STABILITY ANALYSIS

Introduction

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system?

Stability may be defined as the ability of a system to restore its equilibrium position when disturbed or a system which has a bounded response for a bounded output.

Stability Analysis in the Complex Plane

The stability of a linear closed-loop system can be determined from the location of the closed-loop poles in the s-plane. If any of these poles lie in the *Right -Half of the s-plane* (RHS), (either the poles are real or complex as shown in Fig. 1.) then with increasing time, they give rise to the dominant mode, and the transient response increases monotonically or oscillate with increasing amplitude. Either of these systems represents an unstable system.

For such a system, as soon as the power is turned on, the output may increase with time. If no saturation takes place in the system and no mechanical stop is provided, then the system may eventually be damaged and fail, since the response of a real physical system cannot increase indefinitely.

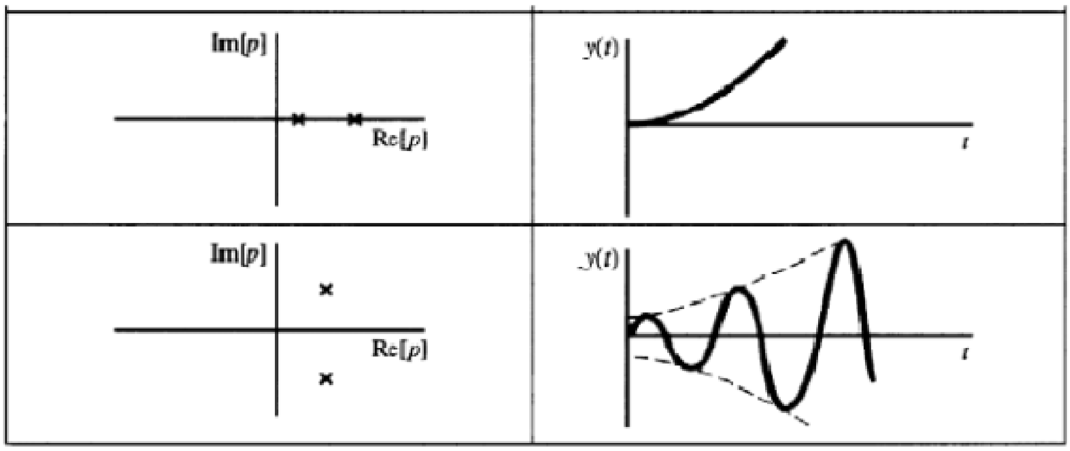

Fig. 1. Poles located in RHS gives unstable response

Consider a simple feedback system shown in Fig. 2.

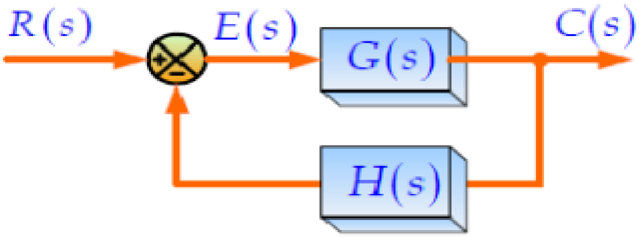

Fig. 2, closed-loop control system

The overall T.F. is given as:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The characteristic equation is of the above system is $1 + G(s)H(s) = 0$

The roots of the characteristic equation are called closed loop poles. The location of such roots or poles on the s-plane will indicate the condition of stability as shown in Fig. 3.

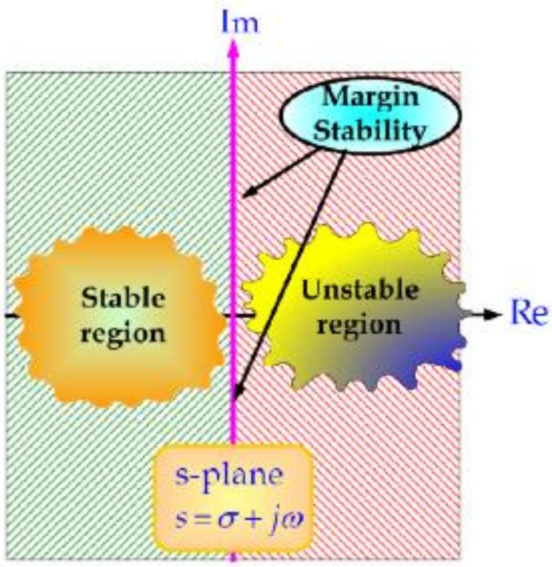

Fig. 3. Stability condition based on the location of the closed loop poles

Routh Stability criterion (***Two Necessary But Insufficient Conditions)***

Since most liner closed loop systems have closed loop transfer function of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)}$$

Where the a's and b's are constant and $m \leq n$, we must first factor the polynomial A(s) in order to find the closed loop poles.

The characteristic equation of the simple feedback system can be written as a polynomial:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$$

There are two necessary but insufficient conditions for the roots of the characteristic equation to lie in Left Hand Side (LHS) of the S-plane (i.e., stable region)

1. All the coefficients a_n, a_{n-1}, a_{n-2}, ..., a_1 *and* a_0 should have the same sign.
2. None of the coefficients vanish (All coefficients of the polynomial should exist).

By this way we judge the ***absolute stability*** of the system (stable or unstable).

Example #1

Given the characteristic equation,

$$a(s) = s^6 + 4s^5 + 3s^4 - 2s^3 + s^2 + 4s + 4$$

Is the system described by this characteristic equation stable?

One coefficient (-2) is negative. Therefore, the system **does not** satisfy the necessary condition for stability.

Example #2

Given the characteristic equation,

$$a(s) = s^6 + 4s^5 + 3s^4 + s^2 + 4s + 4$$

Is the system described by this characteristic equation stable? The term s^3 is missing. Therefore, the system **does not** satisfy the necessary condition for stability.

<u>Hurwitz Stability criterion</u> (Necessary and Sufficient Condition)

Arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$\begin{array}{l|lllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}$$

The process of forming rows continues until we run out of elements. (The total number of rows is n + 1.) The coefficients b_1, b_2, b_3, and so on, are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

The evaluation of the b's is continued until the remaining ones are all zero.

The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, e's, and so on. That is,

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

Note that in developing the array an entire row may be divided or multiplied by a ***<u>positive number</u>*** in order to simplify the subsequent numerical calculation without altering the stability conclusion.

Routh-Hurwitz stability criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

It should be noted that the exact values of the terms in the first column need not be known; instead, only the signs are needed.

The necessary and sufficient condition that all roots of the characteristic equation lie in the left-half **s** plane is that:

a) All the coefficients of the characteristic equation be positive, and
b) All terms in the first column of the array have positive signs.

Example #3

Consider the following polynomial

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Let us follow the procedure just presented and construct the array of coefficients. (The first two rows can be obtained directly from the given polynomial. The remaining terms are obtained from these. If any coefficients are missing, they may be replaced by zeros in the array.)

$$\begin{array}{llll||llll}
s^4 & 1 & 3 & 5 & s^4 & 1 & 3 & 5 \\
s^3 & 2 & 4 & 0 & s^3 & \cancel{2} & \cancel{4} & \cancel{0} \\
 & & & & & 1 & 2 & 0 \\
s^2 & 1 & 5 & & s^2 & 1 & 5 & \\
s^1 & -6 & & & s^1 & -3 & & \\
s^0 & 5 & & & s^0 & 5 & &
\end{array}$$

The second row is divided by 2.

Example #4

Check whether this system is stable or not.

$$H(s) = \frac{2\left(s^2 + 2s + 25\right)}{s^5 + s^4 + 3s^3 + 9s^2 + 16s + 10}$$

The characteristic equation is:

$$s^5 + s^4 + 3s^3 + 9s^2 + 16s + 10 = 0$$

Construct the Routh array

s^5	1	3	16	0
s^4	1	9	10	0
s^3	$\frac{1\times3-9\times1}{1}=-6$	$\frac{1\times16-10\times1}{1}=6$	0	0
s^2	$\frac{-6\times9-6\times1}{-6}=10$	$\frac{-6\times10-0\times1}{-6}=10$	0	0
s^1	$\frac{10\times6-10\times(-6)}{10}=12$	$\frac{10\times0-0\times(-6)}{10}=0$	0	0
s^0	$\frac{12\times10-0\times(10)}{12}=10$	0	0	0

There are 2 sign changes. There are 2 poles on the right half of the S-plane. Therefore, the system is unstable.

Special Cases:

1- If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number **ε** and the rest of the array is evaluated.

Example #5

Consider the following equation:

$$s^3 + 2s^2 + s + 2 = 0$$

The array of coefficients is

$$\begin{array}{lll} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx \epsilon & \\ s^0 & 2 & \end{array}$$

If the sign of the coefficient above the zero *(ε)* is the same as that below it, it indicates that there is a pair of imaginary roots. Actually, the above equation has two roots at

$s = \pm j$.

If, however, the sign of the coefficient above the zero *(ε)* is opposite that below it, it indicates that there is one sign change. For example, for the equation

$$s^3 - 3s + 2$$

The array of coefficients is

One sign change:

$$\begin{array}{lcc} s^3 & 1 & -3 \\ s^2 & 0 \approx \epsilon & 2 \\ s^1 & -3 - \dfrac{2}{\epsilon} & \\ s^0 & 2 & \end{array}$$

One sign change:

There are two sign changes of the coefficients in the first column. This agrees with the correct result indicated by the factored form of the polynomial equation.

$$s^3 - 3s + 2 = (s - 1)^2(s + 2) = 0$$

2- If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the ***s*** plane, that is, two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.

In such a case, the evaluation of the rest of the array can be continued by forming an ***auxiliary polynomial*** with the coefficients of the last row and by using the coefficients of the ***derivative of this auxiliary*** polynomial in the next row. Such roots with equal magnitudes and lying radially opposite in the ***s*** plane can be found by solving the auxiliary polynomial, which is always ***even***.

Example #6

Consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficients is

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \quad \leftarrow \text{Auxiliary polynomial } P(s) \\ s^3 & 0 & 0 & \end{array}$$

The terms in the s^3 row are all zero. (Note that such a case occurs only in an odd numbered row.) The auxiliary polynomial is then formed from the coefficients of the s^4 row. The auxiliary polynomial ***P(s)*** is

$$P(s) = 2s^4 + 48s^2 - 50$$

$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

$$\begin{array}{llll}
s^5 & 1 & 24 & -25 \\
s^4 & 2 & 48 & -50 \\
s^3 & 8 & 96 & \leftarrow \text{Coefficients of } dP(s)/ds \\
s^2 & 24 & -50 & \\
s^1 & 112.7 & 0 & \\
s^0 & -50 & &
\end{array}$$

We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part. By solving for roots of the auxiliary polynomial equation,

$$2s^4 + 48s^2 - 50 = 0$$

$$s^2 = 1, \qquad s^2 = -25$$

$$s = \pm 1, \qquad s = \pm j5$$