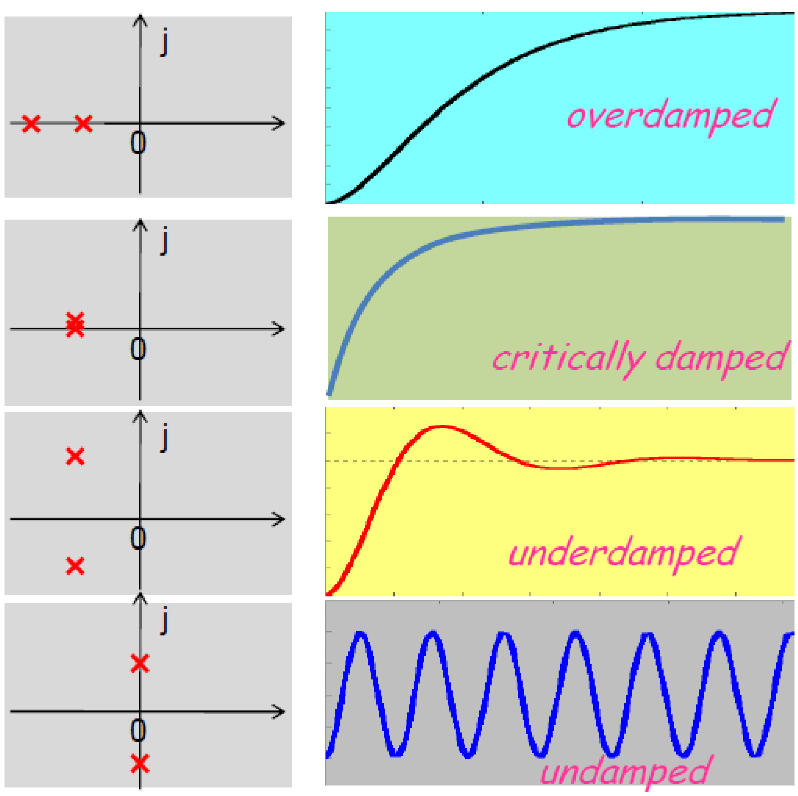
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$\zeta = 1:$ $s_{1,2} = -\omega_n$

$0 < \zeta < 1:$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

$\zeta = 0:$ $s_{1,2} = \pm j\omega_n$



Damped Natural Frequency	$\omega_d = \omega_n\sqrt{1 - \zeta^2}$
	$\sigma = \zeta\omega_n$
	$\beta = \tan^{-1}\frac{\omega_d}{\sigma}$
Rise time	$t_r = \frac{\pi - \beta}{\omega_d}$
Peak time	$t_p = \frac{\pi}{\omega_d}$
Maximum overshoot	$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$ $= \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$
Settling time	$t_s = \frac{4}{\zeta\omega_n}$ (2% criterion)
	$t_s = \frac{3}{\zeta\omega_n}$ (5% criterion)

Example.1 : Consider the second order system where ζ=0.6 and wn=5 rad/sec. find the rise time tr , peak time tp, maximum overshoot *M*p, and settling time ts when the system is subjected to a unit-step input?

Solution:

From the given values of ζ and ω_n, we obtain $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$ and $\sigma = \zeta\omega_n = 3$.

Rise time t_r: The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where β is given by

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

The rise time t_r is thus

$$t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

Peak time t_p: The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

Maximum overshoot M_p: The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

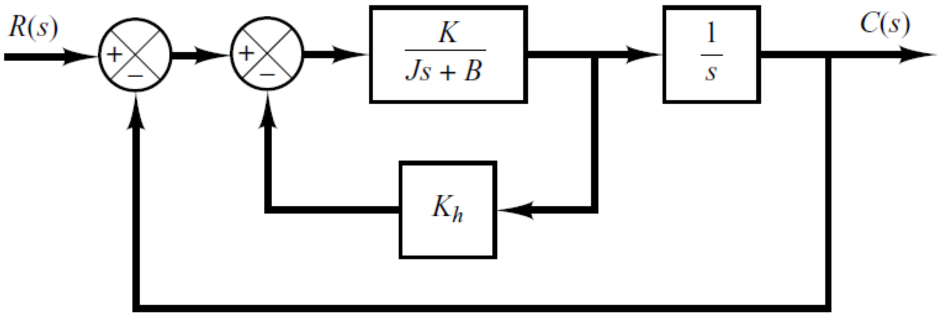
Settling time t_s: For the 2% criterion, the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

Example.2: For the system shown in figure, determine the values of gain K and velocity-feedback constant *K*h so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of Kand *K*h, obtain the rise time and settling time.Assume that J=1kg-m2 and B=1 N-m/rad/sec.



Solution:

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

This value must be 0.2. Thus,

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

or

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.61$$

which yields

$$\zeta = 0.456$$

$$t_p = \frac{\pi}{\omega_d} = 1$$

or

$$\omega_d = 3.14$$

Since ζ is 0.456, ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3.53$$

Since the natural frequency ω_n is equal to $\sqrt{K/J}$,

$$K = J\omega_n^2 = \omega_n^2 = 12.5 \text{ N-m}$$

Then K_h is, from Equation (5–25),

$$K_h = \frac{2\sqrt{KJ}\zeta - B}{K} = \frac{2\sqrt{K}\zeta - 1}{K} = 0.178 \text{ sec}$$

Rise time t_r: From Equation (5–19), the rise time t_r is

$$t_r = \frac{\pi - \beta}{\omega_d}$$

where

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}1.95 = 1.10$$

Thus, t_r is

$$t_r = 0.65 \text{ sec}$$

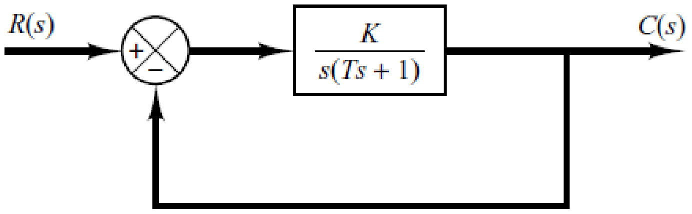
Settling time t_s: For the 2% criterion,

$$t_s = \frac{4}{\sigma} = 2.48 \text{ sec}$$

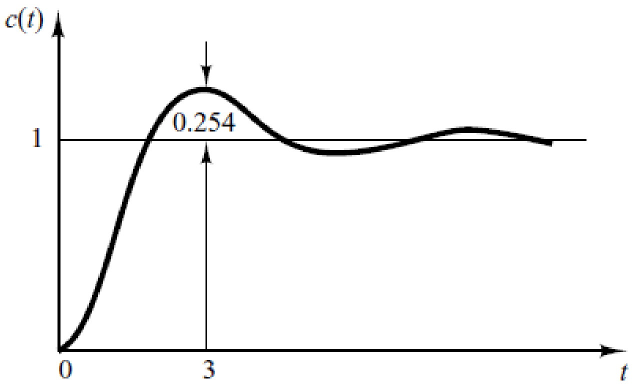
For the 5% criterion,

$$t_s = \frac{3}{\sigma} = 1.86 \text{ sec}$$

Example.3: When the system shown in figure (a) is subjected to a unit-step input, the system output responds as shown in figure (b) . Determine the values of K and T from the response curve.



(a)



(b)

Solution:

The maximum overshoot of 25.4% corresponds to ζ =0.4. From the response curve we have

$$t_p = 3$$

Consequently,

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n\sqrt{1-0.4^2}} = 3$$

It follows that

$$\omega_n = 1.14$$

From the block diagram we have

$$\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K}$$

from which

$$\omega_n = \sqrt{\frac{K}{T}}, \qquad 2\zeta\omega_n = \frac{1}{T}$$

Therefore, the values of T and K are determined as

$$T = \frac{1}{2\zeta\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09$$

$$K = \omega_n^2 T = 1.14^2 \times 1.09 = 1.42$$