EXERCISES 3.6

*3.6.1. Consider

$$f(x) = \begin{cases} 0 & x < x_0 \\ 1/\Delta & x_0 < x < x_0 + \Delta \\ 0 & x > x_0 + \Delta. \end{cases}$$

Assume that $x_0 > -L$ and $x_0 + \Delta < L$. Determine the complex Fourier coefficients c_n.

3.6.2. If $f(x)$ is real, show that $c_{-n} = \bar{c}_n$.

Chapter 4

Wave Equation: Vibrating Strings and Membranes

4.1 Introduction

At this point in our study of partial differential equations, the only physical problem we have introduced is the conduction of heat. To broaden the scope of our discussions, we now investigate the vibrations of perfectly elastic strings and membranes. We begin by formulating the governing equations for a vibrating string from physical principles. The appropriate boundary conditions will be shown to be similar in a mathematical sense to those boundary conditions for the heat equations. Examples will be solved by the method of separation of variables.

4.2 Derivation of a Vertically Vibrating String

A vibrating string is a complicated physical system. We would like to present a simple derivation. A string vibrates only if it is tightly stretched. Consider a horizontally stretched string in its equilibrium configuration, as illustrated in Fig. 4.2.1. We imagine that the ends are tied down in some way (to be described in Sec. 4.3), maintaining the tightly stretched nature of the string. You may wish to think of stringed musical instruments as examples. We begin by tracking the motion of each particle that comprises the string. We let α be the x-coordinate of a particle when the string is in the horizontal equilibrium position. The string moves in time; it is located somewhere other than the equilibrium position at time t, as illustrated in Fig. 4.2.1. The trajectory of particle α is indicated with both horizontal and vertical components.

We will assume the slope of the string is small, in which case it can be shown that the horizontal displacement v can be neglected. As an approximation, **the**

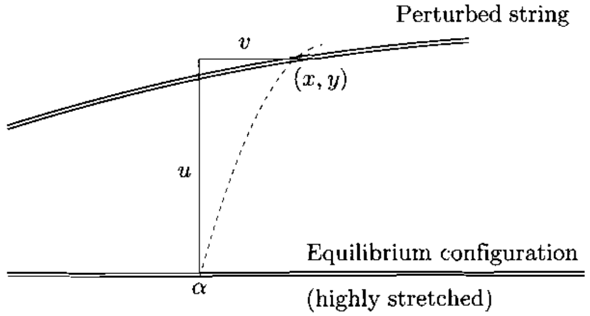

Figure 4.2.1 Vertical and horizontal displacements of a particle on a highly stretched string.

motion is entirely vertical, $x = \alpha$. In this situation, the vertical displacement u depends on x and t:

$$y = u(x, t). \tag{4.2.1}$$

Derivations including the effect of a horizontal displacement are necessarily complicated (see Weinberger [1965] and Antman [1980]). In general ($x \neq \alpha$), it is best to let $y = u(\alpha, t)$.

Newton's law. We consider an infinitesimally thin segment of the string contained between x and $x + \Delta x$ (as illustrated in Fig. 4.2.2). In the unperturbed (yet stretched) horizontal position, we assume that the **mass density** $\rho_0(x)$ is known. For the thin segment, the total mass is approximately $\rho_0(x)\ \Delta x$. Our object is to derive a partial differential equation describing how the displacement u changes in time. Accelerations are due to forces; we must use Newton's law. For simplicity we will analyze Newton's law for a point mass:

$$\boldsymbol{F} = m\boldsymbol{a}. \tag{4.2.2}$$

We must discuss the forces acting on this segment of the string. There are body

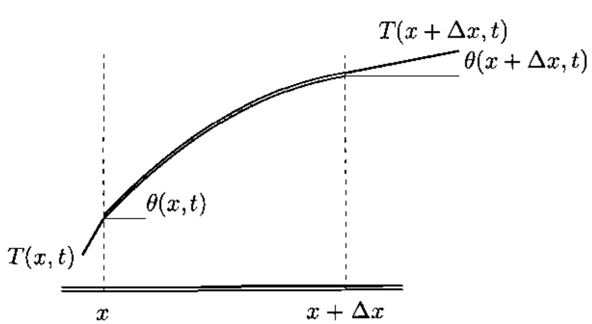

Figure 4.2.2 Stretching of a finite segment of string, illustrating the tensile forces.

forces, which we assume act only in the vertical direction (e.g., the gravitational force), as well as forces acting on the ends of the segment of string. We assume that the string is **perfectly flexible**; it offers no resistance to bending. This means that the force exerted by the rest of the string on the endpoints of the segment of the string is in the direction tangent to the string. This tangential force is known as the **tension** in the string, and we denote its magnitude by $T(x, t)$. In Fig. 4.2.2 we show that the force due to the tension (exerted by the rest of the string) pulls at both ends in the direction of the tangent, trying to stretch the small segment. To obtain components of the tensile force, the angle θ between the horizon and the string is introduced. The angle depends on both the position x and time t. Furthermore, the slope of the string may be represented as either dy/dx or $\tan\theta$:

$$\frac{dy}{dx} = \tan\theta(x, t) = \frac{\partial u}{\partial x}. \tag{4.2.3}$$

The horizontal component of Newton's law prescribes the horizontal motion, which we claim is small and can be neglected. The vertical equation of motion states that the mass $\rho_0(x)\ \Delta x$ times the vertical component of acceleration ($\partial^2 u/\partial t^2$, where $\partial/\partial t$ is used since x is fixed for this motion) equals the vertical component of the tensile forces plus the vertical component of the body forces:

$$\begin{aligned} \rho_0(x)\ \Delta x \frac{\partial^2 u}{\partial t^2} &= T(x + \Delta x, t)\sin\theta(x + \Delta x, t) \\ &\quad -T(x, t)\sin\theta(x, t) + \rho_0(x)\ \Delta x\ Q(x, t), \end{aligned} \tag{4.2.4}$$

where $T(x, t)$ is the magnitude of the tensile force and $Q(x, t)$ is the vertical component of the body force per unit mass. Dividing (4.2.4) by Δx and taking the limit as $\Delta x \to 0$ yields

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}[T(x, t)\sin\theta(x, t)] + \rho_0(x)Q(x, t). \tag{4.2.5}$$

For small angles θ,

$$\frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta,$$

and hence (4.2.5) becomes

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) + \rho_0(x)Q(x, t). \tag{4.2.6}$$

Perfectly elastic strings . The tension of a string is determined by experiments. Real strings are nearly **perfectly elastic**, by which we mean that the magnitude of the tensile force $T(x, t)$ depends only on the local stretching of the string. Since the angle θ is assumed to be small, the stretching of the string is nearly the same as for the unperturbed highly stretched horizontal string, where the tension is constant, T_0 (to be in equilibrium). Thus, **the tension** $T(x, t)$ **may be approximated by a constant** T_0. Consequently, the small vertical vibrations of a highly stretched string are governed by

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x, t)\rho_0(x). \tag{4.2.7}$$

One-dimensional wave equation. If the only body force per unit mass is gravity, then $Q(x,t) = -g$ in (4.2.7). In many such situations, this force is small (relative to the tensile force $\rho_0 g \ll |T_0 \partial^2 u/\partial x^2|$) and can be neglected. Alternatively, gravity sags the string, and we can calculate the vibrations with respect to the sagged equilibrium position. In either way we are often led to investigate (4.2.7) in the case in which $Q(x,t) = 0$,

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} \tag{4.2.8}$$

or

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},} \tag{4.2.9}$$

where $c^2 = T_0/\rho_0(x)$. Equation (4.2.9) is called the **one-dimensional wave equation**. The notation c^2 is introduced because $T_0/\rho_0(x)$ has the dimensions of velocity squared. We will show that c is a very important velocity. For a uniform string, c is constant.

EXERCISES 4.2

4.2.1. (a) Using Equation (4.2.7), compute the sagged equilibrium position $u_E(x)$ if $Q(x,t) = -g$. The boundary conditions are $u(O) = 0$ and $u(L) = 0$.

(b) Show that $v(x,t) = u(x,t) - u_E(x)$ satisfies (4.2.9).

4.2.2. Show that c^2 has the dimensions of velocity squared.

4.2.3. Consider a particle whose x-coordinate (in horizontal equilibrium) is designated by α. If its vertical and horizontal displacements are u and v, respectively, determine its position x and y. Then show that

$$\frac{dy}{dx} = \frac{\partial u/\partial \alpha}{1 + \partial v/\partial \alpha}.$$

4.2.4. Derive equations for horizontal and vertical displacements without ignoring v. Assume that the string is perfectly flexible and that the tension is determined by an experimental law.

4.2.5. Derive the partial differential equation for a vibrating string in the simplest possible manner. You may assume the string has constant mass density ρ_0, you may assume the tension T_0 is constant, and you may assume small displacements (with small slopes).

4.3 Boundary Conditions

The partial differential equation for a vibrating string, (4.2.7) or (4.2.8), has a second-order spatial partial derivative. We will apply one boundary condition at each end, just as we did for the one-dimensional heat equation.

The simplest boundary condition is that of a fixed end, usually fixed with zero displacement. For example, if a string is fixed (with zero displacement) at $x = L$, then

$$u(L,t) = 0. \tag{4.3.1}$$

Alternatively, we might vary an end of the string in a prescribed way:

$$u(L,t) = f(t). \tag{4.3.2}$$

Both (4.3.1) and (4.3.2) are linear boundary conditions; (4.3.2) is nonhomogeneous, while (4.3.1) is homogeneous.

A more interesting boundary condition occurs if one end of the string is attached to a dynamical system. Let us suppose that the left end, $x = 0$, of a string is attached to a spring-mass system, as illustrated in Fig. 4.3.1. We will insist that the motion be entirely vertical. To accomplish this, we must envision the mass to be on a vertical track (possibly frictionless). The track applies a horizontal force to the mass when necessary to prevent the large horizontal component of the tensile force from turning over the spring-mass system. The string is attached to the mass so that if the position of the mass is $y(t)$, so is the position of the left end:

$$u(0,t) = y(t). \tag{4.3.3}$$

However, $y(t)$ is unknown and itself satisfies an ordinary differential equation determined from Newton's laws. We assume that the spring has unstretched length l and obeys Hooke's law with spring constant k. To make the problem even more interesting, we let the support of the spring move in some prescribed way, $y_s(t)$. Thus, the length of the spring is $y(t) - y_s(t)$ and the stretching of the spring is $y(t) - y_s(t) - l$. According to Newton's law (using Hooke's law with spring constant k),

$$m\frac{d^2y}{dt^2} = -k(y(t) - y_s(t) - l) + \text{other forces on mass.}$$

The other vertical forces on the mass are a tensile force applied by the string $T(0,t)\sin\theta(0,t)$ and a force $g(t)$ representing any other external forces on the mass. Recall that we must be restricted to small angles, such that the tension is nearly constant, T_0. In that case, the vertical component is approximately $T_0 \partial u/\partial x$:

$$T(0,t)\sin\theta(0,t) \approx T(0,t)\frac{\sin\theta(0,t)}{\cos\theta(0,t)} = T(0,t)\frac{\partial u}{\partial x}(0,t) \approx T_0 \frac{\partial u}{\partial x}(0,t),$$

since for small angles $\cos\theta \approx 1$. In this way the boundary condition at $x = 0$ for a vibrating string attached to a spring-mass system [with a variable support $y_s(t)$

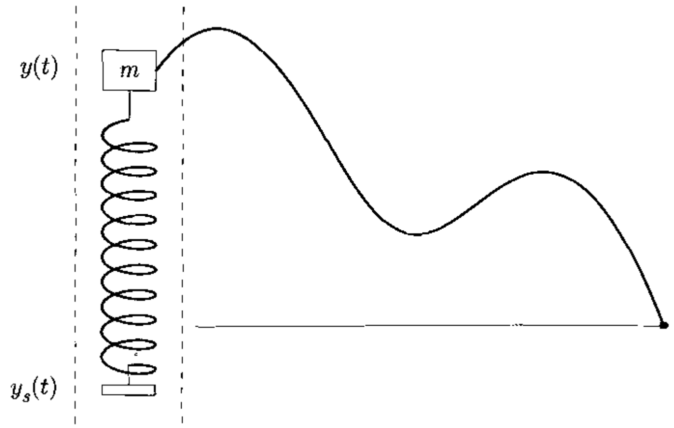

Figure 4.3.1 Spring-mass system with a variable support attached to a stretched string.

and an external force $g(t)$] is

$$\boxed{m\frac{d^2}{dt^2}u(0,t) = -k(u(0,t) - y_s(t) - l) + T_0\frac{\partial u}{\partial x}(0,t) + g(t).} \tag{4.3.4}$$

Let us consider some special cases in which there are no external forces on the mass, $g(t) = 0$. If, in addition, the mass is sufficiently small so that the forces on the mass are in balance, then

$$T_0\frac{\partial u}{\partial x}(0,t) = k(u(0,t) - u_E(t)), \tag{4.3.5}$$

where $u_E(t)$ is the equilibrium position of the mass, $u_E(t) = y_s(t) + l$. This form, known as the nonhomogeneous **elastic** boundary condition, is exactly analogous to Newton's law of cooling [with an external temperature of $u_E(t)$] for the heat equation. If the equilibrium position of the mass coincides with the equilibrium position of the string, $u_E(t) = 0$, the homogeneous version of the elastic boundary condition results:

$$T_0\frac{\partial u}{\partial x}(0,t) = ku(0,t). \tag{4.3.6}$$

$\partial u/\partial x$ is proportional to u. Since for physical reasons $T_0 > 0$ and $k > 0$, the signs in (4.3.6) are prescribed. This is the same choice of signs that occurs for Newton's law of cooling. A diagram (Fig. 4.3.2) illustrates both the correct and incorrect choice of signs. This figure shows that (assuming $u = 0$ is an equilibrium position for both string and mass) if $u > 0$ at $x = 0$, then $\partial u/\partial x > 0$ in order to get a balance of vertical forces on the *massless* spring-mass system. A similar argument shows that there is an important sign change if the elastic boundary condition occurs at $x = L$:

$$T_0\frac{\partial u}{\partial x}(L,t) = -k(u(L,t) - u_E(t)), \tag{4.3.7}$$

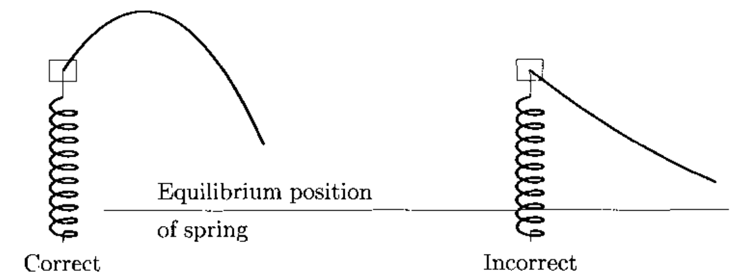

Figure 4.3.2 Boundary conditions for massless spring-mass system.

the same sign change we obtained for Newton's law of cooling.

For a vibrating string, another boundary condition that can be discussed is the **free end**. It is not literally free. Instead, the end is attached to a frictionless vertically moving track as before and is free to move up and down. There is no spring-mass system, nor external forces. However, we can obtain this boundary condition by taking the limit as $k \to 0$ of either (4.3.6) or (4.3.7):

$$T_0\frac{\partial u}{\partial x}(L,t) = 0. \tag{4.3.8}$$

This says that the vertical component of the tensile force must vanish at the end since there are no other vertical forces at the end. If the vertical component did not vanish, the end would have an infinite vertical acceleration. Boundary condition (4.3.8) is exactly analogous to the insulated boundary condition for the one-dimensional heat equation.

EXERCISES 4.3

4.3.1. If $m = 0$, which of the diagrams for the right end shown in Fig. 4.3.3 is possibly correct? *Briefly* explain. Assume that the mass can move only vertically.

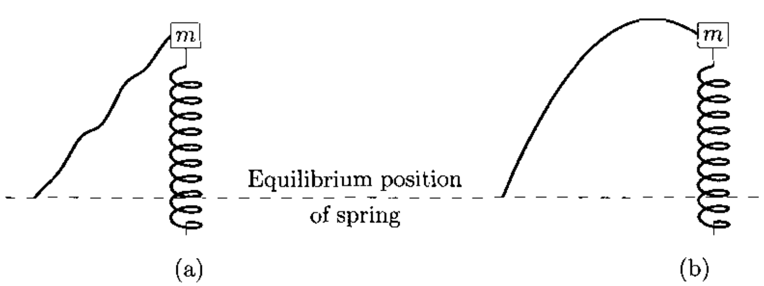

Figure 4.3.3

4.3.2. Consider two vibrating strings connected at $x = L$ to a spring-mass system on a vertical frictionless track as in Fig. 4.3.4. Assume that the spring is

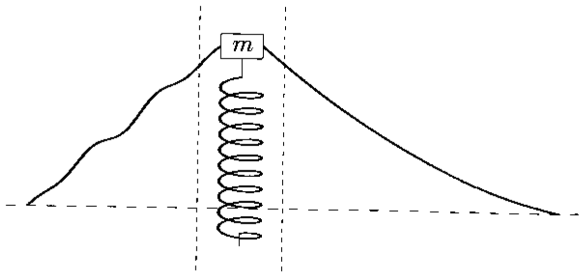

Figure 4.3.4

unstretched when the string is horizontal (the spring has a fixed support). Also suppose that there is an external force $f(t)$ acting on the mass m.

(a) What "jump" conditions apply at $x = L$ relating the string on the left to the string on the right?

(b) In what situations is this mathematically analogous to perfect thermal contact?

4.4 Vibrating String with Fixed Ends

In this section we solve the one-dimensional wave equation, which represents a uniform vibrating string without external forces,

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},} \tag{4.4.1}$$

where $c^2 = T_0/\rho_0$, subject to the simplest homogeneous boundary conditions,

$$\text{BC1:} \quad \boxed{\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0, \end{aligned}} \tag{4.4.2}$$

both ends being fixed with zero displacement. Since the partial differential equation (4.4.1) contains the second time derivative, two initial conditions are needed. We prescribe both $u(x,0)$ and $\partial u/\partial t(x,0)$.

$$\text{IC:} \quad \boxed{\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \end{aligned}} \tag{4.4.3}$$

corresponding to being given the initial position and the initial velocity of each segment of the string. These two initial conditions are not surprising, as the wave equation was derived from Newton's law by analyzing each segment of the string as a particle; ordinary differential equations for particles require both initial position and velocity.

Since both the partial differential equation and the boundary conditions are linear and homogeneous, the method of separation of variables is attempted. As with the heat equation the nonhomogeneous initial conditions are put aside temporarily. We look for special product solutions of the form

$$u(x,t) = \phi(x)h(t). \tag{4.4.4}$$

Substituting (4.4.4) into (4.4.1) yields

$$\phi(x)\frac{d^2h}{dt^2} = c^2 h(t)\frac{d^2\phi}{dx^2}. \tag{4.4.5}$$

Dividing by $\phi(x)h(t)$ separates the variables, but it is more convenient to divide additionally by the constant c^2, since then the resulting eigenvalue problem will not contain the parameter c^2:

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda. \tag{4.4.6}$$

A separation constant is introduced since $(1/c^2)(1/h)(d^2h/dt^2)$ depends only on t and $(1/\phi)(d^2\phi/dx^2)$ depends only on x. The minus sign is inserted *purely for convenience.* With this minus sign, let us explain why we expect that $\lambda > 0$. We need the two ordinary differential equations that follow from (4.4.6):

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \tag{4.4.7}$$

and

$$\frac{d^2\phi}{dx^2} = -\lambda\phi. \tag{4.4.8}$$

The two homogeneous boundary conditions (4.4.2) show that

$$\phi(0) = \phi(L) = 0. \tag{4.4.9}$$

Thus, (4.4.8) and (4.4.9) form a boundary value problem. Instead of first reviewing the solution of (4.4.8) and (4.4.9), let us analyze the time-dependent ODE (4.4.7). If $\lambda > 0$, the general solution of (4.4.7) is a linear combination of sines and cosines,

$$h(t) = c_1 \cos c\sqrt{\lambda}t + c_2 \sin c\sqrt{\lambda}t. \tag{4.4.10}$$

If $\lambda = 0, h(t) = c_1 + c_2 t$, and if $\lambda < 0, h(t)$ is a linear combination of exponentially growing and decaying solutions in time. Since we are solving a vibrating string, it should seem more reasonable that the time-dependent solutions oscillate. This does not prove that $\lambda > 0$. Instead, it serves as an immediate motivation for choosing the minus sign in (4.4.6). Now by analyzing the boundary value problem, we may indeed determine that the eigenvalues are nonnegative.

The boundary value problem is

$$\begin{aligned} \frac{d^2\phi}{dx^2} &= -\lambda\phi \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned}$$

Although we could solve this by proceeding through three cases, we ought to recall that all the eigenvalues are positive. In fact,

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3 \ldots,$$

and the corresponding eigenfunctions are $\sin n\pi x/L$. The time-dependent part of the solution has been obtained previously, (4.4.10). Thus, there are two families of product solutions: $\sin n\pi x/L \sin n\pi ct/L$ and $\sin n\pi x/L \cos n\pi ct/L$. The principle of superposition then implies that we should be able to solve the initial value problem by considering a linear combination of all product solutions:

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\right). \tag{4.4.11}$$

The initial conditions (4.4.3) are satisfied if

$$\begin{aligned} f(x) &= \textstyle\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \\ g(x) &= \textstyle\sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L}. \end{aligned} \tag{4.4.12}$$

The boundary conditions have implied that sine series are important. There are two initial conditions and two families of coefficients to be determined. From our previous work on Fourier sine series, we know that $\sin n\pi x/L$ forms an orthogonal set. A_n will be the coefficients of the Fourier sine series of $f(x)$ and $B_n n\pi c/L$ will be for the Fourier sine series of $g(x)$:

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\ B_n\frac{n\pi c}{L} &= \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx. \end{aligned} \tag{4.4.13}$$

Let us interpret these results in the context of musical stringed instruments (with fixed ends). The vertical displacement is composed of a linear combination of simple product solutions,

$$\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right).$$

These are called the **normal modes** of vibration. The intensity of the sound produced depends on the amplitude,[1] $\sqrt{A_n^2 + B_n^2}$. The time dependence is simple harmonic with **circular frequency** (the number of oscillations in 2π units of time) equaling $n\pi c/L$, where $c = \sqrt{T_0/\rho_0}$. The sound produced consists of the superposition of these infinite number of **natural frequencies** ($n = 1, 2, \ldots$). The normal mode $n = 1$ is called the first harmonic or fundamental. In the case of a vibrating string the fundamental mode has a circular frequency of $\pi c/L$.[2] The larger the natural frequency, the higher the pitch of the sound produced. To produce a desired fundamental frequency, $c = \sqrt{T_0/\rho_0}$ or L can be varied. Usually, the mass density is fixed. Thus, the instrument is tuned by varying the tension T_0; the larger T_0, the higher the fundamental frequency. While playing a stringed instrument the musician can also vary the pitch by varying the effective length L, by clamping down the string. Shortening L makes the note higher. The nth normal mode is called the nth harmonic. For vibrating strings (with fixed ends) the frequencies of the higher harmonics are all integral multiples of the fundamental. It is not necessarily true for other types of musical instruments. This is thought to be pleasing to the ear.

Let us attempt to illustrate the motion associated with each normal mode. The fundamental and higher harmonics are sketched in Fig. 4.4.1. To indicate what these look like, we sketch for various values of t. At each t, each mode looks like a simple oscillation in x. The amplitude varies periodically in time. These are called **standing waves**. In all cases there is no displacement at both ends due to the boundary conditions. For the second harmonic ($n = 2$), the displacement is also zero for all time in the middle $x = L/2$. $x = L/2$ is called a **node** for the second harmonic. Similarly, there are two nodes for the third harmonic. This can be generalized: The nth harmonic has $n - 1$ nodes.[3]

It is interesting to note that the vibration corresponding to the second harmonic looks like two identical strings each with length $L/2$ vibrating in the fundamental mode, since $x = L/2$ is a node. We should find that the frequencies of vibration are identical; that is, the frequency for the fundamental ($n = 1$) with length $L/2$ should equal the frequency for the second harmonic ($n = 2$) with length L. The formula for the frequency $\omega = n\pi c/L$, verifies this observation.

Each standing wave can be shown to be composed of two travelling waves. For example, consider the term $\sin n\pi x/L \sin n\pi ct/L$. From trigonometric identities

$$\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L} = \underbrace{\frac{1}{2}\cos\frac{n\pi}{L}(x-ct)}_{\substack{\text{wave traveling}\\ \text{to the right}\\ \text{(with velocity } c)}} - \underbrace{\frac{1}{2}\cos\frac{n\pi}{L}(x+ct)}_{\substack{\text{wave traveling}\\ \text{to the left}\\ \text{(with velocity } -c)}} \tag{4.4.14}$$

In fact, since the solution (4.4.11) to the wave equation consists of a superposition

[1] $A_n\cos\omega t + B_n\sin\omega t = \sqrt{A_n^2 + B_n^2}\sin(\omega t + \theta)$, where $\theta = \tan^{-1} A_n/B_n$.

[2] Frequencies are usually measured in cycles per second, not cycles per 2π units of time. The fundamental thus has a frequency of $c/2L$, cycles per second.

[3] You can visualize experimentally this result by rapidly oscillating at the appropriate frequency one end of a long rope that is tightly held at the other end. The result appears more easily for an expandable spiral telephone cord or a "slinky."

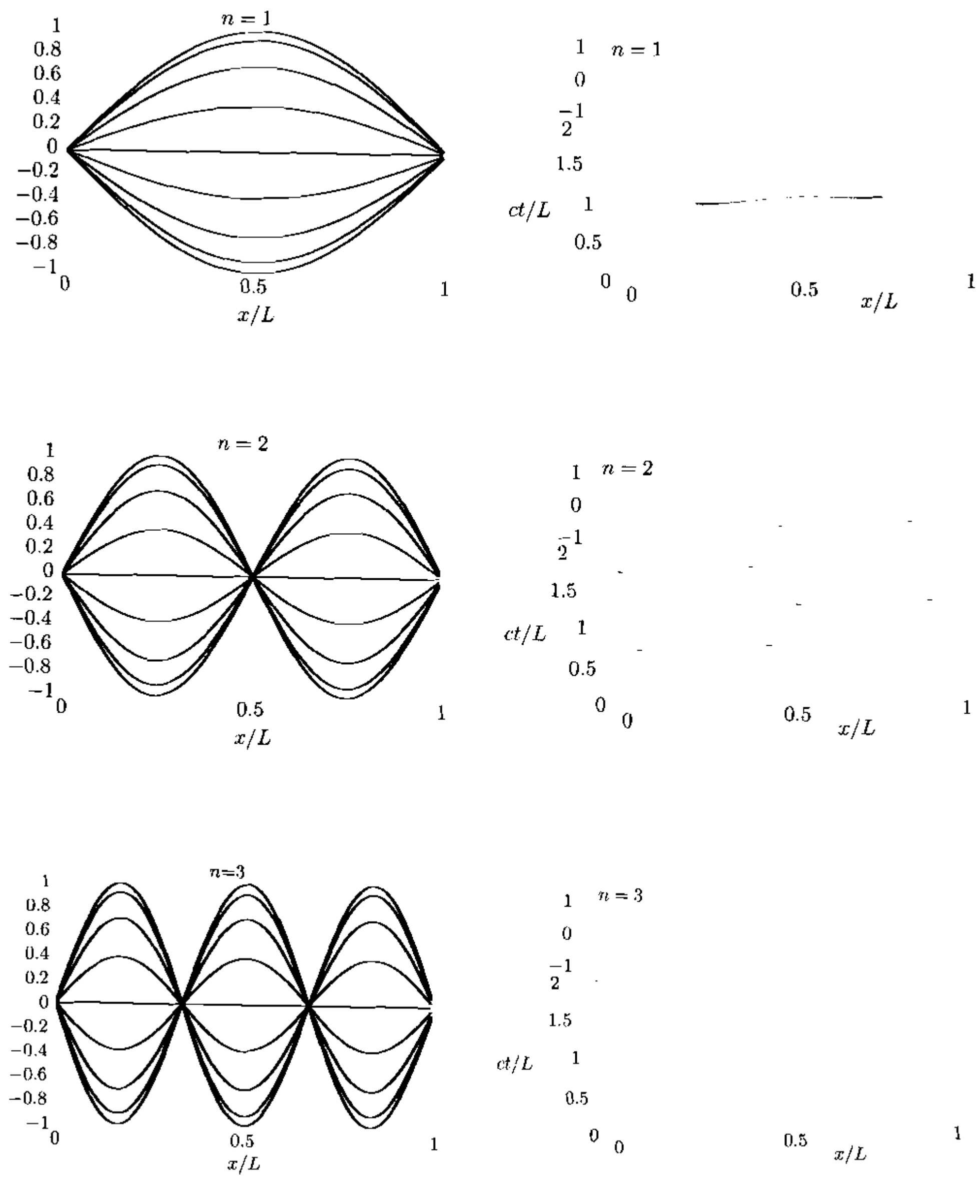

Figure 4.4.1 Normal modes of vibration for a string.

of standing waves, it can be shown that this solution is a combination of just two waves (each rather complicated)—one traveling to the left at velocity $-c$ with fixed shape and the other to the right at velocity c with a different fixed shape. We are claiming that the solution to the one-dimensional wave equation can be written as

$$u(x,t) = R(x-ct) + S(x+ct),$$

even if the boundary conditions are not fixed at $x = 0$ and $x = L$. We will show and discuss this further in the Exercises and in Chapter 12.

EXERCISES 4.4

4.4.1. Consider vibrating strings of uniform density ρ_0 and tension T_0.

*(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?

*(b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H,t) = 0$]? Sketch a few modes of vibration as in Fig. 4.4.1.

(c) Show that the modes of vibration for the *odd* harmonics (i.e., $n = 1, 3, 5, \ldots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

4.4.2. In Sec. 4.2 it was shown that the displacement u of a nonuniform string satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q,$$

where Q represents the vertical component of the body force per unit length. If $Q = 0$, the partial differential equation is homogeneous. A slightly different homogeneous equation occurs if $Q = \alpha u$.

(a) Show that if $\alpha < 0$, the body force is restoring (toward $u = 0$). Show that if $\alpha > 0$, the body force tends to push the string further away from its unperturbed position $u = 0$.

(b) Separate variables if $\rho_0(x)$ and $\alpha(x)$ but T_0 is constant for physical reasons. Analyze the time-dependent ordinary differential equation.

*(c) Specialize part (b) to the constant coefficient case. Solve the initial value problem if $\alpha < 0$:

$$u(0,t) = 0 \qquad u(x,0) = 0$$

$$u(L,t) = 0 \qquad \frac{\partial u}{\partial t}(x,0) = f(x).$$

What are the frequencies of vibration?

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$).

4.4.4. Redo Exercise 4.4.3(b) by the eigenfunction expansion method.

4.4.5. Redo Exercise 4.4.3(b) if $4\pi^2\rho_0 T_0/L^2 < \beta^2 < 16\pi^2 \rho_0 T_0/L^2$.

4.4.6. For (4.4.1)–(4.4.3), from (4.4.11) show that

$$u(x,t) = R(x - ct) + S(x + ct),$$

where R and S are some functions.

4.4.7. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially at rest, $g(x) = 0$, show that

$$u(x,t) = \frac{1}{2}[F(x - ct) + F(x + ct)],$$

where $F(x)$ is the odd periodic extension of $f(x)$. *Hints:*

1. For all x, $F(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$.
2. $\sin a \cos b = \frac{1}{2}[\sin(a + b) + \sin(a - b)]$.

Comment: This result shows that *the practical difficulty of summing an infinite number of terms of a Fourier series may be avoided for the one-dimensional wave equation.*

4.4.8. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially unperturbed, $f(x) = 0$, with the initial velocity given, show that

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(\bar{x})\, d\bar{x},$$

where $G(x)$ is the odd periodic extension of $g(x)$. *Hints:*

1. For all x, $G(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$.
2. $\sin a \sin b = \frac{1}{2}[\cos(a - b) - \cos(a + b)]$.

See the comment after Exercise 4.4.7.

4.4.9 From (4.4.1), derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t}\bigg|_0^L, \tag{4.4.15}$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2} \left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2} \left(\frac{\partial u}{\partial x}\right)^2 dx$.

4.4.10. What happens to the total energy E of a vibrating string (see Exercise 4.4.9)

(a) If $u(0,T) = 0$ and $u(L,t) = 0$

(b) If $\frac{\partial u}{\partial x}(0,t) = 0$ and $u(L,t) = 0$

(c) If $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = -\gamma u(L,t)$ with $\gamma > 0$

(d) If $\gamma < 0$ in part (c)

4.4.11. Show that the potential and kinetic energies (defined in Exercise 4.4.9) are equal for a traveling wave, $u = R(x - ct)$.

4.4.12. Using (4.4.15), prove that the solution of (4.4.1)–(4.4.3) is unique.

4.4.13. (a) Using (4.4.15), calculate the energy of one normal mode.

(b) Show that the total energy, when $u(x,t)$ satisfies (4.4.11), is the sum of the energies contained in each mode.

4.5 Vibrating Membrane

The heat equation in one spatial dimension is $\partial u/\partial t = k\partial^2 u/\partial x^2$. In two or three dimensions, the temperature satisfies $\partial u/\partial t = k\nabla^2 u$. In a similar way, the vibration of a string (one dimension) can be extended to the vibration of a membrane (two dimensions).

The vertical displacement of a vibrating string satisfies the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

There are important physical problems that solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{4.5.1}$$

known as the two- or three-dimensional wave equation. An example of a physical problem that satisfies a two-dimensional wave equation is the vibration of a highly stretched membrane. This can be thought of as a two-dimensional vibrating string. We will give a *brief* derivation in the manner described by Kaplan [1981], omitting

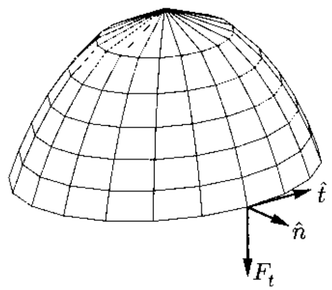

Figure 4.5.1 Perturbed stretched membrane with approximately constant tension T_0. The normal vector to the surface is $\hat{\boldsymbol{n}}$ and the tangent vector to the edge is $\hat{\boldsymbol{t}}$.

some of the details we discussed for a vibrating string. We again introduce the displacement $z = u(x, y, t)$, which depends on x, y and t (as illustrated in Fig. 4.5.1). If all slopes (i.e., $\partial u/\partial x$ and $\partial u/\partial y$) are small, then as an approximation we may assume that the vibrations are entirely vertical and the tension is approximately constant. Then the mass density (mass per unit surface area), $\rho_0(x, y)$, of the membrane in the unperturbed position does not change appreciably when the membrane is perturbed.

The tensile force (per unit arc length), $\boldsymbol{F}_T$, is tangent to the membrane and acts along the entire edge. The direction of the tensile force (see Fig. 4.5.1) is obtained by crossing the unit tangent vector to the edge, $\hat{\boldsymbol{t}}$, with the unit normal vector to the membrane, $\hat{\boldsymbol{n}}$. Since the tensile force has constant magnitude ($|\boldsymbol{F}_T| = T_0$), it follows that

$$\boldsymbol{F}_T = T_0 \hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}},$$

where the vertical component is obtained by $\boldsymbol{F_T} \cdot \hat{\boldsymbol{k}}$.

Newton's law for vertical motion must be applied to each differential section of the membrane and then summed (integrated). The sum (surface integral) of the mass ($\rho_0 \, dA$) times the vertical acceleration ($\partial^2 u/\partial t^2$) equals the total (closed line integral) *vertical* tensile force (ignoring body forces)

$$\iint \rho_0 \frac{\partial^2 u}{\partial t^2} \, dA = \oint T_0 \hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}} \, ds = \oint T_0 (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{t}} \, ds, \tag{4.5.2}$$

where ds is the differential arc length, dA is differential surface area, and the vector triple product relation has been used ($\boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{B} \times \boldsymbol{C} \cdot \boldsymbol{A}$). Stokes' theorem ($\iint \boldsymbol{\nabla} \times \boldsymbol{B} \cdot \hat{\boldsymbol{n}} \, dA = \oint \boldsymbol{B} \cdot \hat{\boldsymbol{t}} \, ds$), will be applied (for the only time in this text):

$$\iint \rho_0 \frac{\partial^2 u}{\partial t^2} dA = \iint T_0 [\nabla \times (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}})] \cdot \hat{\boldsymbol{n}} \, dA. \tag{4.5.3}$$

Since the region is arbitrary, we derive

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 [\nabla \times (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}})] \cdot \hat{\boldsymbol{n}}. \tag{4.5.4}$$

A point on the membrane is described by $z = u(x, y)$. Thus, the unit normal to the vibrating membrane (using the gradient; see Appendix to Sec. 1.5) is calculated as follows:

$$\hat{\boldsymbol{n}} = \frac{-\frac{\partial u}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial u}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 1}} \approx -\frac{\partial u}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial u}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}},$$

since the partial derivatives are assumed to be small. We now begin to calculate the expression needed in (4.5.4):

$$\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ -\dfrac{\partial u}{\partial x} & -\dfrac{\partial u}{\partial y} & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{\partial u}{\partial y}\hat{\boldsymbol{\imath}} + \frac{\partial u}{\partial x}\hat{\boldsymbol{\jmath}}.$$

Continuing, we obtain

$$\nabla \times (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{k}}) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial x} & 0 \end{vmatrix} = \hat{\boldsymbol{k}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

In this way we obtain the partial differential equation for a vibrating membrane,

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Dividing by ρ_0 yields the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{4.5.5}$$

where again $c^2 = T_0/\rho_0$. The solutions of problems for a vibrating membrane are postponed until Chapter 7.

EXERCISES 4.5

4.5.1. If a membrane satisfies an "elastic" boundary condition, show that

$$T_0 \nabla u \cdot \hat{\boldsymbol{n}} = -ku \tag{4.5.6}$$

if there is a restoring force per unit length proportional to the displacement.

4.6 Reflection and Refraction of Electromagnetic (Light) and Acoustic (Sound) Waves

Disturbances in a uniform media frequently satisfy the **three-dimensional wave equation**:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{4.6.1}$$

In a fluid, small displacements u to a uniform mass density ρ and pressure p satisfy (4.6.1), where the coefficient c satisfies $c^2 = \frac{\partial p}{\partial \rho}$. In electrodynamics, each component of a system of vector fields satisfies (4.6.1), where $c^2 = c^2_{\text{light}}/\mu\epsilon$ can be related to the speed of light in a vacuum c^2_{light} and the permeability μ and the dielectric constant ϵ.

Special **plane traveling wave** solutions of (4.6.1) exist of the form

$$u = Ae^{i(k_1x+k_2y+k_3z-\omega t)} = Ae^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}. \tag{4.6.2}$$

(The real or imaginary part have physical significance.) A is the constant amplitude. The vector $\boldsymbol{k}$, called the **wave vector,** is in the direction of the wave (perpendicular to the **wave fronts** $\boldsymbol{k}\cdot\boldsymbol{x}-\omega t =$ constant). The magnitude of the wave vector $k \equiv |\boldsymbol{k}|$ is the **wave number** since it can be shown (see the Exercises) to equal the number of waves in 2π distance in the direction of the wave (in the $\boldsymbol{k}$-direction). (The **wave length** $=\frac{2\pi}{k}$.)

The temporal frequency ω for plane wave solutions of the wave equations is determined by substituting (4.6.2) into (4.6.1):

$$\omega^2 = c^2k^2, \tag{4.6.3}$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$. It is important that ω is a function of k satisfying (4.6.3). As in one dimension, it can be shown that

$$\text{traveling wave or phase velocity } = \frac{\omega}{k} = \pm c. \tag{4.6.4}$$

The plane wave solution corresponds to one component of a multidimensional Fourier series (transform–see Chapter 10) if the wave equation is defined in a finite (infinite) region. As we show in the next subsection, frequently plane waves are considered to be generated at infinity.

4.6.1 Snell's Law of Refraction

We assume that we have two different materials (different mass densities for sound waves or different dielectric constants for electromagnetic waves) extending to infinity with a plane boundary between them. We assume the wave speed is c_+ for $z > 0$ and c_- for $z < 0$, as shown in Fig. 4.6.1. We assume that there is an **incident** plane wave satisfying (4.6.2) (with wave vector $\boldsymbol{k}_I$ and frequency $\omega = \omega_+(k_I) = c_+k_I$) propagating from infinity with $z > 0$ with amplitude $A = 1$, which we normalize to 1. We assume that the incident wave makes an angle θ_I with the normal.

We assume that there is a **reflected** wave in the upper media satisfying (4.6.2) [with unknown wave vector $\boldsymbol{k}_R$ and frequency $\omega = \omega_+(k_R) = c_+k_R$] with unknown complex amplitude R. Due to the wave equation being linear, the solution in the upper media is the sum of the incident and reflected wave:

$$u = e^{i(\boldsymbol{k}_I\cdot\boldsymbol{x}-\omega_+(c_+k_I)t)} + Re^{i(\boldsymbol{k}_R\cdot\boldsymbol{x}-\omega_+(c_+k_R)t)} \text{ for } z > 0. \tag{4.6.5}$$

We will show that the wave vector of the reflected wave is determined by the familiar relationship that the angle of reflection equals the angle of incidence.

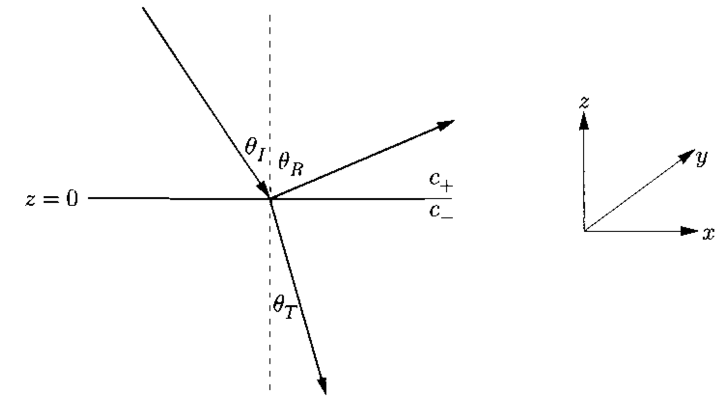

Figure 4.6.1 Reflected and refracted (transmitted) waves.

In the lower media $z < 0$, we assume that a **refracted wave** exists, which we call a **transmitted wave** and introduce the subscript T. We assume that the transmitted wave satisfies (4.6.2) (with unknown wave number vector $\boldsymbol{k}_T$ and frequency $\omega = \omega_-(k_T) = c_-k_T$) with unknown complex amplitude T:

$$u = Te^{i(\boldsymbol{k}_T\cdot\boldsymbol{x}-\omega_-(c_-k_T)t)} \text{ for } z < 0. \tag{4.6.6}$$

In addition, we will show that this refracted wave may or may not exist as prescribed by Snell's law of refraction.

There are two boundary conditions that must be satisfied at the interface between the two materials $z = 0$. One of the conditions is that u must be continuous. At $z = 0$,

$$e^{i(\boldsymbol{k}_I\cdot\boldsymbol{x}-\omega_+(c_+k_I)t)} + Re^{i(\boldsymbol{k}_R\cdot\boldsymbol{x}-\omega_+(c_+k_R)t)} = Te^{i(\boldsymbol{k}_T\cdot\boldsymbol{x}-\omega_-(c_-k_T)t)}. \tag{4.6.7}$$

Since this must hold for all time, the frequencies of the three waves must be the same:

$$\omega_+(k_I) = \omega_+(k_R) = \omega_-(k_T). \tag{4.6.8}$$

From the frequency equation (4.6.3), we conclude that the reflected wave has the same wave length as the incident wave, but the refracted wave has a different wave length:

$$c_+k_I = c_+k_R = c_-k_T. \tag{4.6.9}$$

From (4.6.7), k_1 and k_2 (the projection of $\boldsymbol{k}$ in the x- and y-directions) must be the same for all three waves. Since $k_I = k_R$, the z-component of the reflected wave must be minus the z-component of the incident wave. Thus, the **angle of reflection equals the angle of incidence**,

$$\theta_R = \theta_I, \tag{4.6.10}$$

where the angles are measured with respect to the normal to the surface. Note that $\boldsymbol{k}\cdot\boldsymbol{x} = k\,|\boldsymbol{x}|\cos\phi$ is the same for all three waves, where ϕ is the angle between

$\boldsymbol{k}$ and $\boldsymbol{x}$ with $z = 0$. Thus, for the transmitted wave, the angle of transmission (refraction) satisfies

$$k_I \sin\theta_I = k_T \sin\theta_T.$$

Using (4.6.9), **Snell's law** follows:

$$\boxed{\frac{\sin\theta_T}{\sin\theta I} = \frac{k_I}{k_T} = \frac{c_-}{c_+}.} \tag{4.6.11}$$

Many important and well-known results from optics follow from Snell's law. For example, in the usual case of the upper media (c_+) being air and the lower media (c_-) water, then it is known that $c_+ > c_-$. In this case, from Snell's law (4.6.11) $\sin\theta_T < \sin\theta_I$, so that the transmitted wave is refracted toward the normal (as shown in Fig. 4.6.1).

If $c_+ < c_-$, then Snell's law (4.6.11) predicts in some cases that $\sin\theta_T > 1$, which is impossible. There is a critical angle of incidence $\sin\theta_I = \frac{c_+}{c_-}$ at which total internal reflection first occurs. For larger angles, the transmitted solution is not a plane wave but is an evanescent wave exponentially decaying, as we describe in a later subsection. The refracted plane wave does not exist.

4.6.2 Intensity (Amplitude) of Reflected and Refracted Waves

Here we will assume that a refracted plane wave exists. With these laws for the reflected and refracted waves, the one boundary condition (4.6.7) for the continuity of u becomes simply

$$1 + R = T. \tag{4.6.12}$$

We cannot solve for either amplitude R or T without the second boundary condition. The second boundary condition can be slightly different in different physical applications. Thus, the results of this subsection do not apply to all physical problems, but the method we use may be applied in all cases and the results obtained may be slightly different.

We assume the second boundary condition is $\frac{\partial u}{\partial z} = 0$ at $z = 0$. From (4.6.5) and (4.6.6), it follows that

$$k_{3_I} + k_{3_R} R = k_{3_T} T. \tag{4.6.13}$$

From Fig. 4.6.1, the z-components of the wave number of three waves satisfy,

$$\begin{aligned} k_{3_I} &= -k_I \cos\theta_I \\ k_{3_R} &= k_R \cos\theta_R = k_I \cos\theta_I \\ k_{3_T} &= -k_T \cos\theta_T = -k_I \frac{\sin\theta_I}{\sin\theta_T} \cos\theta_T, \end{aligned} \tag{4.6.14}$$

where we have recalled that the reflected wave satisfies $k_{3_R} = -k_{3_I}$ and we have used Snell's law (4.6.11) to simplify the z-component of the wave number of the transmitted wave. Using (4.6.14), the second boundary condition (4.6.13) becomes (after dividing by $-k_I \cos\theta_I$)

$$1 - R = \frac{\sin\theta_I}{\sin\theta_T} \frac{\cos\theta_T}{\cos\theta_I} T. \tag{4.6.15}$$

The complex amplitudes of reflection and transmission can be determined by adding the two linear equations, (4.6.12) and (4.6.15),

$$\begin{aligned} T &= \frac{2}{1 + \frac{\sin\theta_I}{\sin\theta_T}\frac{\cos\theta_T}{\cos\theta_I}} = \frac{2\sin\theta_T\cos\theta_I}{\sin(\theta_T + \theta_I)} \\ R &= \frac{2\sin\theta_T\cos\theta_I - \sin(\theta_T+\theta_I)}{\sin(\theta_T+\theta_I)} = \frac{\sin(\theta_T - \theta_I)}{\sin(\theta_T+\theta_I)}. \end{aligned}$$

4.6.3 Total Internal Reflection

If $\sin\theta_I \frac{c_-}{c_+} > 1$, then Snell's law cannot be satisfied for a plane transmitted (refracted) wave $u = Te^{i(\boldsymbol{k}_T\cdot\boldsymbol{x}-\omega_-(c_-k_T)t)}$ in the lower media. Because of the boundary condition at $z = 0$, the x- and y-components of the wave number of a solution must be the same as for the incident wave. Thus, the transmitted wave number should satisfy $\overrightarrow{k_T} = (k_{1_I}, k_{2_I}, k_{3_T})$. If we apply Snell's law (4.6.11) and solve for k_{3_T},

$$k_{3_T} = \pm\sqrt{k_I^2\left(\frac{c_+^2}{c_-^2} - \sin^2\theta_I\right)}, \tag{4.6.16}$$

since $k_1^2 + k_2^2 = k^2\sin^2\theta$. We find that k_{3_T} is imaginary, suggesting that there are solutions of the wave equation (in the lower media),

$$\frac{\partial^2 u}{\partial t^2} = c_-^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{4.6.17}$$

which exponentially grow and decay in z. We look for a product solution of the form

$$u(x,y,z,t) = w(z)e^{i(k_1x+k_2y-\omega t)}, \tag{4.6.18}$$

where k_1 and k_2 are the wave numbers associated with the incident wave and ω is the frequency of the incident wave. We insist that (4.6.18) satisfy (4.6.17) so that

$$\frac{d^2w}{dz^2} = \left(k_1^2 + k_2^2 - \frac{\omega^2}{c_-^2}\right)w = k_I^2\left(\sin^2\theta_I - \frac{c_+^2}{c_-^2}\right)w.$$

Thus, $w(z)$ is a linear combination of exponentially growing and decaying terms in z. Since we want our solution to decay exponentially as $z \to -\infty$, we choose the solution to the wave equation in the lower media to be

$$u(x,y,z,t) = Te^{k_I\sqrt{\sin^2\theta_I - \frac{c_+^2}{c_-^2}}\,z} e^{i(k_1x+k_2y-\omega)t)},$$

instead of the plane wave. This is a horizontal two-dimensional plane wave whose amplitude exponentially decays in the $-z$-direction. It is called an **evanescent** wave (exponentially decaying in the $-z$-direction).

The continuity of u and $\frac{\partial u}{\partial z}$ at $z = 0$ is satisfied if

$$1 + R = T \tag{4.6.19}$$

$$ik_{3_I}(1 - R) = Tk_I\sqrt{\sin^2\theta_I - \frac{c_+^2}{c_-^2}}. \tag{4.6.20}$$

These equations can be simplified and solved for the reflection coefficient and T (the amplitude of the evanescent wave at $z = 0$). R (and T) will be complex, corresponding to phase shifts of the reflected (and evanescent) wave.

EXERCISES 4.6

4.6.1. Show that for a plane wave given by (4.6.2), the number of waves in 2π distance in the direction of the wave (the $\boldsymbol{k}$-direction) is $k \equiv |\boldsymbol{k}|$.

4.6.2. Show that the phase of a plane wave stays the same moving in the direction of the wave if the velocity is $\frac{\omega}{k}$.

4.6.3. In optics, the index of refraction is defined as $n = \frac{c_{\text{light}}}{c}$. Express Snell's law using the indices of refraction.

4.6.4. Find R and T for the evanescent wave by solving the simultaneous equations (4.6.19) and (4.6.20).

4.6.5. Find R and T by assuming that $k_{3I} = \pm i\beta$, where β is defined by (4.6.16). Which sign do we use to obtain exponential decay as $z \to -\infty$?

Chapter 5

Sturm-Liouville Eigenvalue Problems

5.1 Introduction

We have found the method of separation of variables to be quite successful in solving some homogeneous partial differential equations with homogeneous boundary conditions. In all examples we have analyzed so far the boundary value problem that determines the needed eigenvalues (separation constants) has involved the simple ordinary differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0. \tag{5.1.1}$$

Explicit solutions of this equation determined the eigenvalues λ from the homogeneous boundary conditions. The principle of superposition resulted in our needing to analyze infinite series. We pursued three different cases (depending on the boundary conditions): Fourier sine series, Fourier cosine series, and Fourier series (both sines and cosines). Fortunately, we verified by explicit integration that the eigenfunctions were orthogonal. This enabled us to determine the coefficients of the infinite series from the remaining nonhomogeneous condition.

In this section we further explain and generalize these results. We show that the orthogonality of the eigenfunctions can be derived even if we cannot solve the defining differential equation in terms of elementary functions [as in (5.1.1)]. Instead, orthogonality is a direct result of the differential equation. We investigate other boundary value problems resulting from separation of variables that yield other families of orthogonal functions. These generalizations of Fourier series will not always involve sines and cosines since (5.1.1) is not necessarily appropriate in every situation.