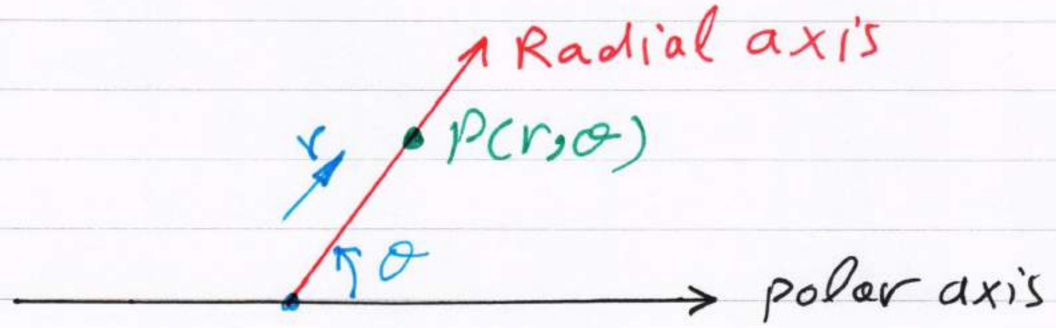
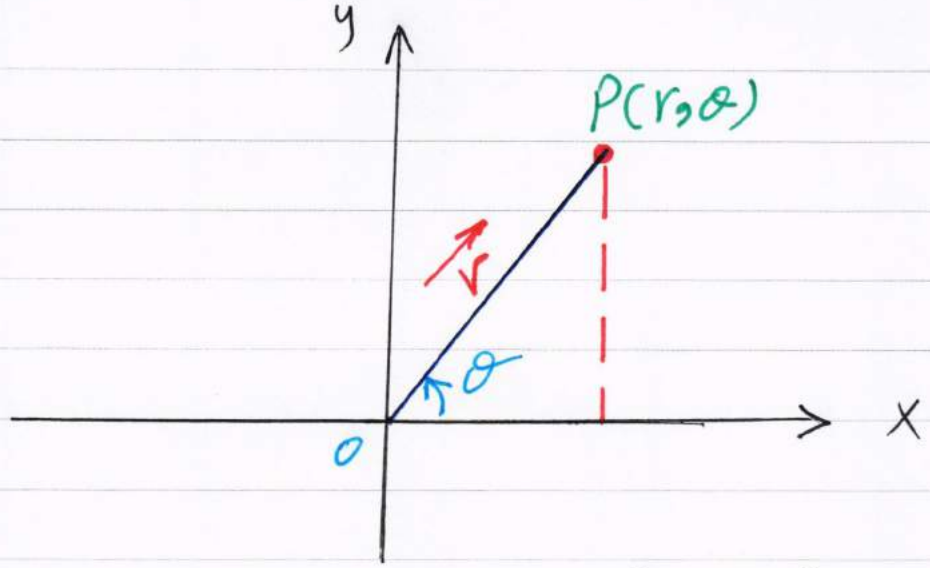
Double Integrals in polar coordinates:

A polar coordinate system consist of a fixed point O called the origin or pole and a line segment Starting from the pole called the polar axis.



r - radial coordinate
θ - polar angle

polar coordinates of a point P is written as (r,θ) where r is the distance of P from the pole and θ is the angle measured from the polar axis to the line OP (radial axis).

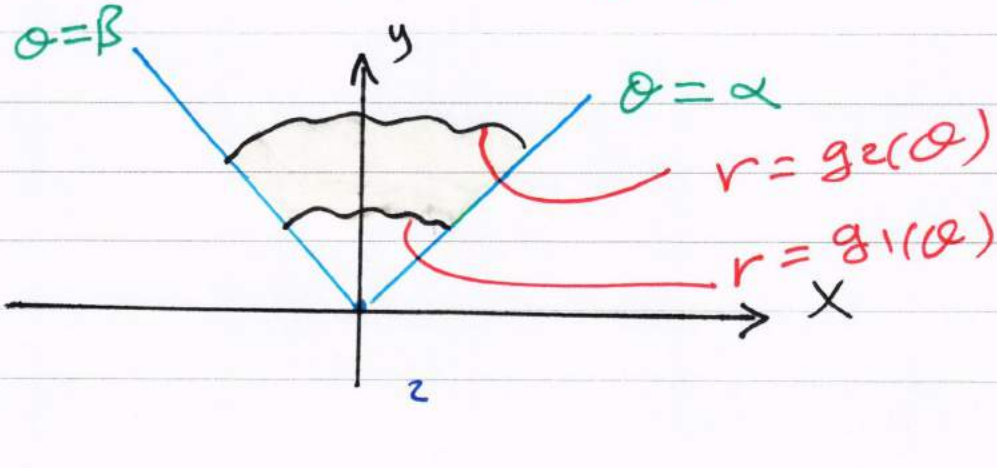


$x = r\cos\theta$ $\qquad$ $x^2 + y^2 = r^2$

$y = r\sin\theta$ $\qquad$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Let R be a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and curves $r = g_1(\theta)$ and $r = g_2(\theta)$. If $f(r, \theta)$ is continuous on R, then:

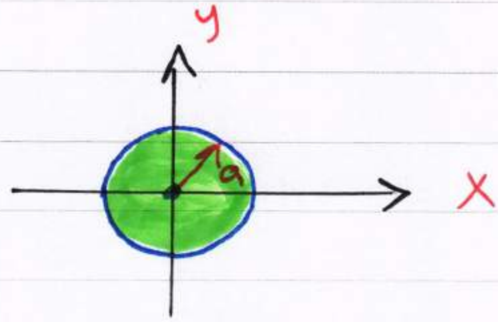
$$\iint_R f(x, y)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta$$



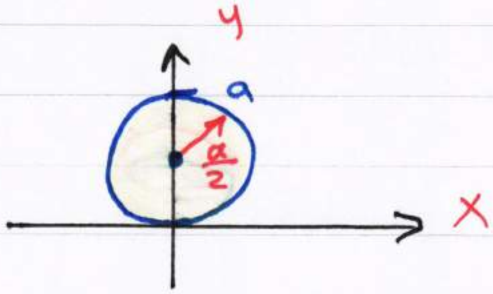
Standared polar form

I- Famlies of circle

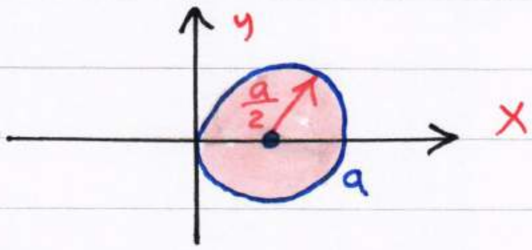
① $r = a$



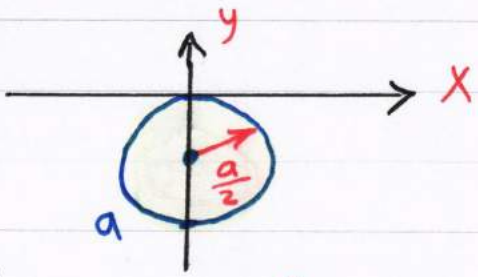
② $r = a\sin\theta$



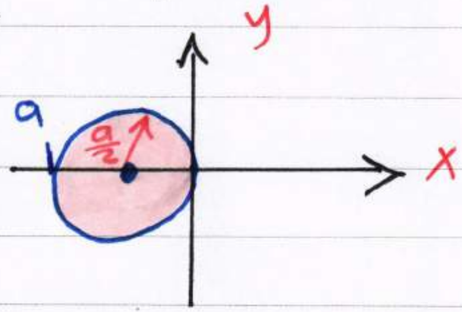
③ $r = a\cos\theta$



④ $r = -a\sin\theta$

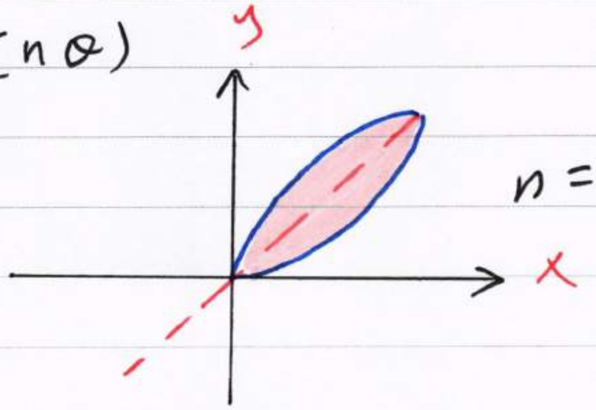


⑤ $r = -a\cos\theta$

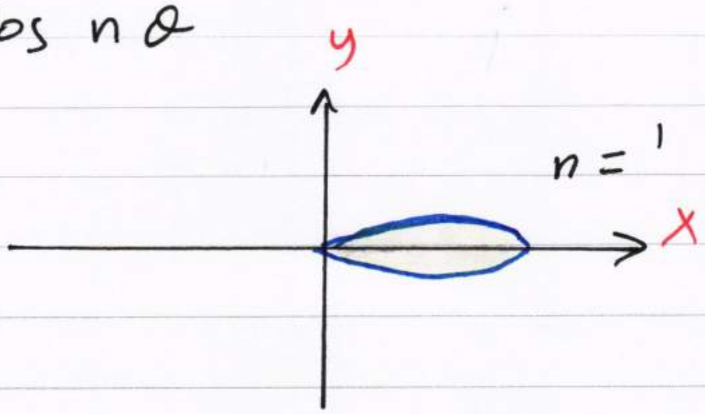


II- Rose curve

① $r = a\sin(n\theta)$

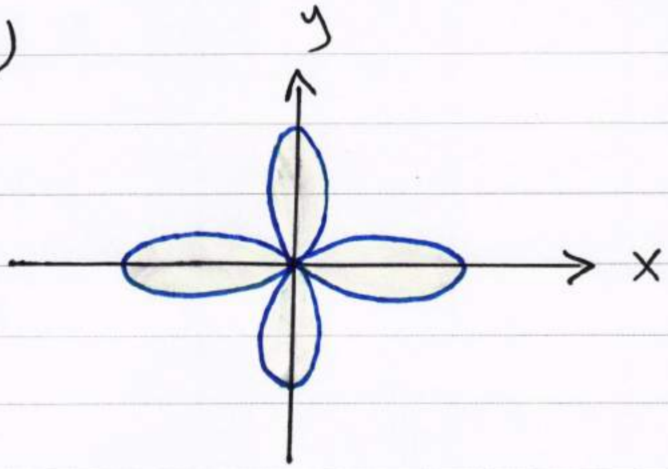


② $r = a\cos n\theta$

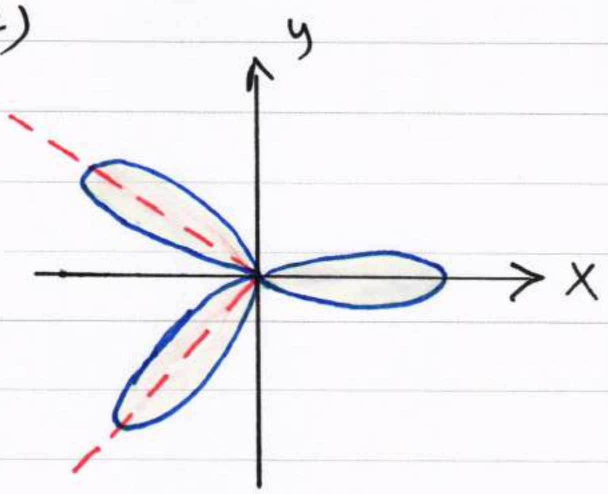


اذا كان n عدد زوجي نضرب $\times 2$.

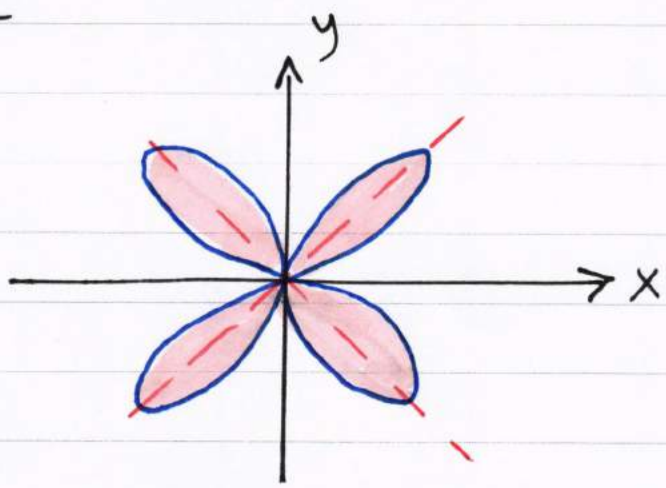
$r = a\cos(2\theta)$



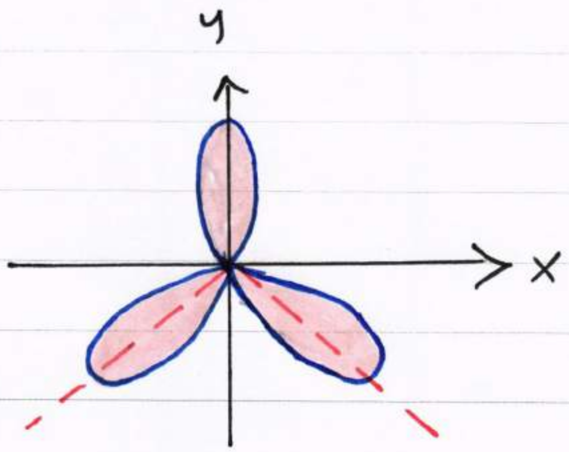
$r = a\cos(3\theta)$



$r = a\sin 2\theta$

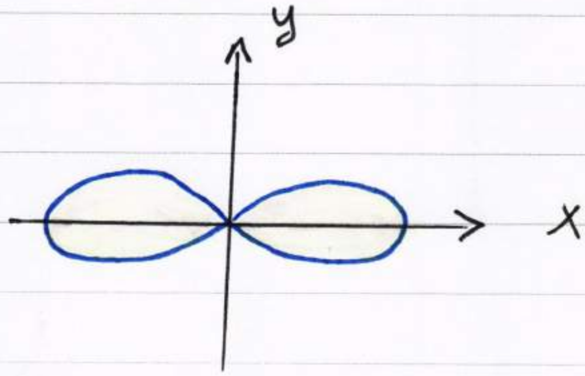


$r = a\sin(3\theta)$

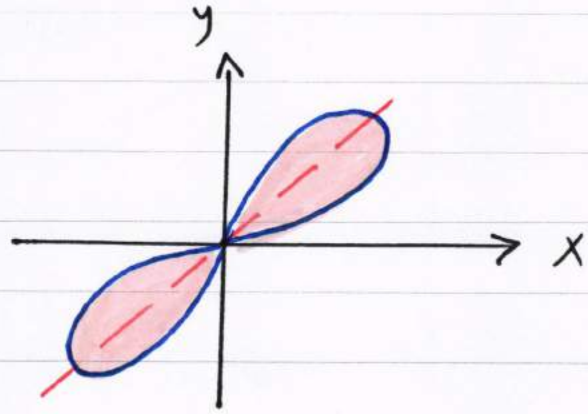


III - Families of Lemnscate

① $r^2 = a^2\cos(2\theta)$



② $r^2 = a^2\sin(2\theta)$



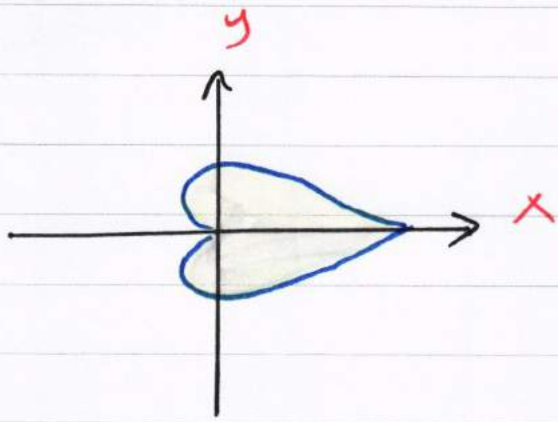
IV - Families of Cardioid and Lima cons

$r = a \mp b\cos\theta$
$r = a \mp b\sin\theta$

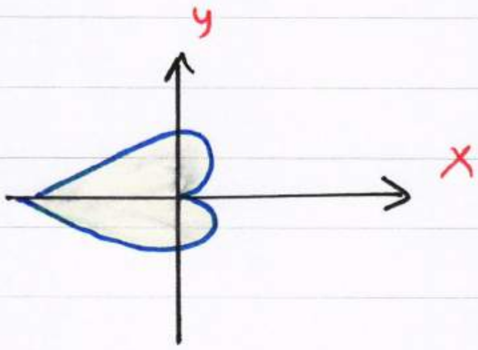
polar curve in which $a > 0$ and $b > 0$

when $a = b$ or $\frac{a}{b} = 1$ Lima cons is called Cardioid because of its the heart shaped appearance so:

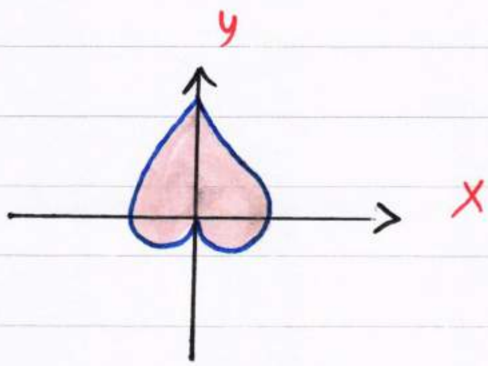
$r = a(1+\cos\theta)$



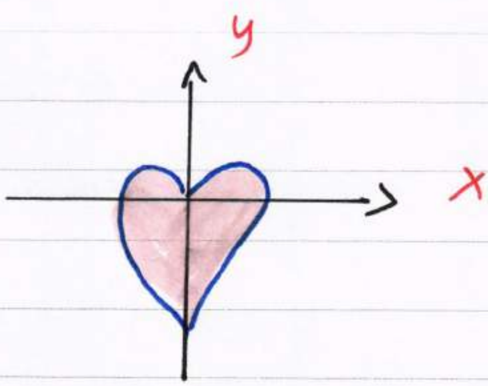
$r = a(1-\cos\theta)$



$r = a(1+\sin\theta)$

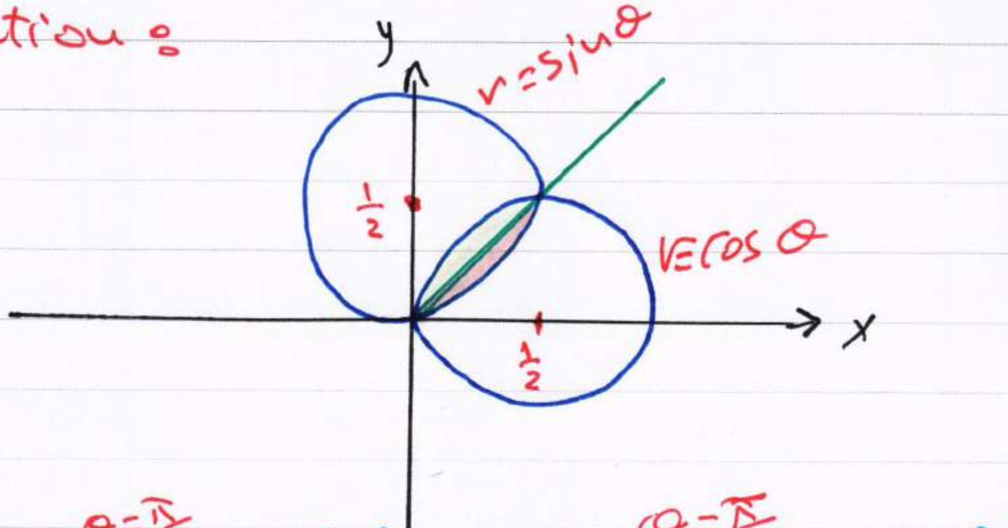


$r = a(1-\sin\theta)$



EX1: Find the area of the region bounded by the two circules $r=\cos(\theta)$ and $r=\sin\theta$.

solution:



$$A=\int_{\theta=0}^{\theta=\frac{\pi}{4}}\int_{r=0}^{r=\sin\theta} r\,dr\,d\theta+\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=\cos\theta} r\,dr\,d\theta$$

وبما انه المساحة متناظرة ممكن ايجاد وحدة وضربها في 2 لايجاد المساحة الكلية.

$$A=2\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sin\theta} r\,dr\,d\theta=2\int_{0}^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_{0}^{\sin\theta}d\theta$$

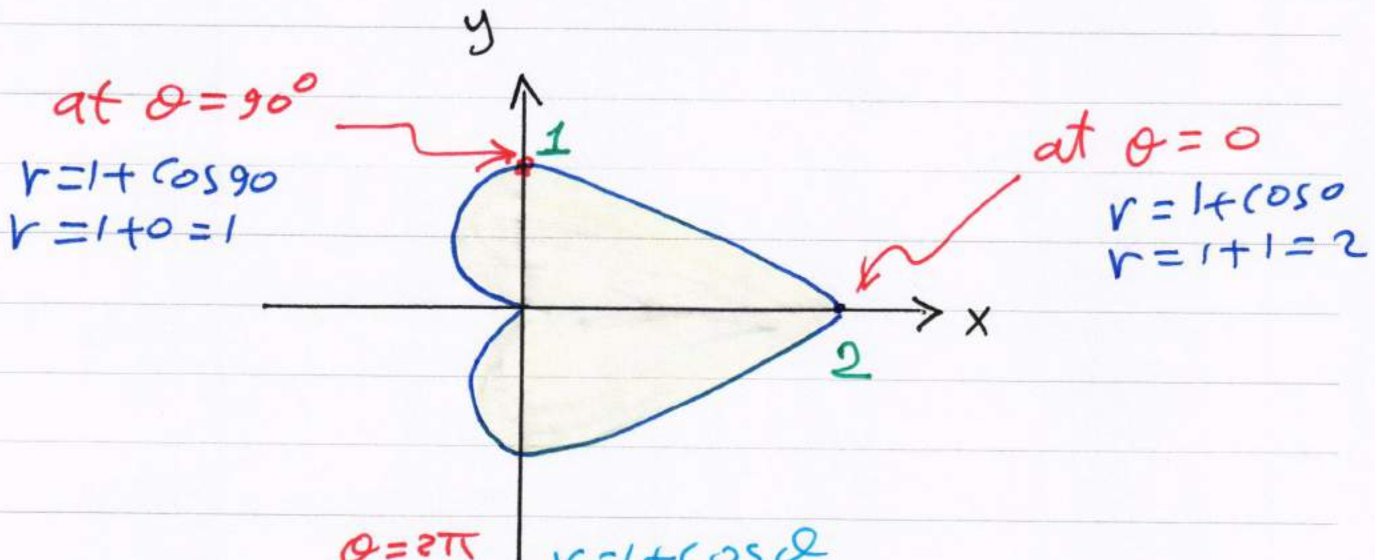
$$=\int_{0}^{\pi/4}\sin^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}(1-\cos(2\theta))\,d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{8} - \frac{1}{4} = \frac{\pi-2}{8}$$

Ex2: Find the area inside the cardioid $r = 1 + \cos\theta$.



$$\iint_R dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1+\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^{1+\cos\theta} d\theta = \int_0^{2\pi} \frac{(1+\cos\theta)^2}{2} d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} + \cos\theta + \frac{\cos^2\theta}{2} \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} + \cos\theta + \frac{1}{2} \frac{1}{2} (1 + \cos 2\theta) \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} + \cos\theta + \frac{1}{4} + \frac{1}{4} \cos 2\theta \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{3}{4} + \cos\theta + \frac{1}{4} \cos 2\theta \right] d\theta$$

$$= \left[\frac{3}{4} \theta + \sin\theta + \frac{1}{8} \sin 2\theta \right]_0^{2\pi}$$

$$= \left(\frac{3}{4} 2\pi + 0 + 0 \right) - (0 + 0 + 0) = \frac{3}{2} \pi$$

EX3: Find the area of the region insid the circle $(x-1)^2+y^2=1$ and outsid circle $x^2+y^2=1$.

solution:

$$(x-1)^2+y^2=1 \longrightarrow x^2-2x+1+y^2=1$$

$$x^2+y^2=2x \longrightarrow r^2=2r\cos\theta$$

$$\therefore r_2=2\cos\theta$$

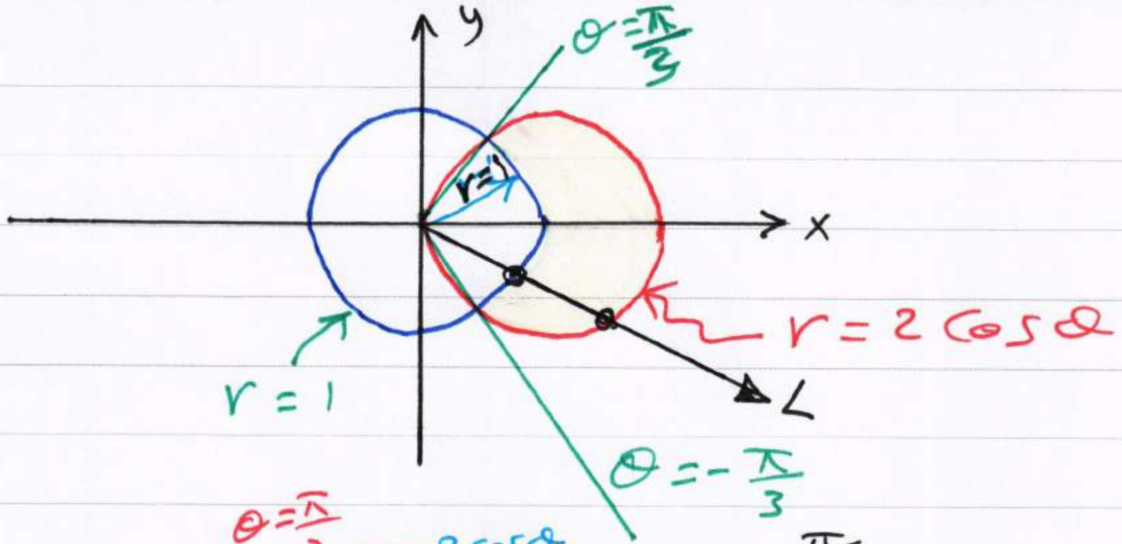
$$x^2+y^2=1 \longrightarrow r^2=1$$

$$\therefore r_1=1$$

$r_1=r_2$ لايجاد زوايا التقاطع

$$1=2\cos\theta \longrightarrow \theta=\cos^{-1}\frac{1}{2}$$

$$\theta=\frac{\pi}{3}, \quad -\frac{\pi}{3}$$



$$\iint_R dA = \int_{\theta=-\frac{\pi}{3}}^{\theta=\frac{\pi}{3}} \int_{r=1}^{r=2\cos\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{r^2}{2}\right]_1^{2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{(2\cos\theta)^2 - (1)^2}{2}\right] d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{4\cos^2\theta - 1}{2}\right] d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[2\cos^2\theta - \frac{1}{2}\right] d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\cancel{2}\,\frac{1}{\cancel{2}}(1+\cos 2\theta) - \frac{1}{2}\right] d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{1}{2} + \cos 2\theta \right] d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{2}\sin 2\theta \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \left[\left(\frac{1}{2}\frac{\pi}{3} + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) \right) - \left(-\frac{1}{2}\frac{\pi}{3} + \frac{1}{2}\sin\left(\frac{-2\pi}{3}\right) \right) \right]$$

$$= \frac{\pi}{6} + \frac{1}{2}\frac{\sqrt{3}}{2} + \frac{\pi}{6} + \frac{1}{2}\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Ex4: Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardiod $r = 1 + \cos\theta$ and outside the circle $r = 1$.

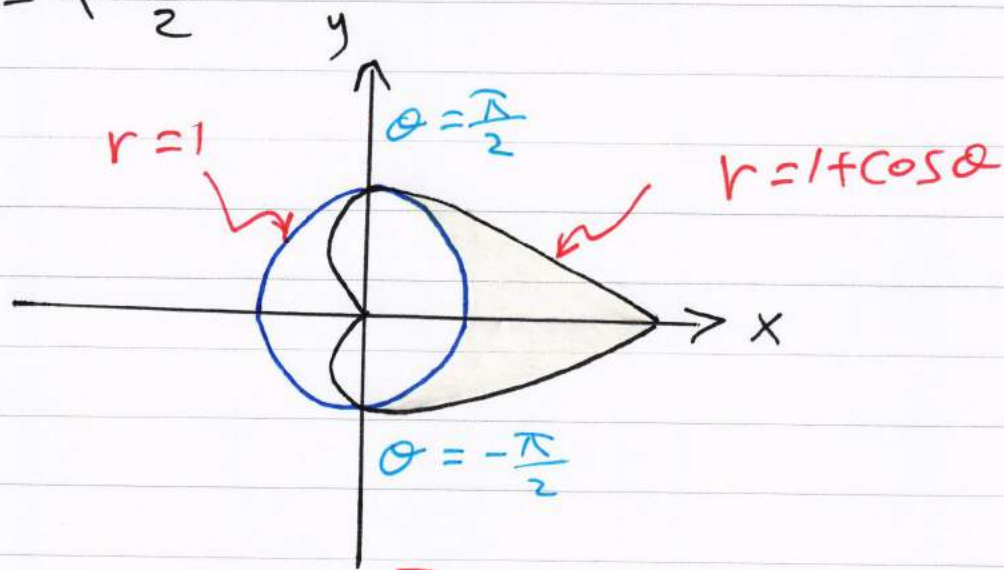
Solution:

$r_1 = 1 , \quad r_2 = 1 + \cos\theta$

$r_1 = r_2$

$1 = 1 + \cos\theta \longrightarrow \cos\theta = 0$

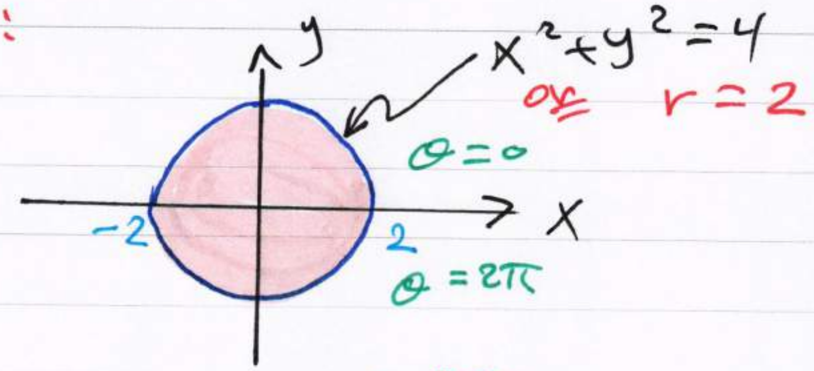
$\therefore \theta = \mp \frac{\pi}{2}$



$$\iint_R f(r,\theta)\, dA = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta$$

Ex 5 : Evaluate $\iint_R (x^2+y^2+1)\,dA$ where R is the region inside the circle $x^2+y^2=4$.

solution :



$$\iint_R (x^2+y^2+1)\,dA = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} (r^2+1)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (r^3+r)\,dr\,d\theta = \int_0^{2\pi} \left(\frac{r^4}{4}+\frac{r^2}{2}\right)\Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \left[\left(\frac{(2)^4}{4}+\frac{(2)^2}{2}\right)-0\right] d\theta$$

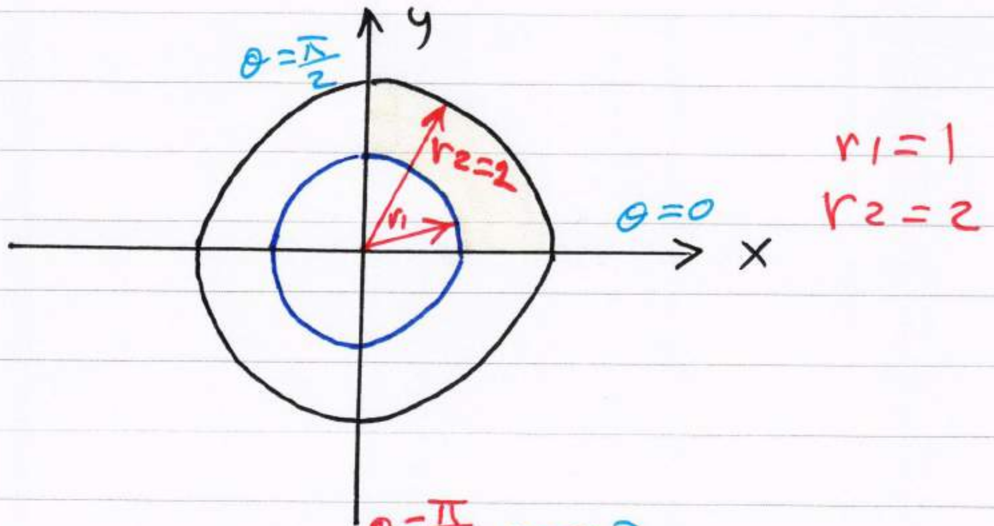
$$= \int_0^{2\pi} (4+2)\,d\theta = \int_0^{2\pi} 6\,d\theta = 6\theta\Big|_0^{2\pi}$$

$$= 12\pi$$

Ex6: Using polar double integral, evaluate

$\iint_R (x+y)\,dA$ whre R is the region

in the first qudrant bounded between the circles $x^2+y^2=1$ and $x^2+y^2=4$.



$$\iint_R (x+y)\,dA = \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=1}^{r=2} (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_1^2 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_1^2 (\cos\theta+\sin\theta)\,d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{(2^3 - 1^3)}{3} (\cos\theta + \sin\theta)\, d\theta$$

$$= \frac{7}{3} \int_{0}^{\frac{\pi}{2}} (\cos\theta + \sin\theta)\, d\theta$$

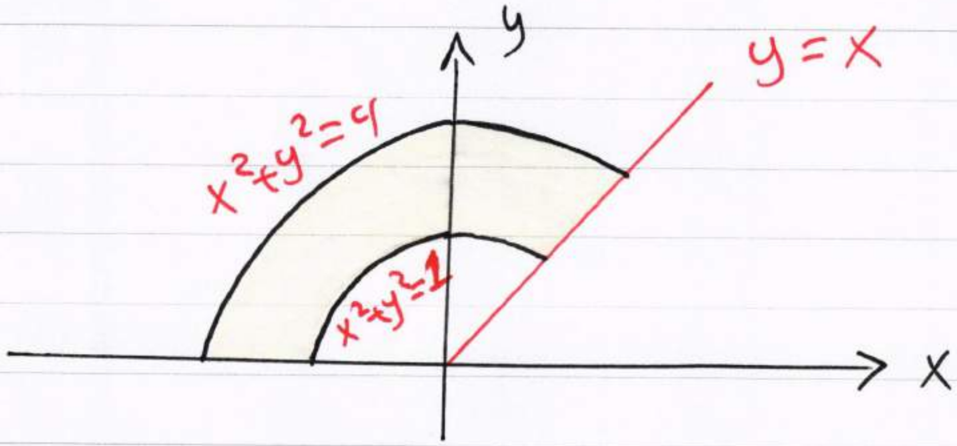
$$= \frac{7}{3} \left[\sin\theta - \cos\theta \right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{7}{3} \left[(\sin 90 - \cos 90) - (\sin 0 - \cos 0) \right]$$

$$= \frac{7}{3} \left[1 - 0 - 0 - (-1) \right]$$

$$= \frac{14}{3}$$

Ex 7: consider the region R shown below which is enclosed by $x^2+y^2=1$, $x^2+y^2=4$, $y=x$ and x-axis.



Fill in the missing limits of integration:

$$\iint_R f(x,y)\,dA = \int_{\square}^{\square}\int_{\square}^{\square} f(r,\theta)\, r\,dr\,d\theta$$

solution:

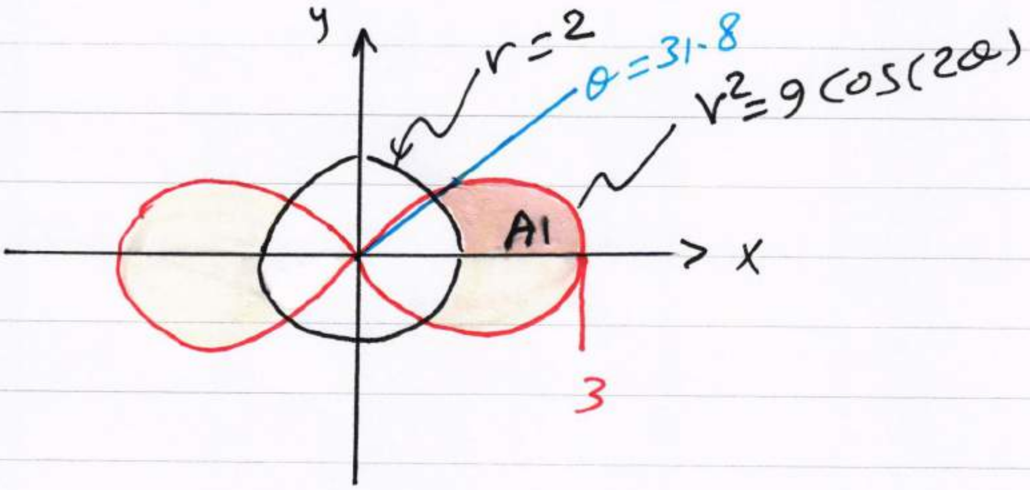
$$\iint_R f(x,y)\,dA = \int_{\theta=\frac{\pi}{4}}^{\theta=\pi}\int_{r=1}^{r=2} f(r,\theta)\, r\,dr\,d\theta$$

EX 8: Find the area of the given region R by calculating $\iint_R r\,dr\,d\theta$ where R is the region outside the circle $r=2$ and inside the lemniscate $r^2=9\cos(2\theta)$

solution: ايجاد زاوية نقطة التقاطع

$$r_1^2 = 4 \quad , r_2^2 = 9\cos(2\theta)$$

$$4 = 9\cos(2\theta) \longrightarrow \theta = \frac{1}{2}\cos^{-1}\left(\frac{4}{9}\right) = 31.8^\circ$$



$$r^2 = 9\cos(2\theta)$$
at $\theta = 0$
$$r^2 = 9\cos(0)$$
$$r = 3$$

$$A = 4A1$$

$A = 4A_1$

$31.8^{\circ} = 0.555$ rad

$$A = 4\int_{\theta=0}^{\theta=0.555}\int_{r=2}^{r=\sqrt{9\cos 2\theta}} r\,dr\,d\theta = 4\int_{\theta=0}^{\theta=0.555}\left[\frac{r^2}{2}\right]_2^{\sqrt{9\cos 2\theta}} d\theta$$

$$= 2\int_0^{0.555}\left[9\cos(2\theta) - 4\right] d\theta$$

$$= 2\left[\frac{9}{2}\sin(2\theta) - 4\theta\right]_0^{0.555}$$

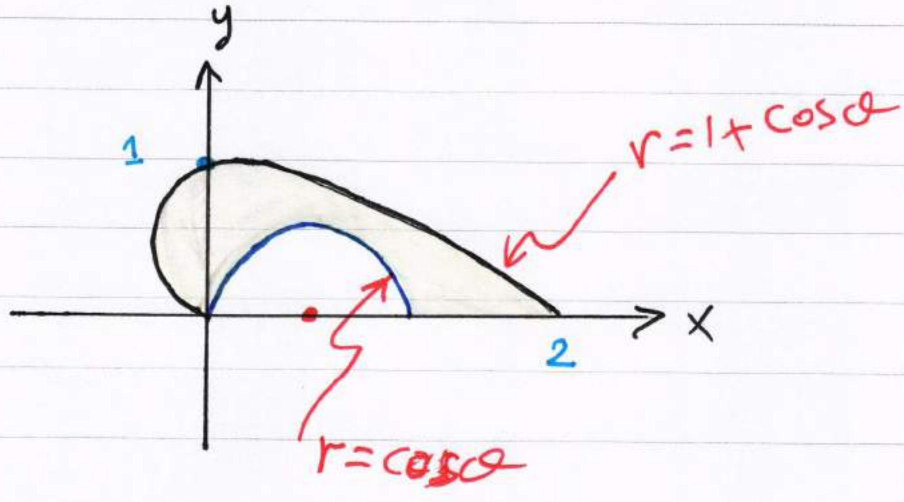
$$= 2\left[\left(\frac{9}{2}\sin(2\times 0.555) - 4\times 0.555\right) - (0-0)\right]$$

$$= 2\left[4.03 - 2.22\right]$$

$$= 3.62$$

Ex 9: Find the area of the top half of the region inside the cardiod $r = 1 + \cos\theta$ and outsid the circle $r = \cos\theta$.

solution:



The top half of the circle $r = \cos\theta$ is swept out as θ goes from 0 to $\frac{\pi}{2}$. The area of this region is:

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=\cos\theta} r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_0^{\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{4}\left[\frac{\pi}{2} + \frac{1}{2}\cancelto{0}{\sin\pi} - 0 - 0\right]$$

$$= \frac{\pi}{8}$$

The top half of the cardiod is swept out by $r = 1 + \cos\theta$ as θ goes from 0 to π so its area is

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1+\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi} (1+\cos\theta)^2\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \left(1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{\cos(2\theta)}{2}\right)d\theta$$

$$= \frac{1}{2}\left(\frac{3\theta}{2} + 2\sin\theta + \frac{\sin(2\theta)}{4}\right)\Big|_0^{\pi}$$

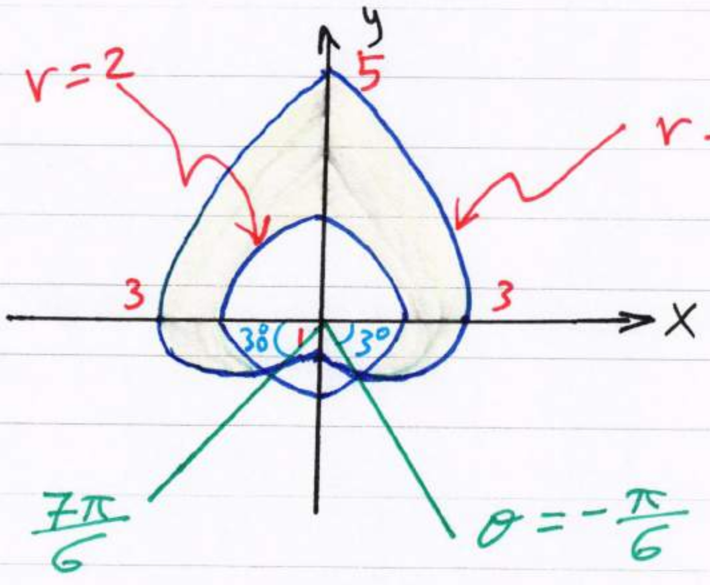
$$= \frac{1}{2}\left(\left(\frac{3\pi}{2} + 2\cancel{\sin\pi}^{0} + \frac{\cancel{\sin(2\pi)}^{0}}{4}\right) - \left(0 + 2\cancel{\sin 0}^{0} + \frac{\cancel{\sin 0}^{0}}{4}\right)\right)$$

$$= \frac{3\pi}{4}$$

Thus the area in question is :

$$\frac{3\pi}{4} - \frac{\pi}{8} = \frac{5\pi}{8}$$

Ex 10: Find the area inside the cardiod $r = 3 + 2\sin\theta$ and outside circle $r = 2$.



$r = 3 + 2\sin\theta$

at $\theta = 0 \longrightarrow r = 3 + 2\sin 0 = 3 + 0 = 3$

$\theta = 90^\circ \longrightarrow r = 3 + 2\sin 90 = 3 + 2 = 5$

$\theta = 180 \longrightarrow r = 3 + 2\sin 180 = 3 + 0 = 3$

$\theta = 270 \longrightarrow r = 3 + 2\sin 270 = 3 + 2(-1) = 1$

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$r_1 = 3 + 2\sin\theta$

$r_2 = 2$

$r_1 = r_2$

$3 + 2\sin\theta = 2 \longrightarrow 2\sin\theta = -1$

$\theta = \sin^{-1}\left(-\frac{1}{2}\right) = -30^\circ = -\frac{\pi}{6} \,,\, 210^\circ = \frac{7\pi}{6}$

$$A = \int_{\theta=-\frac{\pi}{6}}^{\theta=\frac{7\pi}{6}} \int_{r=2}^{r=3+2\sin\theta} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \left.\frac{r^2}{2}\right|_{2}^{3+2\sin\theta} d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \left[\frac{(3+2\sin\theta)^2}{2} - \frac{2^2}{2}\right] d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \left[\frac{9 + 12\sin\theta + 4\sin^2\theta}{2} - 2\right] d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \left[\frac{5}{2} + 6\sin\theta + 2\sin^2\theta\right] d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \left[\frac{5}{2} + 6\sin\theta + \frac{1}{2}2(1-\cos 2\theta))\right] d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \left[\frac{7}{2} + 6\sin\theta - \cos 2\theta\right] d\theta$$

$$= \left[\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \left[\left(\frac{7}{2}\times\frac{7\pi}{6} - 6\cos\frac{7\pi}{6} - \frac{1}{2}\sin\frac{7\pi}{3}\right)\right.$$

$$\left. - \left(-\frac{7}{2}\times\frac{\pi}{6} - 6\cos\left(-\frac{\pi}{6}\right) - \frac{1}{2}\sin\left(-\frac{\pi}{3}\right)\right)\right]$$

$$= \left[\frac{49\pi}{12} + 3\sqrt{3} - \frac{\sqrt{3}}{4} + \frac{7\pi}{12} + 3\sqrt{3} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{14\pi}{3} + \frac{11\sqrt{3}}{2} = 24.187$$