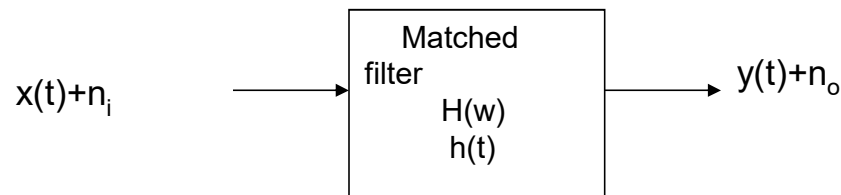


Matched Filter

Detection problem: Detection of the signal $x(t)$ embedded in AWGN noise $n(t)$ will produce the output $y(Tb)$ which was the signal used at the decision block (comparator block). The device that carries out the function of detection is called a *Matched Filter*. This matched filter is simply a linear system with impulse response $h(t)$ (or frequency response $H(\omega)$) that acts as a filter. The job of this filter is to maximize the ratio $\frac{|y(Tb)|^2}{n_o^2(t)}$.



This filter is called “*Matched*” since for a certain signal $x(t)$, there exists a filter with impulse response $h(t)$ matched to it (maximizes the above ratio).

After long derivation, the following two equations are obtained:

1- $h(t) = x(T_b - t)$.

Which gives the impulse response $h(t)$ of a matched filter matched to the signal $x(t)$. Hence the impulse response of a matched filter matched to the signal $x(t)$ is the negative time of $x(t)$ shifted by T_b .

2-
$$\left[\frac{|y(T_b)|^2}{n_o^2(t)} \right]_{\max} = \frac{E}{(\eta_o / 2)}$$

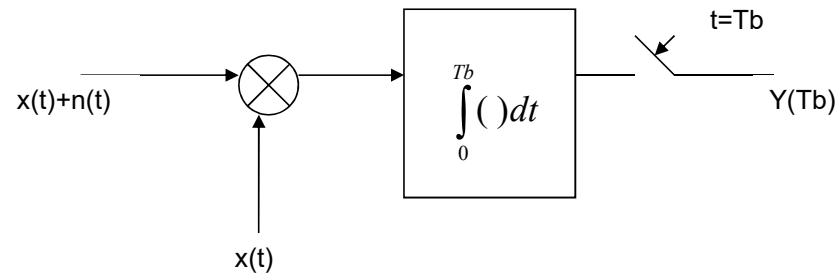
Which gives Max SNR at matched filter output. where

η_o = one-sided AWGN spectral density (Watt/Hz) and

and $E = \int_0^{T_b} [x(t)]^2 dt$ = energy of the signal over bit duration T_b .

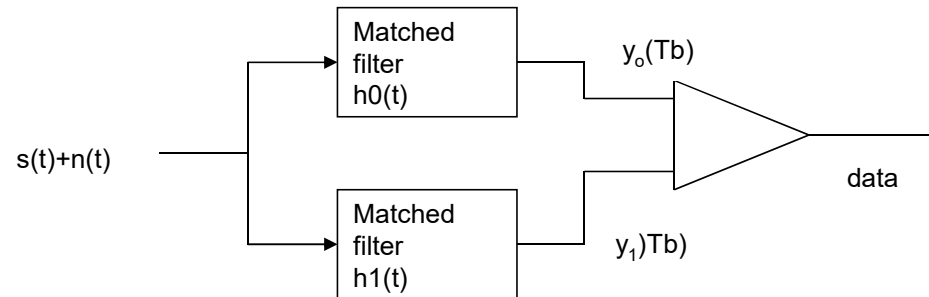
Practical Implementation of the Matched filter:

The practical implementation of a matched filter can be deduced as the product of the input $x(t)+n(t)$ and $x(t)$ then integrate the result over $t=0$ to $t=T_b$. The figure below shows the practical implementation of the matched filter:



Error probability of binary signal detection using matched filter:

Assume that the matched filter is used to detect the binary signal waveforms $s_0(t)$ and $s_1(t)$ embedded in AWGN. Here, we need two matched filters, one matched to $s_0(t)$ with impulse response $h_0(t)=s_0(T_b-t)$ and the other matched to $s_1(t)$ with impulse response $h_1(t)=s_1(T_b-t)$



Where: $E_o = \int_0^{Tb} s_o^2(t) dt$ = energy of $s_o(t)$ = o/p of $h_o(t)$ if $s(t) = s_o(t)$ and

$E_1 = \int_0^{Tb} s_1^2(t) dt$ = energy of $s_1(t)$ = o/p of $h_1(t)$ if $s(t) = s_1(t)$ also:

$E_{o1} = \int_0^{Tb} s_o(t)s_1(t) dt$ = cross energy between $s_o(t)$ and $s_1(t)$
 = o/p of $h_o(t)$ if $s(t) = s_1(t)$
 = o/p of $h_1(t)$ if $s(t) = s_o(t)$

For equiprobable $p(0_T)=p(1_T)=0.5$, and for Gaussian case:

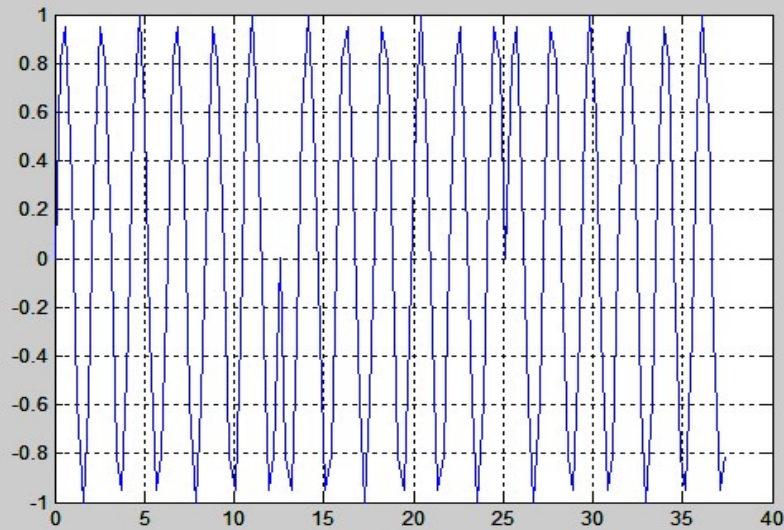
$$pe = Q\left(\sqrt{\frac{\int_0^{Tb} [s_o(t) - s_1(t)]^2 dt}{2\eta_o}}\right) = Q\left(\sqrt{\frac{E_o + E_1 - 2E_{01}}{2\eta_o}}\right) \text{ -----(1)}$$

Error prob for digital carrier systems:

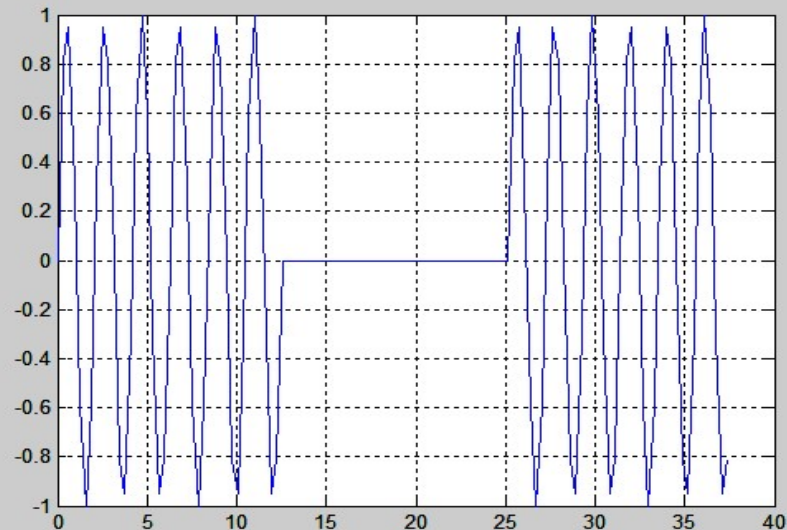
Review of digital carrier systems:

These systems are used to digitally modulate a binary data “0” and “1”. Depending on the choice of $s_o(t)$ and $s_1(t)$, there are three basic of such systems:

1- ASK (Amplitude Shift Keying) or OOK (On Off Keying): where, in OOK $s_o(t)=0$ and $s_1(t)=A \cos\omega t$ where the OOK signal is as shown



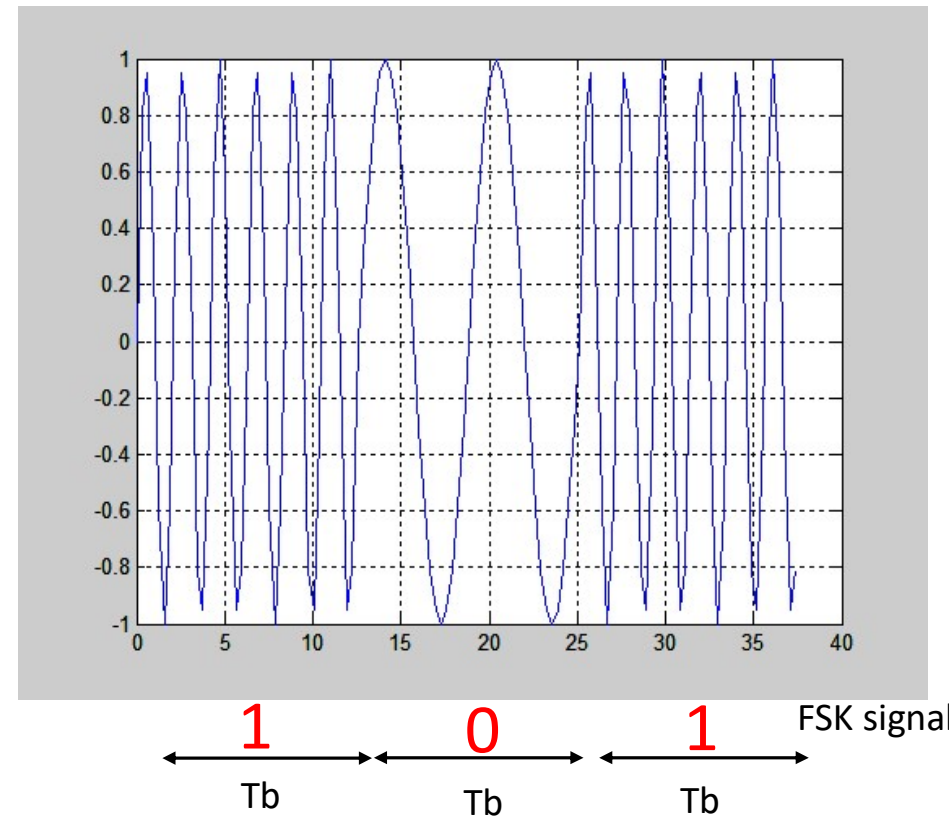
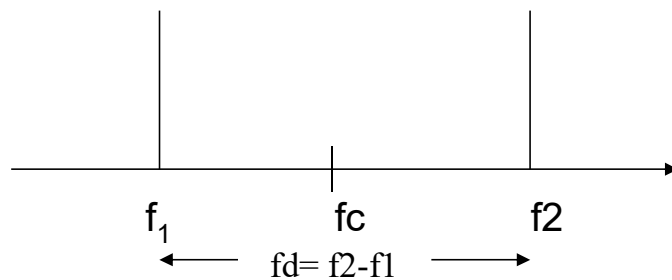
1 0 1 PSK signal
 $\longleftrightarrow$ $\longleftrightarrow$ $\longleftrightarrow$
 T_b T_b T_b



1 0 1 OOK signal
 $\longleftrightarrow$ $\longleftrightarrow$ $\longleftrightarrow$
 T_b T_b T_b

2- PSK (Phase Shift Keying): where in PSK, the phase of the carrier is inverted every "0" to "1" or "1" to "0" data change, i.e., $S_1(t) = A \cos \omega t$ and $S_0(t) = -A \cos \omega t$ as shown.

3-FSK (Frequency Shift Keying): where two frequencies are used, f_2 for logic "1" and f_1 for logic "0", i.e: $S_1(t) = A \cos \omega_2 t$ and $S_0(t) = A \cos \omega_1 t$ as shown in the figure where $f_2 > f_1$



Error prob for digital carrier systems using matched filter(coherent detection):

1- ASK (OOK) (ON-OFF) keying: Here the two signals are $s_0(t)=0$, $s_1(t)=A \cos \omega t$ over bit duration T_b . We use the general equation:

$$p_e = Q\left(\sqrt{\frac{E_o + E_1 - 2E_{01}}{2\eta_o}}\right)$$

where for OOK, $E_o = E_{o1} = 0$ (since $s_o(t) = 0$), and:

$E_1 = (A^2 T_b) / 2$ ($s_1(t)$ is a sinusoid whose normalized power $= A^2 / 2$), putting E_o , E_{o1} and E_1 then:

$$pe = Q\left(\sqrt{\frac{A^2 T_b}{4 \eta_o}}\right)$$

and express in terms of average signal power, then:

$S = [0.5 E_o + 0.5 E_1] / T_b = A^2 / 4$, then: $pe = Q\left(\sqrt{\frac{S}{\eta_o R b}}\right)$ and if $\lambda = \frac{S}{\eta_o R b} = \frac{E_b}{\eta_o} = \text{energy / bit}$

Then: $pe = Q(\sqrt{\lambda})$

2-BPSK signals(Binary Phase Shift Keying): Here the two signals are $s_o(t) = -A \cos \omega t$, $s_1(t) = A \cos \omega t$ over bit duration T_b . Again, using the same general equation, then:

$E_o = E_1 = [A^2 T_b] / 2$ since both $s_o(t)$ and $s_1(t)$ are two sinusoids, also:

$$E_{o1} = \int_0^{T_b} -A^2 \cos^2 \omega t dt = -[A^2 T_b] / 2.$$

Also $S = [0.5E_0 + 0.5E_1]/T_b = A^2/2 = \text{average signal power}$, then one can show that after putting E_0 , E_1 and E_{01} in the general equation:

$$p_e = Q\left(\sqrt{\frac{2S}{\eta_o R b}}\right) = Q(\sqrt{2\lambda}) \quad \text{which is better than OOK}$$

3-FSK (frequency shift keying): Here the two signals are $s_0(t) = A \cos \omega_1 t$, $s_1(t) = A \cos \omega_2 t$ over bit duration T_b . Note that p_e depends also on

$$\Delta\omega = \omega_d = \omega_2 - \omega_1$$

After a similar longer derivation, one can show that:

$$p_e = Q\left(\sqrt{\lambda \left(1 - \frac{\sin \omega_d T_b}{\omega_d T_b}\right)}\right) \quad \text{which is a general formula used for matched filter detection of FSK signals. (where } \lambda = \frac{S}{\eta_o R b} = \frac{E_b}{\eta_o} = \text{energy/bit as before)}$$

Special cases in FSK:

1-if $\omega_d T_b = \pi, 2\pi, 3\pi, \dots$, or $f_d = 0.5R_b, R_b, 1.5R_b, \dots$ this gives $E_{o1} = 0$ (orthogonal FSK), and $pe = Q(\sqrt{\lambda})$ (similar to OOK)

2-if $\omega_d T_b = 4.49 \text{ rad}$, or $f_d = 0.715R_b$, then: $\frac{\sin \omega_d T_b}{\omega_d T_b} = -0.217$

And this gives $pe = Q(\sqrt{1.217\lambda})$ which is the best (optimum) performance of FSK.

(Note: Use the general equation $pe = Q\left(\sqrt{\lambda\left(1 - \frac{\sin \omega_d T_b}{\omega_d T_b}\right)}\right)$ for FSK in solving problems, do not mix with above special cases.)

Example: Matched_filter detection is used to detect BPSK signals at a rate of 600bps. If transmitted power is 5KW over an HF channel having estimated path losses of 150dB, find the error prob if the noise at detector input has one sided spectral density of 10^{-15} Watt/Hz .

- Solution:

P_T =trans power=5000 W, then, $P_R=S=P_T * 10^{-0.1 * \text{losses in dB}} = 5000 * 10^{-0.1 * 150}$

$S=5 * 10^{-12}$ W=average signal power.

$$\lambda = \frac{S}{\eta_o R b} = \frac{5 * 10^{-12}}{10^{-15} * 600} = 8.333 \text{ then } pe = Q(\sqrt{2\lambda}) = Q(\sqrt{2 * 8.333}) = Q(4.08) \cong 2 * 10^{-5}$$

Example: Repeat previous example for FSK signals with $f_1=700\text{Hz}$, $f_2=2000\text{Hz}$.

Solution:

Here $\omega_d = 2\pi(2000-700) = 2600\pi$, and $\omega_d T b = 2600\pi / 600 = 4.3333\pi$, then and for the same $\lambda = 8.333$:

$$pe = Q\left(\sqrt{\lambda\left(1 - \frac{\sin \omega_d T b}{\omega_d T b}\right)}\right) = Q\left(\sqrt{8.333\left(1 - \frac{\sin 4.333\pi}{4.333\pi}\right)}\right) = Q(2.84) \cong 2.3 * 10^{-3}$$

Which is worst than BPSK for the same λ . (remember that both BPSK and FSK have the same average power of $A^2 / 2$, but the performance of BPSK is better)