Derivatives and Integrals of the Inverse Hyperbolic Functions

Derivatives of the Inverse Hyperbolic Functions

$y = \sinh^{-1}(x) \rightarrow \sinh(y) = \sinh(\sinh^{-1}(x))$

$x = \sinh(y)$ $\qquad \therefore x = \sinh(y)$

$1 = \cosh(y)\dfrac{dy}{dx}$ $\qquad \boxed{\sinh(y) = x}$

$$\frac{dy}{dx} = \frac{1}{\cosh(y)}$$

$\cosh^2(y) - \sinh^2(y) = 1$

$\cosh^2(y) = \sinh^2(y) + 1$ جذر الطرفين $\Rightarrow \cosh(y) = \sqrt{\sinh^2(y) + 1}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\sinh^2(y) + 1}} \qquad \boxed{x = \sinh(y)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\sinh^2(\sinh^{-1}(x)) + 1}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

$y = \cosh^{-1}(x)$

$\cosh(y) = x$

$$\sinh(y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh(y)} \quad — 1$$

$\sinh^2(y) = \cosh^2(y) - 1$

$$\sinh(y) = \sqrt{\cosh^2(y) - 1} \quad — 2$$

نعوض 2 في 1

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2(y) - 1}} \rightarrow y = \cosh^{-1}(x) \text{ نعوض في } y$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2(\cosh^{-1}(x)) - 1}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}} \rightarrow$$

[illegible]

و ; $x^2 > 1$

$|x| > 1$

$$y = \tanh^{-1}(x)$$

$$\tanh(y) = x$$

$$\operatorname{sech}^2(y)\,\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2(y)} \quad \text{—} ①$$

$$\operatorname{sech}^2(y) = 1 - \tanh^2(y) \quad \text{—} ②$$

~~substitute~~ ① و ② نعوض

$$\frac{dy}{dx} = \frac{1}{1 - \tanh^2(y)}$$

$$\frac{dy}{dx} = \frac{1}{1 - \tanh^2(\tanh^{-1}(x))} \Rightarrow \frac{dy}{dx} = \frac{1}{1 - x^2} \;;\; |x| < 1$$

$$-1 < x < 1$$

$$y = \coth^{-1}(x)$$

$$\frac{d}{dx}\coth = -\operatorname{csch}^2$$

$$\coth(y) = x \Rightarrow -\operatorname{csch}^2(y)\,\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{csch}^2(y)}$$

$$\boxed{1 -> x}$$

$$\coth^2(y) = 1 + \operatorname{csch}^2(y) \Rightarrow \boxed{\operatorname{csch}^2(y) = \coth^2(y) - 1}$$

[illegible]

$$\frac{dy}{dx} = \frac{-1}{\coth^2(y) - 1} \Rightarrow \frac{dy}{dx} = \frac{1}{1 - \coth^2(y)}$$

$$\frac{dy}{dx} = \frac{1}{1 - x^2} \;;\; |x| > 1 \rightarrow x > 1 \text{ or } x < -1$$

$y = \operatorname{sech}^{-1}(x)$

$\operatorname{sech}(y) = x \quad \Rightarrow \quad -\operatorname{sech}(y)\tanh(y)\dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{-1}{\underset{x}{\underset{\shortparallel}{\operatorname{sech}(y)}}\tanh(y)}$$

$\tanh^2(y) = 1 - \operatorname{sech}^2(y)$

$\tanh(y) = \mp\sqrt{1-\operatorname{sech}^2(y)}$ ميل tanhy دائما موجب لذلك نهمل (-)

$$= \frac{-1}{x\sqrt{1-\operatorname{sech}^2 y}} \quad \operatorname{sech} y \to x$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}}} \qquad 0 < x < 1$$

$y = \operatorname{csch}^{-1}(x)$

$\operatorname{csch}(y) = x$

$$-\underset{x}{\underset{\shortparallel}{\operatorname{csch}(y)}}\coth(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\underset{x}{\underset{\shortparallel}{\operatorname{csch}(y)}}\coth(y)}$$

$\coth^2(y) = 1 + \operatorname{csch}^2(y)$

$\coth(y) = \mp\sqrt{1+\operatorname{csch}^2(y)}$

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{1+\operatorname{csch}^2(y)}} \Rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1+\operatorname{csch}^2(\operatorname{csch}^{-1}(x))}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{|x|\sqrt{1+x^2}}} \qquad x \neq 0$$

$$\frac{d}{dx}(\sinh^{-1}(\frac{x}{3})) = \frac{1}{3\sqrt{1+\dfrac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$$

$$\frac{d}{dx}(\cosh^{-1}(3x)) = \frac{3}{\sqrt{9x^2-1}}$$

Integrals of the Inverse Hyperbolic Functions

التكامل باستخدام معكوس الدوال الزائدية

$$I = \int \frac{du}{\sqrt{u^2+a^2}}$$

$$\sqrt{u^2+a^2} \Rightarrow a\sqrt{\left(\tfrac{u}{a}\right)^2 - \tfrac{a^2}{a^2}}$$

$$\Rightarrow a\sqrt{\left(\tfrac{u}{a}\right)^2+1} = \sqrt{a^2\left[\left(\tfrac{u^2}{a^2}\right)+1\right]} = \sqrt{u^2+a^2}$$

$$= \int \frac{du}{a\sqrt{\left(\frac{u}{a}\right)^2+1}} \Rightarrow \frac{1}{a}\int \frac{du}{\sqrt{\left(\frac{u}{a}\right)^2+1}}$$

$s = \frac{u}{a}$

$u = sa$

$du = a\,ds$

$$= \frac{1}{\cancel{a}}\int \frac{\cancel{a}\,ds}{\sqrt{s^2+1}} \Rightarrow \int \frac{ds}{\sqrt{s^2+1}}$$

$u = x, y, w$

$$= \sinh^{-1}(s) + c$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + c$$

$$\boxed{\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + c}$$

$$I = \int \frac{du}{\sqrt{u^2-a^2}}$$

$\boxed{\frac{u}{a} = s}$

$$= \frac{1}{a}\int \frac{du}{\sqrt{\left(\frac{u}{a}\right)^2-1}} \Rightarrow \int \frac{ds}{\sqrt{s^2-1}} \quad ; \quad s > 1$$

$\frac{u}{a} > 1$

$u > a$

$$= \cosh^{-1}(s) + c$$

$$= \cosh^{-1}\left(\frac{u}{a}\right) + c$$

$$\boxed{\therefore \int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + c \quad ; \quad \boxed{u > a}}$$

$$I = \int \frac{du}{a^2 - u^2}$$

$$= \frac{1}{a^2} \int \frac{du}{1 - \left(\frac{u}{a}\right)^2}, \quad s = \frac{u}{a} \Rightarrow du = a\,ds$$

$$= \frac{1}{a} \int \frac{ds}{1 - s^2}$$

If $s^2 < 1$

$\frac{u^2}{a^2} < 1$

$u^2 < a^2$

$|u| < a$

$\therefore = \frac{1}{a} \tanh^{-1}\left(\frac{u}{a}\right) + c$

If $s^2 > 1$

$\frac{u^2}{a^2} > 1$

$u^2 > a^2$

$|u| > a$

$\therefore = \frac{1}{a} \coth^{-1}\left(\frac{u}{a}\right) + c$

$$\therefore \int \frac{du}{a^2 - u^2} = \begin{cases} \frac{1}{a} \tanh^{-1}\left(\frac{u}{a}\right) + c & \text{If } |u| < a \\ \frac{1}{a} \coth^{-1}\left(\frac{u}{a}\right) + c & \text{If } |u| > a \end{cases}$$

$$I = \int \frac{du}{u\sqrt{a^2 - u^2}}$$

شرط الدالة

$$\boxed{\begin{array}{l} 0 \le s \le 1 \\ 0 \le \frac{u}{a} \le 1 \quad \times a \\ 0 \le u \le a \end{array}}$$

$$= \int \frac{du}{u\sqrt{a^2\left(1 - \left(\frac{u}{a}\right)^2\right)}} \Rightarrow \frac{1}{a} \int \frac{du}{u\sqrt{1 - \left(\frac{u}{a}\right)^2}} \;; \quad s = \frac{u}{a} \to u = as, \quad du = a\,ds$$

$$= \frac{1}{a} \int \frac{a\,ds}{as\sqrt{1 - s^2}} \Rightarrow = \frac{-1}{a} \int \frac{-ds}{s\sqrt{1 - s^2}} \;; \quad \boxed{0 \le s \le 1}$$

* وضعت (-) العلامة لان ناتج تكامل هذه الدالة يجب ان يكون سالبة

$$I = -\frac{1}{a} \operatorname{sech}^{-1}(s) + c$$

$$I = -\frac{1}{a} \operatorname{sech}^{-1}\left(\frac{u}{a}\right) + c$$

$$I = \int \frac{du}{u\sqrt{a^2+u^2}}$$

$$= \frac{1}{a}\int \frac{du}{u\sqrt{1+(\frac{u}{a})^2}} \qquad s = \frac{u}{a} \to u = as \to du = a\,ds$$

$$= \frac{1}{a}\int \frac{\cancel{a}\,ds}{\cancel{a}s\sqrt{1+s^2}} \qquad \boxed{s \neq 0}$$

$$= -\frac{1}{a}\int \frac{-\,ds}{s\sqrt{1+s^2}} \qquad \leftarrow \text{نضيف (-) للطرفين}$$

$$= -\frac{1}{a}\,\mathrm{csch}^{-1}(|s|) + C$$

$$= \frac{-1}{a}\,\mathrm{csch}^{-1}(\frac{u}{a}) + C$$

$$s \neq 0$$

$$\frac{u}{a} \neq 0$$

$$a \neq 0$$

$$u \neq 0$$

$$\boxed{\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\,\mathrm{csch}^{-1}(|\frac{u}{a}|) + C} \quad a \neq 0,\ u \neq 0$$

$$\int \frac{1}{\sqrt{4x^2-1}}\,dx = \int \frac{1}{2\sqrt{u^2-1}}\,du$$

$$= \frac{1}{2}\cosh^{-1} u + C$$

$$= \frac{1}{2}\cosh^{-1}(2x) + C.$$

$$\int \frac{1}{2x\sqrt{1-9x^2}}\,dx = \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}\,du$$

$$= -\frac{1}{2}\mathrm{sech}^{-1}|u| + C$$

$$= -\frac{1}{2}\mathrm{sech}^{-1}|3x| + C$$

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \cosh^{-1}(\frac{x}{2}) + C$$

$$\int \frac{1}{\sqrt{1-e^{2x}}}\,dx = -\operatorname{sech}^{-1}(e^{x}) + C$$

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$