Equal spacing Newton's Forward difference formula.
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Central Difference Notation.

❖ Equal spacing Newton's Forward difference formula:

Previous general formula is valid for any arbitrary spaced nodes,

if the x_i are equally spaced:

$$x_0, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad \ldots, \quad x_n = x_0 + nh$$

Then previous formula may be simplified .to get started, define

the first forward difference of g at x_i by:

$\Delta g_j = g_{j+1} - g_j$ And the second forward diff. of g_i

$$\Delta^2 g_j = \Delta g_{j+1} - \Delta g_j$$

Continuing in this way, the kth.

Forward diff. of g at x_i

$$\Delta^k g_j = \Delta^{k-1} g_{j+1} - \Delta^{k-1} g_j$$

Theorem: **the kth. Divided difference of an equally spaced (h) forward interpolation is given by**:

$$g[x_0, x_1, \ldots, x_k] = \frac{1}{k!\, \mathrm{h}^{\mathrm{k}}} \Delta^k g_0$$

Proof: we prove a above theorem by induction.

It is true for k=1 so that:

$$g[x_0, x_1] = \frac{g_1 - g_0}{x_1 - x_0} = \frac{1}{h}(g_1 - g_0) = \frac{1}{1!\, \mathrm{h}^1} \Delta g_0$$. **Now**:

$$g[x_0, x_1, \ldots, x_{k+1}] = \frac{g[x_1, x_2, \ldots, x_{k+1}] - g[x_0, x_2, \ldots, x_k]}{(k+1)h}$$

$$= \frac{1}{(k+1)h}\left[\frac{1}{k!\, h^k} \Delta^k g_1 - \frac{1}{k!\, \mathrm{h}^{\mathrm{k}}} \Delta^k g_0\right] = \frac{1}{(k+1)!\, h^{k+1}} \Delta^{k+1} g_0$$

Replacing (k+1) by k then:

$$g[x_0, x_1, \ldots, x_k] = \frac{1}{k!\,h^k} \Delta^k g_0$$

Next we write $f_n(x)$: first $x = x_0 + rh \Rightarrow r = \frac{x - x_0}{h}$

$$x - x_1 = (r-1)h$$

$$f_n(x) = g_0 + (x - x_0)\, g[x_0, x_1] + (x - x_0)(x - x_1) g[x_0, x_1, x_2] + \ldots$$
$$+ (x - x_0)\ldots(x - x_n)\, g[x_0, x_1, \ldots\ldots, x_n]$$

$$= g_0 + \frac{rh}{h}\Delta g_0 + \frac{rh(r-1)h}{2!\,h^2}\Delta^2 g_0 + \frac{rh(r-1)h(r-2)h}{3!\,h^3}\Delta^3 g_0 + \ldots$$

Simplifying get:

$$f_n(x) = g_0 + r\Delta g_0 + \frac{r(r-1)}{2!}\Delta^2 g_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 g_0 + \ldots + \frac{r(r-1)\ldots(r-n+1)}{n!}\Delta^n g_0$$

Or in compact form:

$$f_n(x) = \sum_{s=0}^{n} C_s^r \Delta^s g_0 \quad , C_s^r = \frac{r(r-1)(r-2)\ldots(r-s+1)}{s!}$$

Ex: given the following data, approximate the functional value at x = 3.75 using **Newton's forward interpolating** formula:

X	-1	0	1	2	3	4
Y	2	-1	1	3	5	2

Solution: equally spaced (h =1):

j	x_j	y_j	Δy_j	$\Delta^2 y_j$	$\Delta^3 y_j$	$\Delta^4 y_j$	$\Delta^5 y_j$
0	-1	2					
			-3				
1	0	-1		5			
			2		-5		
2	1	1		0		5	
			2		0		-10
3	2	3		0		-5	
			2		-5		
4	3	5		-5			
			-3				
5	4	2					

$$f_5(x) = \sum_{s=0}^{5} C_s^r \Delta^s y_0 \quad , \quad r = \frac{x - x_0}{h} = \frac{3.75 - (-1)}{1} = 4.75$$

$$f_5(3.75) = 2 + 4.75(-3) + \frac{5(4.75)(4.75-1)}{2!} + \frac{-5(4.75)(4.75-1)(4.75-2)}{3!}$$
$$+ \frac{5(4.75)(4.75-1)(4.75-2)(4.75-3)}{4!} + \frac{-10(4.75)(4.75-1)\ldots(4.75-4)}{5!}$$
$$= 3.96215 \quad \text{(check)}$$

❖ Equal spacing: Newton`s backward Difference formula:

In stead of forward –sloping differences. We may also employ backward –sloping differences .the difference table remains the same as before (same numbers in the same positions) except for the running subscript j. we define the **1st. backward difference** of g at x_j by:

$\nabla g_j = g_j - g_{j-1}$ and the 2nd. backward diff.

$\nabla^2 g_j = \nabla g_j - \nabla g_{j-1}$ and continuing for the kth diff

$\nabla^k g_j = \nabla^{k-1} g_j - \nabla^{k-1} g_{j-1}$

And in a similar procedure the **Newton's backward** diff. interpolating formula will be:

$$\mathrm{f_n(x)} = \sum_{s=0}^{n} C_s^{r+s-1} \nabla^s g_0 \quad , \ \mathrm{r} = \frac{\mathrm{x - x_0}}{\mathrm{h}}$$

$$\mathrm{f_n(x)} = g_0 + r\nabla g_0 + \frac{r(r+1)}{2!}\nabla^2 g_0 + \frac{r(r+1)(r+2)}{3!}\nabla^3 g_0 + \ldots + \frac{r(r+1)\ldots(r+n-1)}{n!}\nabla^n g_0$$

Ex: compute $j_0(1.72)$ from the values of **Bessel's fn**.

x	1.7	1.8	1.9	2
$j_0(x)$	0.397985	0.339986	0.281819	0.223891

Using **(a)** Newton's **forward** **(b)** Newton's **backward** formula:

Solution:

Equal spacing **h = 0.1**

$j_{forward}$	j_{back}	x_j	$j_0(x_j)$	1^{st} diff.	2^{nd} diff	3^{rd} diff.
0	-3	1.7	**0.397985**			
				-0.057999		
1	-2	1.8	0.339986		**-0.000168**	
				-0.058167		**0.000409**
2	-1	1.9	0.281819		**0.00024**	
				-0.057928		
3	0	2.0	**0.223891**			

(a) $\mathrm{r} = \frac{\mathrm{x - x_0}}{\mathrm{h}} = \frac{1.72 - 1.7}{0.1} = 0.2$

$$j_0(1.72) = 0.397985 + 0.2(-0.057999) + \frac{0.2(0.2-1)(-0.000168)}{2!}$$

$$+ \frac{0.2(0.2-1)(0.2-2)(0.000409)}{3!} = 0.386418$$

(b) $r = \dfrac{x - x_0}{h} = \dfrac{1.72 - 2}{0.1} = -2.8$

$$j_0(1.72) = 0.223891 - 2.8(-0.057928) + \frac{-2.8(-2.8+1)(0.00024)}{2!}$$
$$+ \frac{-2.8(-2.8+1)(-2.8+2)(-0.000409)}{3!} = 0.386418$$

Ex: show that (a) $\nabla^4 g_j = g_j - 4g_{j-1} + 6g_{j-2} - 4g_{j-3} + g_{j-4}$

(b) $\Delta^4 g_j = g_{j+4} - 4g_{j+3} + 6g_{j+2} - 4g_{j+1} + g_j$

(a) Start with: $\nabla g_j = g_j - g_{j-1}$

$$\nabla^2 g_j = \nabla(g_j - g_{j-1}) = \nabla g_j - \nabla g_{j-1} = g_j - g_{j-1} - (g_{j-1} - g_{j-2})$$
$$= g_j - 2g_{j-1} + g_{j-2}$$

$$\nabla^3 g_j = \nabla(g_j - 2g_{j-1} + g_{j-2}) = \nabla g_j - 2\nabla g_{j-1} + \nabla g_{j-2}$$
$$= g_j - g_{j-1} - 2(g_{j-1} - g_{j-2}) + (g_{j-2} - g_{j-3})$$
$$= g_j - 3g_{j-1} + 3g_{j-2} - g_{j-3}$$

$$\nabla^4 g_j = \nabla(g_j - 3g_{j-1} + 3g_{j-2} - g_{j-3}) = \nabla g_j - 3\nabla g_{j-1} + 3\nabla g_{j-2} - \nabla g_{j-4}$$
$$= g_j - g_{j-1} - 3(g_{j-1} - g_{j-2}) + 3(g_{j-2} - g_{j-3}) - g_{j-3} + g_{j-4}$$
$$= g_j - 4g_{j-1} + 6g_{j-2} - 4g_{j-3} + g_{j-4}$$

(b) H.w. (similarly except that $\Delta g_j = g_{j+1} - g_j$ forward

❖ Central Difference Notation:

There is a 3rd. notation for difference which is useful in numerical differentiation and connection with differential eqs. This is the central difference notation given by at x_j

$\delta g_j = g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}}$ The difference table

x_{-1}	g_{-1}			
		$\delta g_{-\frac{1}{2}}$		
x_0	g_0		$\delta^2 g_0$	
		$\delta g_{\frac{1}{2}}$		$\delta^3 g_{\frac{1}{2}}$
x_1	g_1		$\delta^2 g_1$	
		$\delta g_{\frac{3}{2}}$		
x_2	g_2			

Relationship between difference operators:

Forward difference $\Delta g_j = g_{j+1} - g_j$

Backward difference $\nabla g_j = g_j - g_{j-1}$

Central difference $\delta g_j = g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}}$

In fact these are related:

Define the shift operator **E** as g(x+h) = **E** g(x) [shift one space h .Hence: $g_{j+1} = E\ g_j$ .now the shift operator E will be applied to forward, backward &central diff. and in general $g_{j+n} = E^n\ g_j$.

a) **For forward diff.**

$$\Delta g_j = g_{j+1} - g_j = E\, g_j - g_j = (E-1)g_j \quad \therefore \Delta = E - 1$$

And in general

$$\Delta^n g_j = (E-1)^n g_j \quad \text{or} \quad \Delta^n = (\mathrm{E}-1)^n$$

b) **For backward diff.**

$$\nabla g_j = g_j - g_{j-1}, \textit{and} \text{ since } g_j = E g_{j-1} \Rightarrow g_{j-1} = \frac{1}{E} g_j$$

$$\therefore \nabla g_j = (1-\frac{1}{E}) g_j \text{ or simply } \nabla = \frac{\mathrm{E}-1}{\mathrm{E}}$$

Therefore the relationship between Δ and ∇ is :

$$\nabla = \frac{\Delta}{\Delta+1} \quad \& \quad \nabla^n = \frac{\Delta^n}{(\Delta+1)^n}$$

c) **For central diff.**

$$\delta g_j = g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}} = (\mathrm{E}^{1/2} - E^{-1/2})\, g_j$$

$$\therefore \delta = \mathrm{E}^{1/2} - E^{-1/2} = (\Delta+1)^{1/2} - (\Delta+1)^{-1/2}$$

$$or \quad \delta = \frac{\Delta}{(\Delta+1)^{1/2}} \quad \text{and} \quad \delta^n = \frac{\Delta^n}{(\Delta+1)^{n/2}}$$