Transient Heat Flow in Semi-Infinite Solids

The general equ is $\frac{d^2T}{dx^2} = \frac{1}{\alpha}\frac{dT}{dt} \Rightarrow T = f(x,t)$.

The types of Boundary conditions are:

1. Constant wall temp.
2. constant heat flux.
3. Convection boundary condition.

① Constant wall temp. B.C.

1. I.C. $T(x,0) = T_i$
2. B.C. $T(0,t) = T_o$
3. B.C $T(\infty,t) = T_i$

The final solution for the semi-infinite solid is:

$$\boxed{\frac{T - T_o}{T_i - T_o} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}$$

where erf is the error function defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2} \cdot d\eta \quad ; \eta = \text{dummy variable.}$$

α = Thermal diffusivity = $k/\rho c_p$ (m^2/s)

use Figure 4-4a (P. 145) or Table A-1

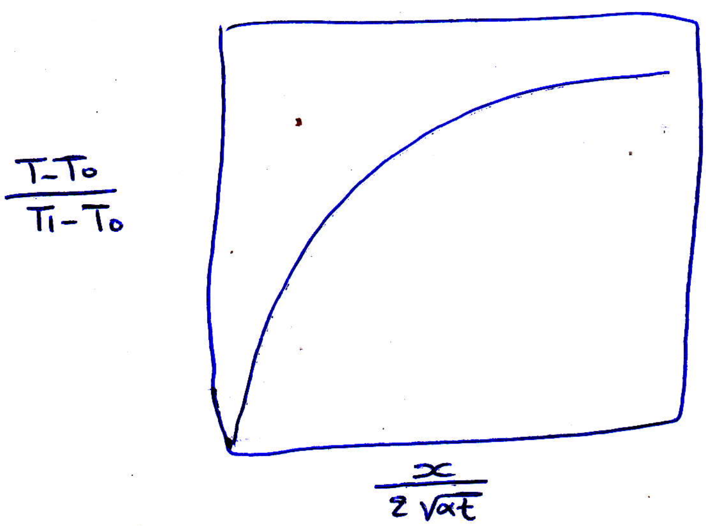
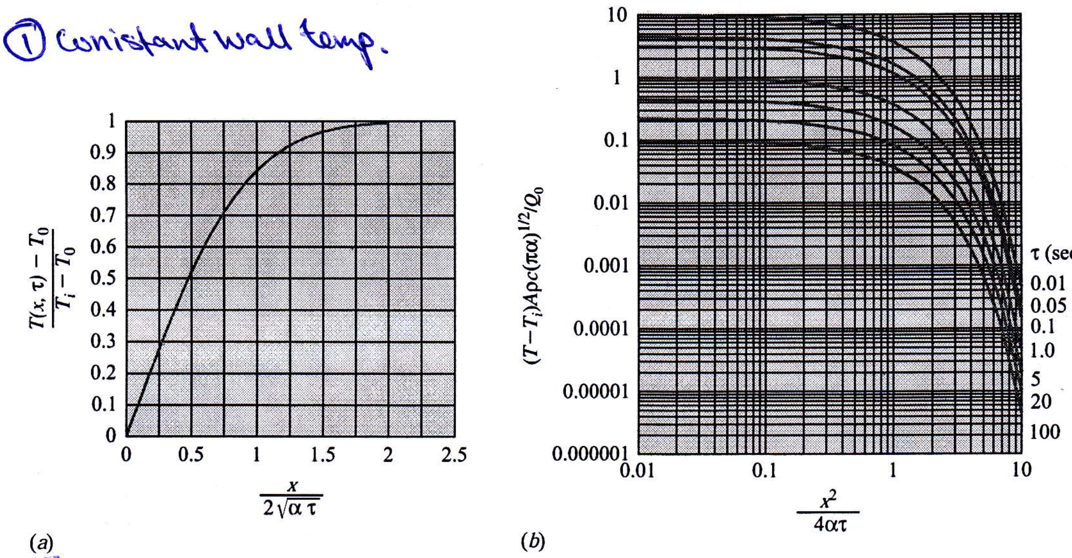


Figure 4-4 | Response of semi-infinite solid to (*a*) sudden change in surface temperature and (*b*) instantaneous surface pulse of Q_0/A J/m^2.

① Constant wall temp.



At the surface ($x = 0$) the heat flow is

$$q_0 = \frac{kA(T_0 - T_i)}{\sqrt{\pi\alpha\tau}} \tag{4-12}$$

The surface heat flux is determined by evaluating the temperature gradient at $x = 0$ from Equation (4-11). A plot of the temperature distribution for the semi-infinite solid is given in Figure 4-4. Values of the error function are tabulated in Reference 3, and an abbreviated tabulation is given in Appendix A.

Constant Heat Flux on Semi-Infinite Solid

For the same uniform initial temperature distribution, we could suddenly expose the surface to a constant surface heat flux q_0/A. The initial and boundary conditions on Equation (4-7) would then become

$$T(x, 0) = T_i$$

$$\frac{q_0}{A} = -k \left.\frac{\partial T}{\partial x}\right]_{x=0} \qquad \text{for } \tau > 0$$

The solution for this case is

$$T - T_i = \frac{2q_0\sqrt{\alpha\tau/\pi}}{kA} \exp\left(\frac{-x^2}{4\alpha\tau}\right) - \frac{q_0 x}{kA}\left(1 - \operatorname{erf}\frac{x}{2\sqrt{\alpha\tau}}\right) \tag{4-13a}$$

Energy Pulse at Surface

Equation (4-13*a*) presents the temperature response that results from a surface heat flux that remains constant with time. A related boundary condition is that of a short, instantaneous pulse of energy at the surface having a magnitude of Q_0/A. The resulting temperature response is given by

$$T - T_i = [Q_0/A\rho c(\pi\alpha\tau)^{1/2}] \exp(-x^2/4\alpha\tau) \tag{4-13b}$$

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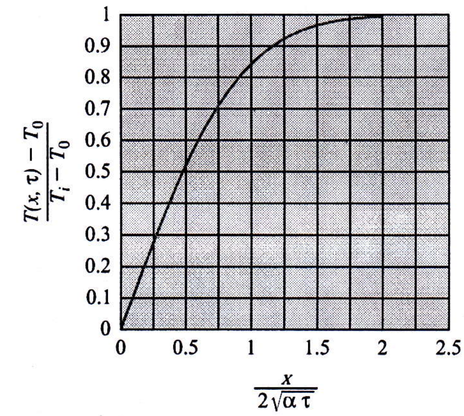
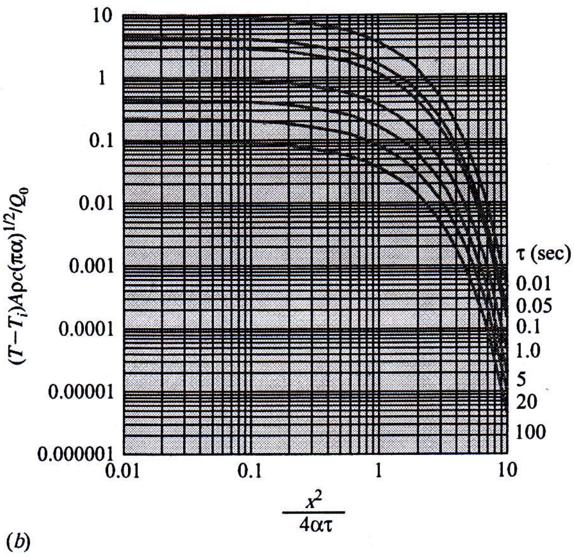
① Constant wall temp.



(*a*)



(*b*)

At the surface ($x=0$) the heat flow is

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The surface heat flux is determined by evaluating the temperature gradient at $x=0$ from Equation (4-11). A plot of the temperature distribution for the semi-infinite solid is given in Figure 4-4. Values of the error function are tabulated in Reference 3, and an abbreviated tabulation is given in Appendix A.

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For the same uniform initial temperature distribution, we could suddenly expose the surface to a constant surface heat flux q_0/A. The initial and boundary conditions on Equation (4-7) would then become

$$T(x,0) = T_i$$

$$\frac{q_0}{A} = -k\left.\frac{\partial T}{\partial x}\right]_{x=0} \qquad \text{for } \tau > 0$$

The solution for this case is

$$T - T_i = \frac{2q_0\sqrt{\alpha\tau/\pi}}{kA}\exp\left(\frac{-x^2}{4\alpha\tau}\right) - \frac{q_0 x}{kA}\left(1 - \operatorname{erf}\frac{x}{2\sqrt{\alpha\tau}}\right) \qquad \textbf{[4-13}\boldsymbol{a}\textbf{]}$$

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Equation (4-13*a*) presents the temperature response that results from a surface heat flux that remains constant with time. A related boundary condition is that of a short, instantaneous pulse of energy at the surface having a magnitude of Q_0/A. The resulting temperature response is given by

$$T - T_i = [Q_0/A\rho c(\pi\alpha\tau)^{1/2}]\exp(-x^2/4\alpha\tau) \qquad \textbf{[4-13}\boldsymbol{b}\textbf{]}$$

APPENDIX

A Tables

Table A-1 | The error function.

$\frac{x}{2\sqrt{\alpha\tau}}$	$\operatorname{erf}\frac{x}{2\sqrt{\alpha\tau}}$	$\frac{x}{2\sqrt{\alpha\tau}}$	$\operatorname{erf}\frac{x}{2\sqrt{\alpha\tau}}$	$\frac{x}{2\sqrt{\alpha\tau}}$	$\operatorname{erf}\frac{x}{2\sqrt{\alpha\tau}}$
0.00	0.00000	0.76	0.71754	1.52	0.96841
0.02	0.02256	0.78	0.73001	1.54	0.97059
0.04	0.04511	0.80	0.74210	1.56	0.97263
0.06	0.06762	0.82	0.75381	1.58	0.97455
0.08	0.09008	0.84	0.76514	1.60	0.97636
0.10	0.11246	0.86	0.77610	1.62	0.97804
0.12	0.13476	0.88	0.78669	1.64	0.97962
0.14	0.15695	0.90	0.79691	1.66	0.98110
0.16	0.17901	0.92	0.80677	1.68	0.98249
0.18	0.20094	0.94	0.81627	1.70	0.98379
0.20	0.22270	0.96	0.82542	1.72	0.98500
0.22	0.24430	0.98	0.83423	1.74	0.98613
0.24	0.26570	1.00	0.84270	1.76	0.98719
0.26	0.28690	1.02	0.85084	1.78	0.98817
0.28	0.30788	1.04	0.85865	1.80	0.98909
0.30	0.32863	1.06	0.86614	1.82	0.98994
0.32	0.34913	1.08	0.87333	1.84	0.99074
0.34	0.36936	1.10	0.88020	1.86	0.99147
0.36	0.38933	1.12	0.88079	1.88	0.99216
0.38	0.40901	1.14	0.89308	1.90	0.99279
0.40	0.42839	1.16	0.89910	1.92	0.99338
0.42	0.44749	1.18	0.90484	1.94	0.99392
0.44	0.46622	1.20	0.91031	1.96	0.99443
0.46	0.48466	1.22	0.91553	1.98	0.99489
0.48	0.50275	1.24	0.92050	2.00	0.995322
0.50	0.52050	1.26	0.92524	2.10	0.997020
0.52	0.53790	1.28	0.92973	2.20	0.998137
0.54	0.55494	1.30	0.93401	2.30	0.998857
0.56	0.57162	1.32	0.93806	2.40	0.999311
0.58	0.58792	1.34	0.94191	2.50	0.999593
0.60	0.60386	1.36	0.94556	2.60	0.999764
0.62	0.61941	1.38	0.94902	2.70	0.999866
0.64	0.63459	1.40	0.95228	2.80	0.999925
0.66	0.64938	1.42	0.95538	2.90	0.999959
0.68	0.66278	1.44	0.95830	3.00	0.999978
0.70	0.67780	1.46	0.96105	3.20	0.999994
0.72	0.69143	1.48	0.96365	3.40	0.999998
0.74	0.70468	1.50	0.96610	3.60	1.000000

The heat flow $q_x = -KA\frac{dT}{dx}$

$\frac{dT}{dx}$ can be obtained by differentiating the eqn $\frac{T-T_o}{T_i-T_o} = erf\,\frac{x}{2\sqrt{\alpha t}}$

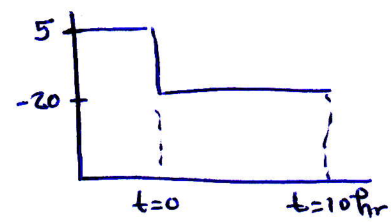
since $\frac{d}{dx}\,erf(x) = \frac{2}{\sqrt{\pi}}\, e^{-x^2} . dx$

$$\therefore \frac{dT}{dx} = (T_i - T_o)\,\frac{2}{\sqrt{\pi}}\, e^{-\frac{x^2}{4\alpha t}} \cdot \frac{d}{dx}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \frac{T_i - T_o}{\sqrt{\pi\alpha t}}\, e^{-\frac{x^2}{4\alpha t}}$$

$$\Rightarrow \boxed{q_x = -KA\,\frac{T_i - T_o}{\sqrt{\pi\alpha t}}\, e^{-\frac{x^2}{4\alpha t}}}$$

at the surface ($x=0$) $\Rightarrow$ $\boxed{q_x = -KA\,\frac{T_i - T_o}{\sqrt{\pi\alpha t}}}$

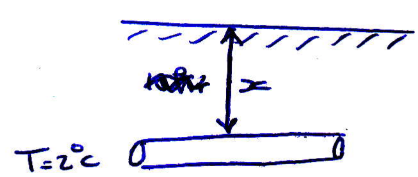
Example: At what depth should a water pipe be buried in wet soil ($\alpha = 7.75 * 10^{-7}\ m^2/s$) initially at a uniform soil temp. of 5°c, for the surrounding ~~temp~~. soil temp. to remain above 2°c, if the soil surface temp. suddenly drops to (-20°c) & remains at this temp. for 10 hrs ?



Soln

$$\frac{2-(-20)}{5-(-20)} = erf\left(\frac{x}{2\sqrt{(7.75*10^{-7})(10)(36000)}}\right)$$

$$\Rightarrow \frac{x}{2\sqrt{\alpha t}} = 1.1 \Rightarrow x = 0.37\ m$$



Example: A large slabe of aluminium at a uniform temp. of 200°C. Suddenly has its surface temp. lowered to 70°C. What is the total heat removed from the slab per unit surface area when the temp. at a depth 40mm has dropped to 120°C?

$\alpha = 8.4 * 10^{-5}\ m^2/s$, $k = 215\ W/mK$

Sol^n/ Semi Infinite – case – 1

$$\frac{T - T_o}{T_i - T_o} = \frac{120 - 70}{200 - 70} = 0.387$$

$$0.387 = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

From table $\Rightarrow \left(\frac{x}{2\sqrt{\alpha t}}\right) = 0.3553$

$$\therefore t = \frac{(0.04)^2}{4(0.3553)(8.4 * 10^{-5})} = 37.72\ \text{sec.}$$

$$q/A = \frac{-k(T_i - T_o)}{\sqrt{\pi \alpha t}} \quad (\text{heat flux at the surface})$$

$\therefore$ The total heat removed is obtained by integration

$$q_o/A = -\frac{k(T_i - T_o)}{\sqrt{\pi \alpha t}}$$

$$Q_o/A = \int_0^t (\dot{q}/A)\, dt = \int_0^t \frac{k(T_o - T_i)}{\sqrt{\pi \alpha t}}\, dt = 2k(T_o - T_i)\sqrt{\frac{t}{\alpha t}}$$

$$\frac{Q_o}{A} = 2(215)(70 - 200)\left[\frac{37.72}{\pi(8.4 * 10^{-5})}\right]^{1/2} = 21.13 * 10^{6}\ J/m^2$$