Second Moments or Moments of Inertia

The second moment or moment of inertia of an element of area such as dA in Figure 1, with respect to any axis is defined as the product of the area of the element and the square of the distance from the axis to the element.

Rectangular and Polar Moments of Inertia

Consider the area A in the x-y plane, Figure 1. The moments of inertia of the element dA about the x- and y-axes are, by definition, $dI_x = y^2 dA$ and $dI_y = x^2 dA$ respectively.

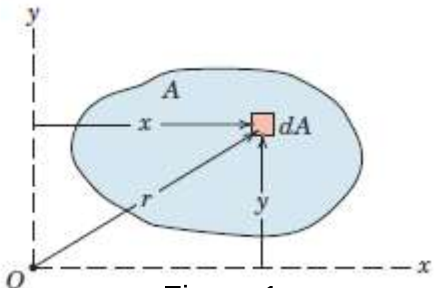


Figure 1

The sum of the second moments of all the elements of an area is defined as the moment of inertia of the area A, that is,

$$I_x = \int y^2 \, dA$$

$$I_y = \int x^2 \, dA$$

Where we carry out the integration over the entire area

The second moment of the element of area in Figure 1 with respect to an axis through O perpendicular to the plane of area is

$$dJ_o = r^2 dA = (x^2 + y^2) dA$$

The polar moment of inertia of the area is

$$J_o = \int (x^2 + y^2) \, dA = \int x^2 \, dA + \int y^2 \, dA = I_x + I_y$$

Thus, the polar moment of inertia of an area is equal to the sum of the rectangular moments of inertia with respect to any two perpendicular axes intersecting the polar axis. The second moment of an area has dimensions of length raised to the fourth power, L^4, and common units of in^4, ft^4, cm^4, and so on.

An element of area is inherently positive, since the square of the length of its moment arm is also positive, the second moment of an element of area is always a positive quantity. The moment of inertia of an area is the sum of the second moments of the elements of the area; consequently, it is always positive.

The Parallel-Axis Theorem for Areas

The parallel-axis theorem can be used to determine the moment of inertia of the area with respect to a parallel axis. The parallel-axis theorem (sometimes called the transfer formula) provides a convenient relationship between the moments of inertia of an area with respect to two parallel axes, one of which passes through the centroid of the area.

The parallel-axis theorem can be stated as follows: The moment of inertia of an area with respect to any axis is equal to the moment of Inertia with respect to a parallel axis through the centroid of the area plus the product of the area and the square of the distance between the two axes. Thus,

$$I_b = Ad^2 + I_c$$

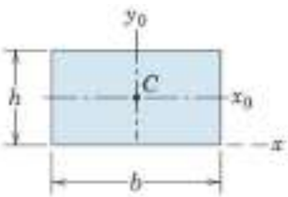
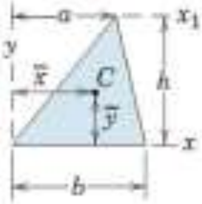
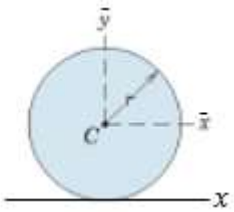
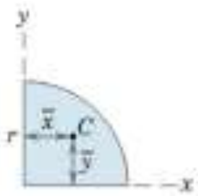
Where I_c is the second moment of the area with respect to an axis through the centroid parallel to the axis b, A is the area, and d is the distance between the two axes. Similarly,

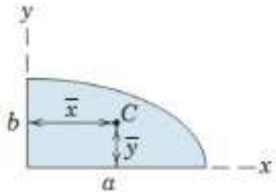
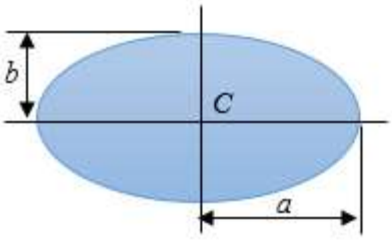
$$J_b = Ad^2 + J_c$$

Moments of Inertia of Composite Areas

A composite area consists of two or more simple areas, such as rectangles, triangles, and circles. The cross-sectional areas of standard structural elements, such as channels, I beams, and angles, are frequently included in composite areas. The moment of inertia of a composite area with respect to any axis is equal to the sum of the moments of inertia of its component areas with respect to the same axis. When an area, such as a hole, is removed from a larger area, its moment of inertia is subtracted from the moment of inertia of the larger area to obtain the net moment of inertia. Table 1 illustrates Moment of inertia of common geometric areas.

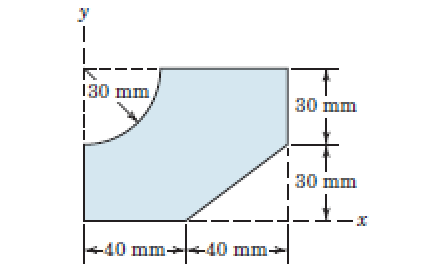
Table 1. Moment of inertia of common geometric areas.

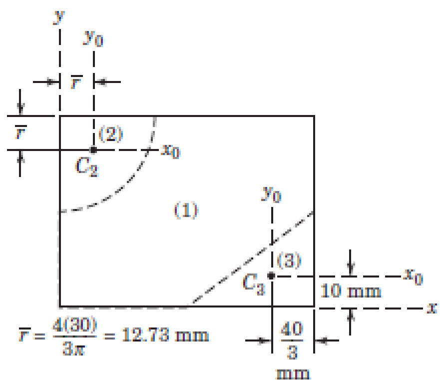
Geometric Area	Moment of Inertia
 Rectangle	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$
 Triangle	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$
 Circle	$\bar{I}_x = \bar{I}_y = \frac{\pi r^4}{4}$ $I_x = \frac{5\pi r^4}{4}$
 Quarter Circle	$\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4 = 0.0549 r^4$ $I_x = \frac{\pi r^4}{16} = 0.1963 r^4$

	Area of Elliptical Quadrant	$I_x = \frac{\pi ab^3}{16}$, $\bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) ab^3$ $I_y = \frac{\pi a^3 b}{16}$, $\bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) a^3 b$
	Ellipse	$\bar{I}_x = \frac{\pi ab^3}{4}$

Example 1. Determine the moments of inertia about the *x*- and *y*-axes for the shaded area.

Solution. The given area is subdivided into the three subareas shown a rectangular (1), a quarter-circular (2), and a triangular (3) area. Two of the subareas are “holes” with negative areas.





PART	A mm^2	d_x mm	d_y mm	Ad_x^2 mm^3	Ad_y^2 mm^3	$\bar{I}_x$ mm^4	$\bar{I}_y$ mm^4
1	$80(60)$	30	40	$4.32(10^6)$	$7.68(10^6)$	$\frac{1}{12}(80)(60)^3$	$\frac{1}{12}(60)(80)^3$
2	$\frac{1}{4}\pi(30)^2$	$(60-12.73)$	12.73	$-1.579(10^6)$	$-0.1146(10^6)$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$
3	$-\frac{1}{2}(40)(30)$	$\frac{30}{3}$	$\left(80-\frac{40}{3}\right)$	$-0.06(10^6)$	$-2.67(10^6)$	$\frac{1}{36}40(30)^3$	$\frac{1}{36}(30)(40)^3$
TOTALS	3490			$2.68(10^6)$	$4.90(10^6)$	$1.366(10^6)$	$2.46(10^6)$

$[I_x = \Sigma \bar{I}_x + \Sigma A d_x^2]$ $\qquad I_x = 1.366(10^6) + 2.68(10^6) = 4.05(10^6)\ \text{mm}^4$ *Ans.*

$[I_y = \Sigma \bar{I}_y + \Sigma A d_y^2]$ $\qquad I_y = 2.46(10^6) + 4.90(10^6) = 7.36(10^6)\ \text{mm}^4$ *Ans.*