Chapter-3

Unsteady State Heat Conduction

The general form of heat transfer by conduction is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{K} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

For one dimension, unsteady state heat conduction with no heat generation ($\dot{q}=0$) $\Rightarrow$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \Rightarrow T = f(x,t) \;;\; \alpha = K/\rho c$$

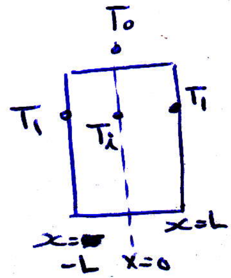
There are three main types of boundary conditions

1. Constant wall temp.
2. Constant heat flux.
3. Convection boundary condition

Geometries:

1. lumped heat capacity system.
2. Semi-Infinite solid.
3. Infinite Plate.
4. Infinite cylinder.
5. Short Cylinder.
6. Sphere.

Infinite Plate: The differential Eqn is $\frac{d^2\theta}{dx^2} = \frac{1}{\alpha}\frac{d\theta}{dt}$; $\theta = T - T_1$



IC. @ $t=0$, $T=T_i$, $\theta=\theta_i=T_i-T_1$

BC(1) @ $x=L$, $\theta=0$; $t>0$

BC(2) @ $x=-L$, $\theta=0$; $t>0$

The solution is

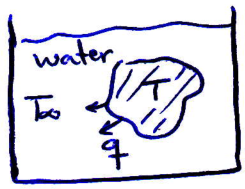
$$\frac{\theta}{\theta_i} = \frac{T-T_1}{T_i-T_1} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} e^{-\left[\frac{n\pi}{2L}\right]^2 \alpha t} \cdot \sin\frac{n\pi}{2L}x$$

Lumped Heat Capacity System

The lumped heat capacity method of analysis is used for systems in which a uniform temperature exists, ie, no temperature gradient is found.

For a hot body immersed in a cool vessel of water, the energy balance is:

In − out = Acc.



$$0 - hA(T - T_\infty) = \rho V c_p \frac{dT}{dt}$$

where A = surface area of convection (m^2)

V = volume of the body (m^3)

ρ = Density of solid (kg/m^3)

c_p = specific heat of the body ($J/kg.K$)

h = convective heat transfer coefficient. (W/m^2K) ($J/s.m^2K$)

$$\frac{dT}{dt} + \frac{hA}{\rho V c_p}(T - T_\infty) = 0 \Rightarrow \frac{d\theta}{dt} + \frac{hA}{\rho V c_p}\theta = 0 \quad ; \quad \theta = T - T_\infty$$

By Integration factor, the solution is: $\theta = c\, e^{-\left(\frac{hA}{\rho V c_p}\right)t}$

IC @ $t = 0$, $T = T_i \Rightarrow \theta = \theta_i$

$\Rightarrow \theta_i = c$

$\Rightarrow$ The solⁿ is $\boxed{\dfrac{\theta}{\theta_i} = e^{-\left(\frac{hA}{\rho V c_p}\right)t}}$

The expression $\dfrac{\rho c_p V}{hA}$ has units of time, and is called the time constant of the system (τ).

$$\tau = \text{Resistance} \times \text{capacitance} = \frac{1}{hA} \times \rho V c_p$$

$$\Rightarrow \boxed{\frac{\theta}{\theta_i} = e^{\frac{-t}{\tau}}}$$

Application of lumped heat capacity

The criteria of applying lumped heat capacity for a system is by obtaining the value of Biot number (Bi)

$$Bi = \frac{h\,(V/A)}{K} \leq 0.1$$

Example: A steel ball ($c_p = 0.46$ KJ/kg.K }, $K = 35$ w/mK), 50mm in diameter and initially at a uniform temp. of 450°C is suddenly placed in a controlled environment in which the temp. is maintained at 100°C. The convection heat transfer coefficient is 10 w/m².K. Calculate the time required for the ball to attain a temp. of 150°C ($\rho_{steel} = 7800$ Kg/m³).

Sol^n $r = d/2 = 25*10^{-3}$ m

$$Bi = \frac{hS}{K} \quad ; \quad S = V/A = \frac{\frac{4}{3}\pi\,(25*10^{-3})^3}{4\pi\,(25*10^{-3})^2} = 0.0083$$

$$Bi = \frac{10\,(0.0083)}{35} = 0.0023 \quad < 0.1 \Rightarrow \text{lumped heat system}$$

$$\tau = \frac{\rho c_p V}{hA} = \frac{(7800)(460)\left(\frac{4}{3}\pi\,(0.0025)^3\right)}{10 * 4\pi\,(0.0025)^2} = 2988 \text{ sec.}$$

$$\theta = \theta_i\, e^{-t/\tau} \quad ; \quad \theta_i = T_i - T_\infty = 450 - 100 = 350°C$$

~~θ = 150 − 100~~

$$\theta = T - T_\infty = 150 - 100 = 50°C$$

$$\Rightarrow 50 = 350\, e^{-t/2988} \quad \Rightarrow t = 5818 \text{ sec.} = 1.616 \text{ hr.}$$

Example: Determine the time required for a 1.25 cm diameter steel sphere ($K = 40\ W/mK$) to cool from $T_i = 500°c$ to $100°c$ if exposed to a cooling air flow at $T_\infty = 25°c$ resulting in $h = 110\ w/m^2K$, $\rho_{steel} = 7801\ Kg/m^3$, $C_p = 473\ KJ/KgK$.

Sol^n
$$Bi = \frac{h(V/A)}{K} = \frac{(110)(0.0125)}{(40)(6)} = 0.0057 \quad < 0.1 \Rightarrow \text{lump}$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = e^{-t/\tau} \quad ; \quad \tau = \frac{\rho C_p V}{hA}$$

$$\frac{100-25}{500-25} = 0.1579 = \text{Exp}\left[\frac{-t}{\frac{(7801)(473)(0.0125)}{(6)(110)}}\right] \Rightarrow t = 2.16\ min$$

Example For the steel of the above example, determine:

a The instantoueous heat transfer rate 2min after the start of cooling.

b/ The total energy transfered from the sphere during the first two minutes.

Sol^n ⓐ
$$q = hA(T - T_\infty) = hA(T_i - T_\infty)\, e^{-t/\tau}$$

$$= 110 * \pi(0.0125)^2 (500-25)\, e^{\left[\frac{-120}{\frac{7801*473*0.0125}{6*110}}\right]} =$$

$$= 0.405\ w$$

ⓑ
$$Q = \int_0^t q \cdot dt = \int_0^t hA(T_i - T_\infty)\, e^{\frac{-t}{\tau}} \cdot dt = hA(T_i - T_\infty)\int_0^t e^{-t/\tau} \cdot dt$$

$$= -\tau hA(T_i - T_\infty) \cdot \left. e^{-t/\tau}\right]_0^t = \tau hA(T_i - T_\infty)(1 - e^{-t/\tau})$$

$$= \rho C_p V (T_i - T_\infty)(1 - e^{-t/\tau})$$

$$= 7801 * 473 * \frac{\pi}{6}(0.0125)^2 \left[1 - e^{\left[\frac{-120}{\frac{(7801)(473)(0.0125)}{6*110}}\right]}\right](500-25)$$

$$= 1.467\ kJ$$

Example: An aluminium sphere weighing 7 kg and initially at a temp. of ~~…~~ 260°C is suddenly immersed in a fluid at 10°C. If $h = 50\ W/m^2K$, determine the time required to cool the sphere to 90°C. given $\rho = 2707\ kg/m^3$, $C_p = 900\ J/kgK$, $k = 204\ W/mK$.

Sol^n:-

$$V = \frac{mass}{\rho} = \frac{7\ kg}{2707\ \frac{kg}{m^3}} = 2.58 \times 10^{-3}\ m^3$$

$$V = \frac{4}{3}\pi r_o^3 \Rightarrow r_o = \left(\frac{3V}{4\pi}\right)^{1/3} = \left(\frac{3(2.58 \times 10^{-5})}{4\pi}\right)^{1/3} = 0.085\ m$$

$$\frac{V}{A} = \frac{r_o}{3} = 0.028\ m$$

$$Bi = \frac{h r_o}{3k} = \frac{50(0.085)}{3(204)} = 0.007 < 0.1$$

∴ Internal resistance can be neglected.

$$\frac{T - T_\infty}{T_o - T_\infty} = e^{-t/\tau}$$

$$\tau = \frac{\rho C_p V}{hA} = \frac{(2707)(900)(0.085)}{3(50)} = 1388.89\ sec$$

$$\therefore \frac{90-10}{260-10} = e^{-t/1388.89} \Rightarrow t = 1580\ sec.$$

Example: A thermocouple junction, which may be approximated as a sphere, is to be used for temp. measurement in a gas stream. The convection coefficient between the junction surface & the gas is known to be 400 W/m²K and the junction thermophysical properties are $K = 20$ W/mK, $c_p = 400$ J/kgK, and $\rho = 8500$ kg/m³. Determine the junction diameter needed for the thermocouple to have a time constant of 1 sec. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?

Sol

Assume lumped heat capacity system

$$\tau = \frac{\rho c_p V}{hA} = \frac{\rho c_p \frac{\pi}{6} D^3}{h \pi D^2}$$

$$\Rightarrow D = \frac{6 h \tau}{\rho c_p} = \frac{6 * 400 * 1}{8500 * 400} = 7.06 * 10^{-4} \text{ m} \quad r = \frac{D}{2}$$

$$L_c = \frac{D}{6} = r_o/3 \Rightarrow Bi = \frac{h r_o}{3K} = \frac{400 * \left(\frac{7.06*10^{-4}}{2}\right)}{3(20)} = 2.35 * 10^{-4} < 0.1$$

$\Rightarrow$ lumped assumption is correct

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-t/\tau}$$

$$\frac{199 - 200}{25 - 200} = e^{-t/1} \Rightarrow t = 5.16 \text{ sec.}$$