Construction of Root Locus, Stability, and Dominant Poles.

Root locus is a graphical presentation of the closed-loop poles as a system parameter k is varied.
The graph of all possible roots of this equation (K is the variable parameter) is called the root locus.
The root locus gives information about the stability and transient response of feedback control systems.

Properties and general rules for construction of the Root Loci is as follows:

Rule no.	**Statement**	**Comments**
Rule 1	The root locus is symmetric about the real axis	Root locus that we construct will always be symmetric about the real axis (irrespective of system)
Rule 2	Total number of loci	The total number of loci will be equal to max(p,z) where p is no: of the open loop poles and z is number of open loop zeros
Rule 3	Real axis loci	A point that lies on real axis basically lies on the root locus only if the total number of real open loop poles and open loop zeros present in the RHS of this point is odd
Rule 4	Angle of asymptotes	Total number of branches of the root locus tending towards infinity is equal to p – z. The angle of asymptotes gives us the direction along which these p-z branches approach infinity.
Rule 5	centroid	Centroid is a point on real axis , through which the asymptotes pass.
Rule 6	Breakaway point	A break away point on the root locus is a point where the two poles will meet. Once they meet, they divide (split) i.e break away from the real axis.
Rule 7	Angle of departure / arrival	This gives us angles along which complex poles will depart(and complex zeros arrive) from their original position
Rule 8	Intersection with the imaginary axis	This gives us points on imaginary axis which the given root locus cut through while moving to right half of the s plane

Root Locus and Stability.

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system?

How do you determine the stability of a root locus?

The root locus procedure should produce a graph of where the poles of the system are for all values of gain K. When any or all of the roots of D (denominator) are in the unstable region, the system is unstable. When any of the roots are in the marginally stable region, the system is marginally stable (oscillatory). When all of the roots of D are in the stable region, then the system is stable.

It is important to note that a system that is stable for gain K_1 may become unstable for a different gain K_2. Some systems may have poles that cross over from stable to unstable multiple times, giving multiple gain values for which the system is unstable.

The roots of the characteristic equation are called closed loop poles. The location of such roots or poles on the s-plane will indicate the condition of stability as shown in Fig. 1.

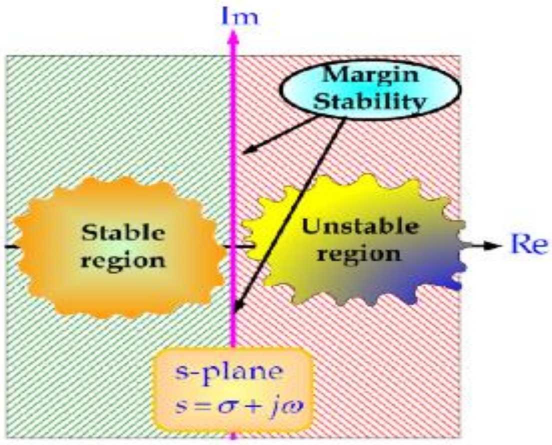


Fig. 1. Stability condition based on the location of the closed loop poles

Example 1: First-Order System

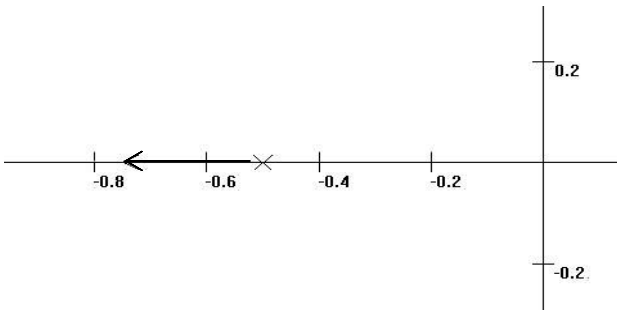
Find the root-locus of the open-loop system:

$$G(s)H(s) = \frac{K}{1+2s}$$

If we look at the characteristic equation, we can quickly solve for the single pole of the system:

S= -1/2

We plot that point on our root-locus graph, and everything on the real axis to the left of that single point is on the root locus (from the rules, above). Therefore, the root locus of our system looks like this:



From this image, we can see that <u>for *all* values *of gain* (K)</u> this system is stable.

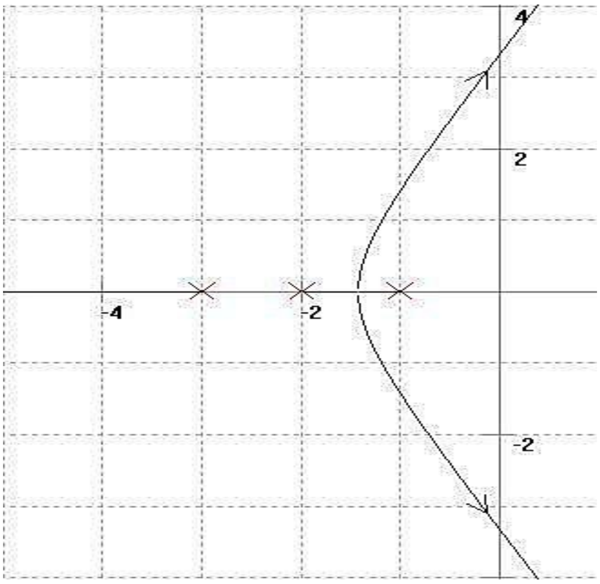
Example 2: Third Order System

$$G(s)H(s) = \frac{K}{(s+1)(s+2)(s+3)}$$

Is this system stable?

To answer this question, we can plot the root-locus. First, we draw the poles on the graph at locations -1, -2, and -3. The real-axis between the first and second poles is on the root-locus, as well as the real axis to the left of the third pole. We know also that there is going to be breakaway from the real axis at some point. The origin of asymptotes is located at: -2

We know that the breakaway occurs between the first and second poles, so we will estimate the exact breakaway point. Drawing the root-locus gives us the graph below.



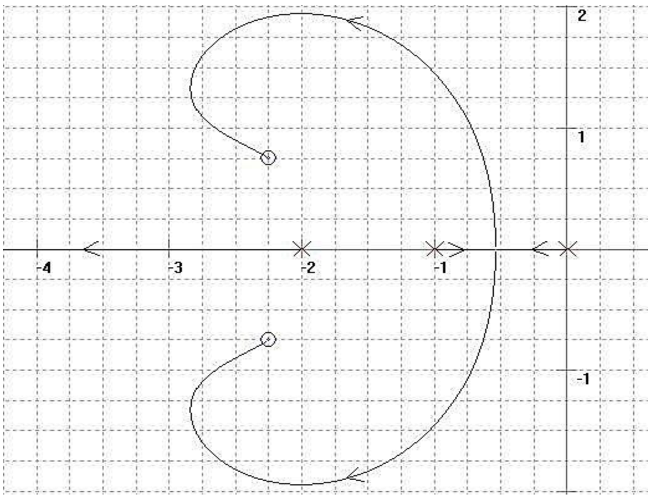
We can see that for low values of gain the system is stable, but for higher values of gain, the system becomes unstable.

Example: Complex-Conjugate Zeros

$$G(s)H(s) = \frac{K(s2 + 4.5s + 5.625)}{s(s+1)(s+2)}$$

If we look at the denominator, we have poles at the origin, -1, and -2. Following , we know that the real-axis between the first two poles, and the real axis after the third pole are all on the root-locus. We also know that there is going to be a breakaway point between the first two poles, so that they can approach the complex conjugate zeros. If we use the quadratic equation on the numerator, we can find that the zeros are located at:

S= (-2.25± j0.75)



We can see from this graph that the system is stable for <u>all values of K.</u>

<u>Dominant pole :</u>

What is the dominant pole?

Dominant pole is a pole which is more near to origin than other poles in the system.

The **poles** near to the jw axis are called the **dominant poles**. Or, get the closed-loop TF from Open loop TF. Determine the poles of the denominators.

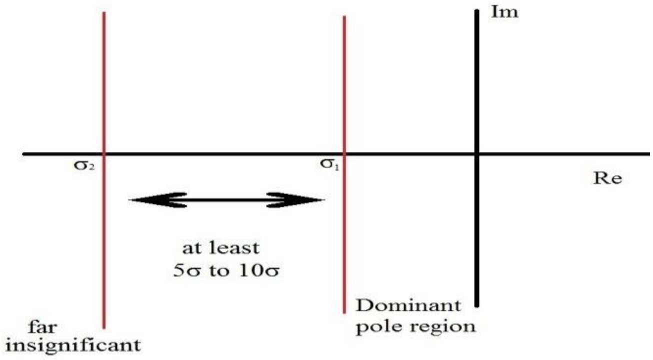
The **poles** which have very small real parts or near to the jw axis have small damping ratio. These **poles** are the **dominant poles** of the **system**

The ***dominant pole*** *approximation* is a method for approximating a (more complicated) high order system with a (simpler) system of lower order if the location of the real part of some of the system **poles** are sufficiently close to the origin compared to the other **poles**.

Why is a dominant pole required in control systems?

Dominant pole is significantly required in stability analysis, because it is that location which gives an idea where the root locus is progressing- towards right or towards left. It is also called near poles.

In fact, there is a region of boundary for considering the significant and insignificant region. If we considering a pole at σ1, then its insignificant pole, say σ2, must be 5 or 10 times far away from of that σ1



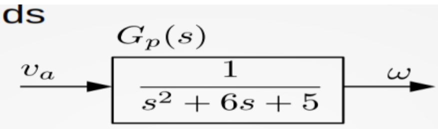
In root locus plot, contribution of σ1 is more helpful than the farther σ2 pole. Adding more nearer poles stabilizes the system.

For example

The transfer function representing the system is then

$$G_p(s) = \frac{\omega(s)}{v_a(s)} = \frac{1}{s^2+6s+5}$$

Which corresponds

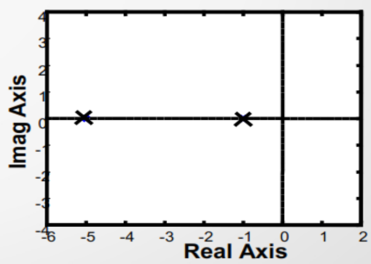


open loop ch. eq.

$$\Delta(s) = s^2+6s+5 = (s+1)(s+5)$$

Poles are located at $s=1$ and $s=5$

Dominant pole



Def : Poles closest to the $j\omega$ axis are the dominant poles (if the system is stable). Dominant pole corresponds to the slowest mode

Reduction of a second order system to first order:

Consider an overdamped second order system (and its step response).

$H(s) = K \frac{\alpha\cdot\beta}{(s+\alpha)(s+\beta)}$ If the magnitude of β is very large compared to α

we can write approximations for the transfer function (assuming s is sufficiently small comparted to β), as well as an approximation for the step response.

$$H(s) \approx K \frac{\alpha \cdot \beta}{(s+\alpha)(\beta)} = K \frac{\alpha}{(s+\alpha)}$$

$$H_{dp}(s) = K \frac{\alpha}{(s+\alpha)}$$

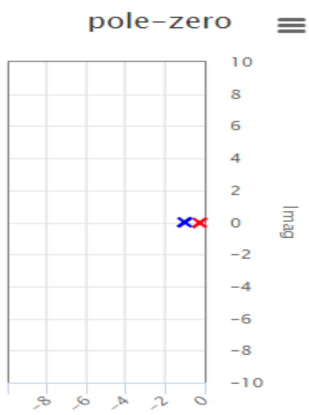
Note that $H(0)$ is unchanged for the exact and approximate transfer functions. where $H_{dp}(s)$ represents the dominant pole approximation. Note that the numerator of the approximation is chosen such that $H(0)=H_{dp}(0)$.

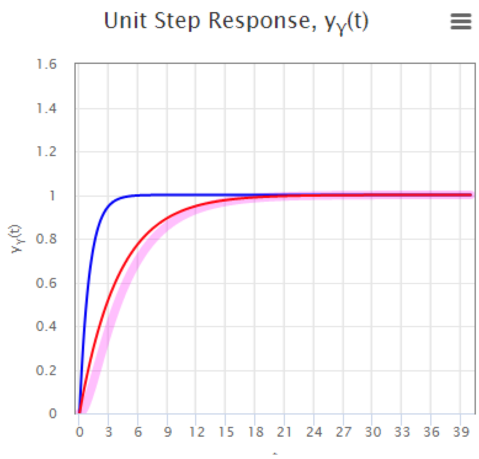
Example 2, Second order:

$$H(s) = K \frac{0.2}{(s+0.2)(s+1)}$$

$$H_{dp}(s) = K \frac{0.2}{(s+0.2)}$$

Pole-Zero plot: On the pole-zero plot the pole at $s=-\alpha$ is shown in red, and that for $s=-\beta$ is blue. Since α is so much closer to the origin in dominates the response. Note that when we say the poles are far apart, it is not physical distance that is of interest, but the ratio of the pole locations.



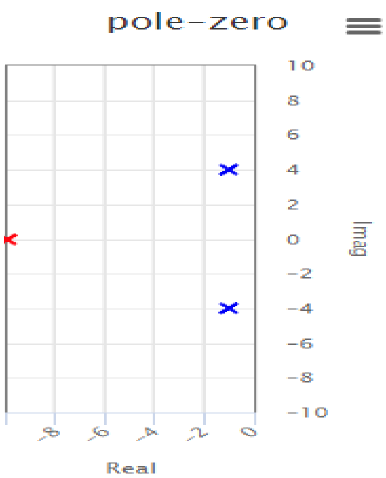


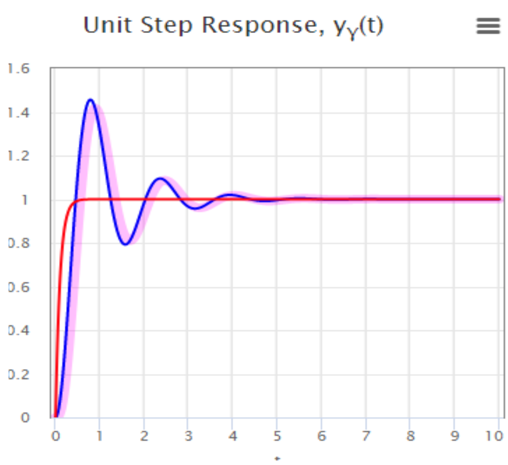
Time domain: The step response plot shows three plots: the magenta plot is the exact response, the red plot is the approximation assuming the pole at $-\alpha$ dominates (note that the red and magenta plots are very close to each other, so the dominant pole approximation is a good one), and the blue plot is the approximation assuming that the pole at $-\beta$ dominates. The exact response has two exponentials, a fast one with a relatively short time constant of $1/\beta$ and a much slower exponential with time constant $1/\alpha$. If we look at the overall resonse, the fast exponential comes to equilibrium much more quickly than the slow explonential. From the perspective of the overall response, the faster exponential comes to equilibrium (i.e., has decayed to zero) instantaneously compared to the slower exponential. Therefore, the slower response (due to the pole closer to the origin — at $s=-\alpha$) *dominates*.

Example 6: Third order, complex poles dominate

$$H(s) = \frac{10*17}{(s+10)(s^2+2s+17)}, \quad H_{dp}(s) = \frac{17}{(s^2+2s+17)}$$

Note that the numerator of the approximation is chosen such that $H(0)=H_{dp}(0)$.





Numerical example:

Consider the transfer function $H(s) = \frac{20}{(s+3)(s+30)}$

Since we have one pole at s=-3, and one pole at s=-30, the pole at s=-3 will dominate, so the denominator of the transfer function is (s+3) and the dominant pole approximation has the form

$$H_{dp}(s) = \frac{?}{(s+3)}$$

In order to find the correct value for the numerator we set $H(0)=H_{dp}(0)$.

$$H(0) = \frac{20}{(3)(30)} = \frac{2}{9}$$

Note: this ensures that the final value to a step input is equal for the exact and approximate systems.
In order for this equation to hold, the numerator of the approximation must be equal to 2/3, so

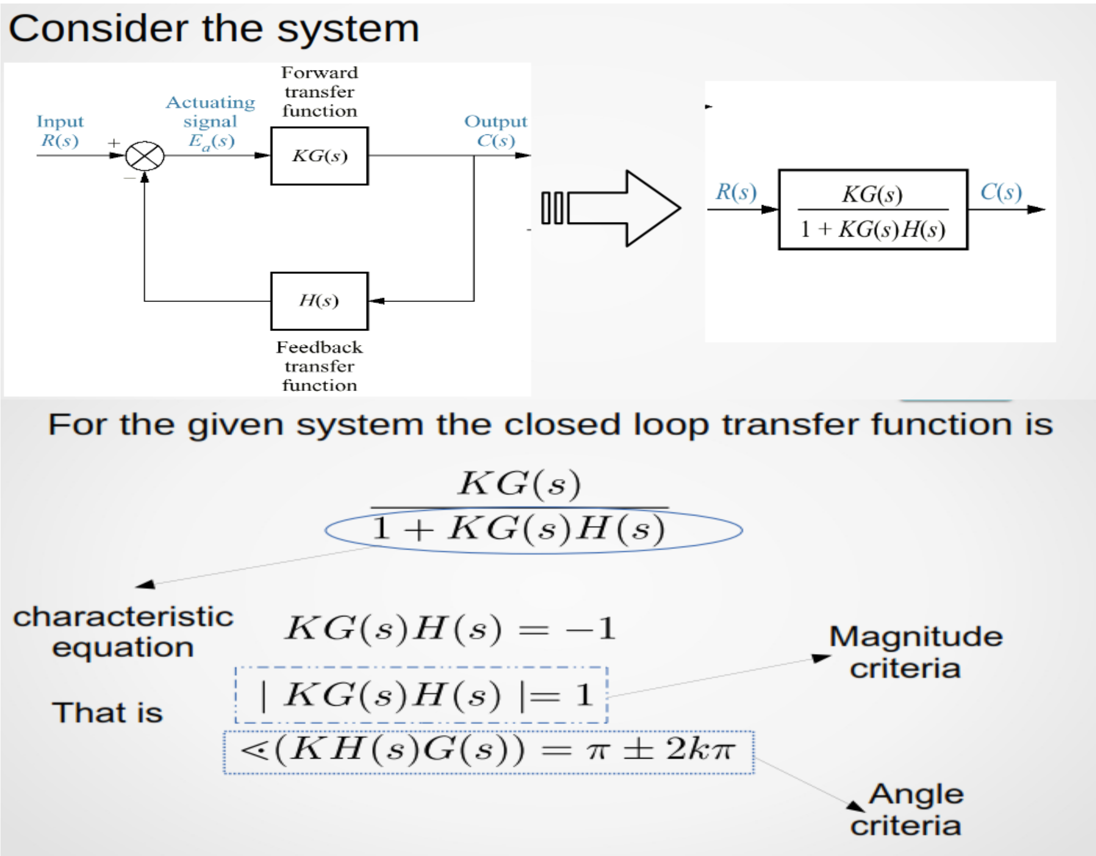
$$H_{dp}(s) = \frac{2}{(3)(s+3)}, \quad H_{dp}(0) = \frac{2}{(3)(3)}$$

<u>Application Of Root Locus.</u>

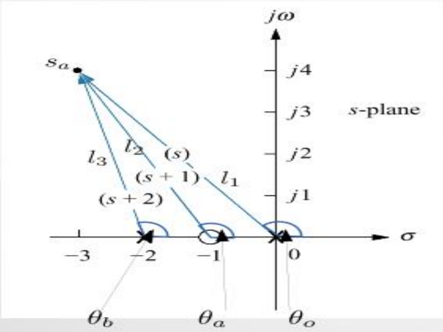
What is the application of root locus?

This is a technique used as a stability criterion in the field of classical control theory developed by Walter R. Evans which can determine stability of the system. The root locus plots the poles of the closed **loop** transfer function in the complex s-plane as a function of a gain parameter (pole–zero plot)

The effects of gains on the system response, overshoot and the stability can be determined.



Consider $G(s)H(s) = \dfrac{s+1}{s(s+2)}$



Magnitude criteria $\Big\}$ $\dfrac{|s_a+1|}{|s_a||(s_a+2)|} = 1$

Angle criteria $\Big\}$ $\theta_a - \theta_o - \theta_b = \pi \pm 2k\pi$

Design using mag. and angle cond.

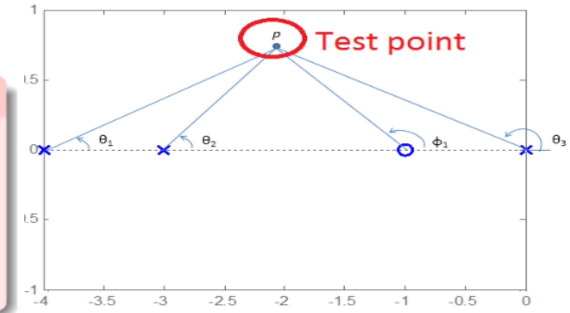
For Example.

$$G(s)H(s) = \frac{s+1}{s(s+3)(s+4)}$$

Angle Condition

If angle of $G(s)H(s)$ at $s = p$ is equal to $\pm 180^o(2k+1)$, the point p is on root locus.

$$\angle G(s)H(s)\big|_{s=p} = \phi_1 - \theta_1 - \theta_2 - \theta_3$$

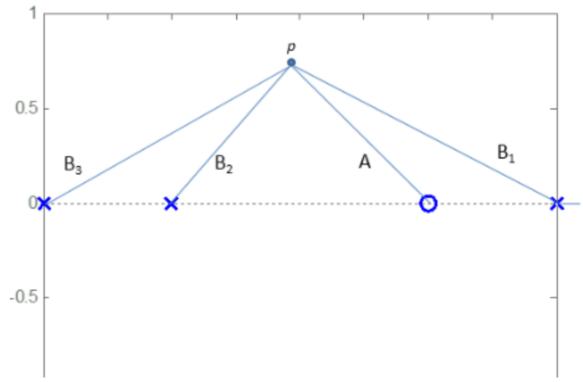


Angles are measured **counterclockwise**

Magnitude Condition

If lengths A, B_1, B_2 and B_3 measured from the test point to the poles and zeros satisfy the magnitude condition, the point p is on root locus.

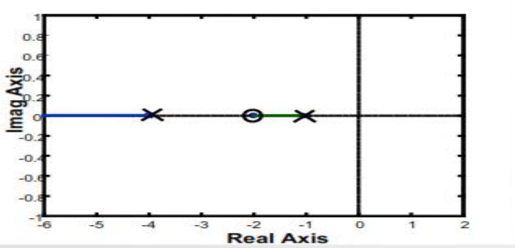
$$|G(s)H(s)|_{s=p} = \frac{|s+1|_{s=p}}{|s|_{s=p}\,|s+3|_{s=p}\,|s+4|_{s=p}} = \frac{A}{B_1B_2B_3}$$



Example : Find the value of K which places closed loop pole at -5 for the system

$$G_p(s) = \frac{s+2}{(s+1)(s+4)}$$

Characteristic polynomial $1 + K\dfrac{s+2}{(s+1)(s+4)}$



The root locus

Note that -5 lies on the root locus

Magnitude Condition

$$\boxed{K \mid G_p(s) \mid_{s=-5} = 1} \Rightarrow K \left| \frac{s+2}{(s+1)(s+4)} \right|_{s=-5} = 1$$

$$\Rightarrow K \left| \frac{-3}{(-4)(-1} \right| = 1 \quad \Rightarrow K = \frac{4}{3}$$

Angle condition

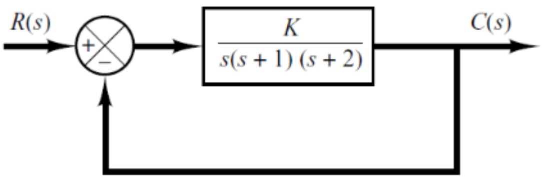
$$\boxed{Angle[G_p(s)] \mid_{s=-5} = -180}$$

$$\Rightarrow Angle(s+2) - Angle(s+4) - Angle(s+1) \mid_{s=-5} = -180$$

$$\Rightarrow Angle(-3) - Angle(-1) - Angle(-4) = -180$$

$$\Rightarrow 180 - (180 + 180) = -180 \quad \text{Angle condition satisfied}$$

(Example):



$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Angle Condition:

$$\angle G(s)H(s) = \angle \frac{K}{s(s+1)(s+2)}$$

$$\angle G(s)H(s) = \angle K - \angle s - \angle(s+1) - \angle(s+2)$$

$$\boxed{\angle K - \angle s - \angle(s+1) - \angle(s+2) = \pm 180^\circ(2k+1)}$$

Magnitude Condition:

$$\boxed{|G(s)H(s)| = \left| \frac{K}{s(s+1)(s+2)} \right| = 1}$$

(Example):

check whether $s = -0.25$ is on the root locus or not

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Angle Condition

$$\angle G(s)H(s)\big|_{s=-0.25} = \angle K\big|_{s=-0.25} - \angle s\big|_{s=-0.25} - \angle(s+1)\big|_{s=-0.25} - \angle(s+2)\big|_{s=-0.25}$$

$$\angle G(s)H(s)\big|_{s=\ 0.25} = -\angle(-0.25) - \angle(0.75) - \angle(1.75)$$

$$\angle G(s)H(s)\big|_{s=-0.25} = -180^\circ - 0^\circ - 0^\circ$$

$$\angle G(s)H(s)\big|_{s=-0.25} = \pm 180^\circ(2k+1)$$

$s = -0.25$ is on the root locus

Magnitude Condition

Now we know from angle condition that the point $s = -0.25$ is on the rot locus. But we will find the gain that satisfy the magnitude condition:

$$\left|\frac{K}{s(s+1)(s+2)}\right|_{s=-0.25} = 1$$

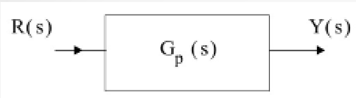
$$\left|\frac{K}{(-0.25)(-0.25+1)(-0.25+2)}\right|_{s=-0.25} = 1$$

$$\left|\frac{K}{(-0.25)(0.75)(1.75)}\right| = 1$$

$$K = 0.328$$

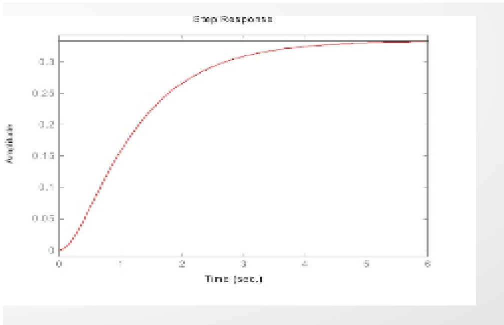
Open Loop Response

Note that for the open loop system



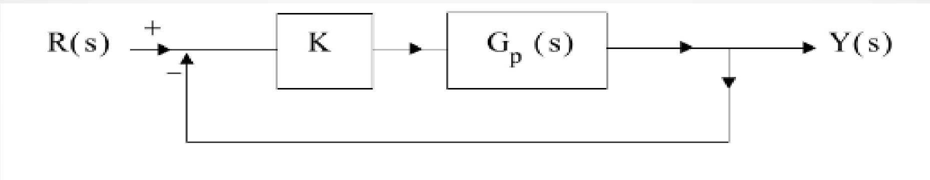
$$G_p(s) = \frac{1}{(s+1)(s+3)}$$

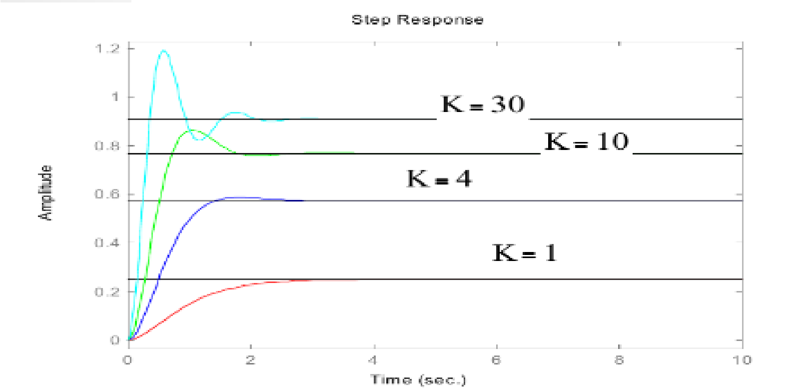
For the unit step input we have



Closed Loop Response

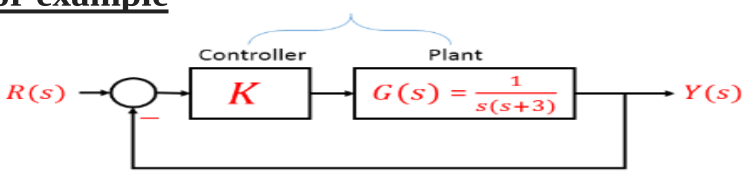
The step response of the closed loop system is





As the values of K changes the transient response changes

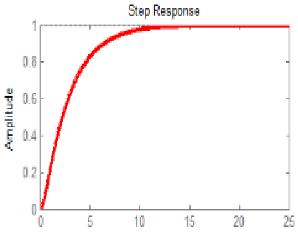
For example



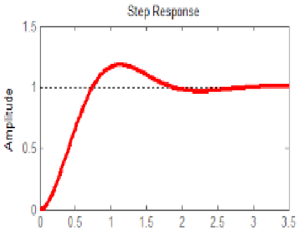
Proportional feedback controller

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)} = \frac{K}{s^2+3s+K}$$

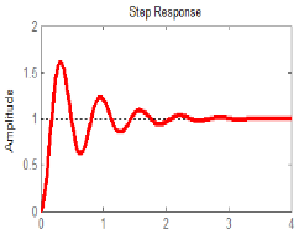
We want to examine how the behavior of the system varies as K changes, so let's try several values of K. Let's arbitrarily try K=1, 10 and 100 so that we have a wide range of K values.



$$\frac{Y(s)}{R(s)} = \frac{1}{s^2+3s+1} \quad \textbf{(K=1)}$$



$$\frac{Y(s)}{R(s)} = \frac{10}{s^2+3s+10} \quad \textbf{(K=10)}$$



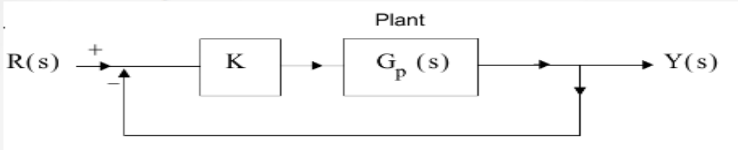
$$\frac{Y(s)}{R(s)} = \frac{100}{s^2+3s+100} \quad \textbf{(K=100)}$$

<u>Root Locus In Matlab:</u>

What is root locus in Matlab?

rlocus(sys) calculates and plots the **root locus** of the SISO model sys . The **root locus** returns the closed-loop pole trajectories as a function of the feedback gain k (assuming negative feedback). **Root loci** are used to study the effects of varying feedback gains on closed-loop pole locations.

Consider the system



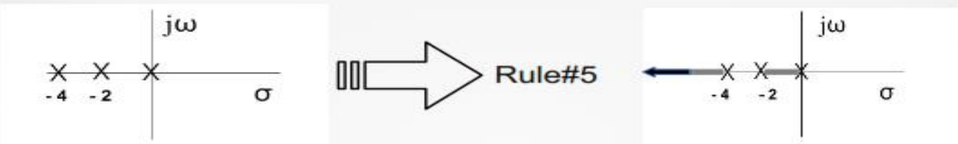
where $$G_p(s) = \frac{1}{s(s+2)(s+4)}$$

Open loop poles are at $s=0, s=-2, s=4$

Closed loop transfer function $$\frac{Y(s)}{R(s)} = \frac{K\frac{1}{s(s+2)(s+4)}}{1+K\frac{1}{s(s+2)(s+4)}}$$

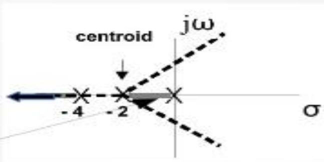
Characteristic Eq. $$\Delta(s) = 1 + K\frac{1}{s(s+2)(s+4)}$$

<u>Step 1 :</u> Pole Zero plot



<u>Step 2 :</u> Centroids and Asymptotes (RD=3)

$$\text{Centroid} = \frac{0-2-4-0}{3} = -2$$



<u>Step 3 :</u> Break away point

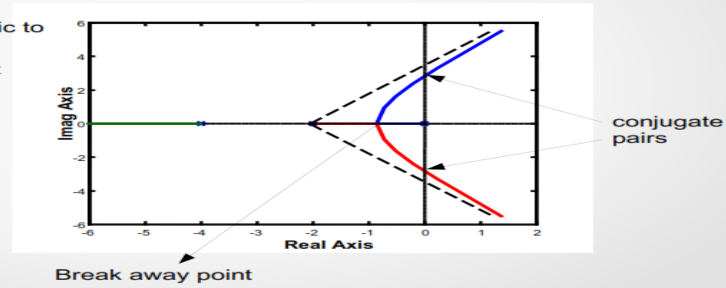
(We might not need this :) .. Why?)

Step 4 : Plot the root locus

- Locus must be symmetric to real axis
- 3 open loop zeros are at infinity

matlab code

```
figure;
num = [1];
denum = [1 6 8 0];
rlocus(num,denum);
```



Another matlab code
num = [1];
den = conv(conv([1 0],[1 2]),[1 4]);
sys = tf(num,den);
rlocus (sys);

Same system with

$$G_p(s) = \frac{1}{(s+1)(s+3)}$$

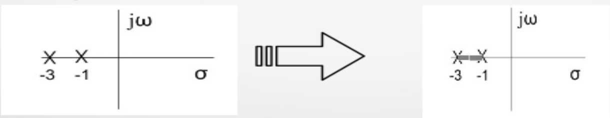
open loop poles at $s = -1, -3$

closed loop transfer function $\frac{Y(s)}{R(s)} = \frac{K\frac{1}{(s+1)(s+3)}}{1+K\frac{1}{(s+1)(s+3)}}$

characteristic equation

$$\Delta(s) = 1 + K\frac{1}{(s+1)(s+3)}$$

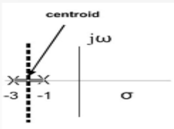
pole zero plot



Centroid and asymptotes

centroid = $\frac{-3-1}{2} = -2$

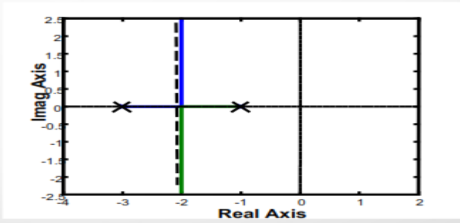
RD = 2 – 0 = 2



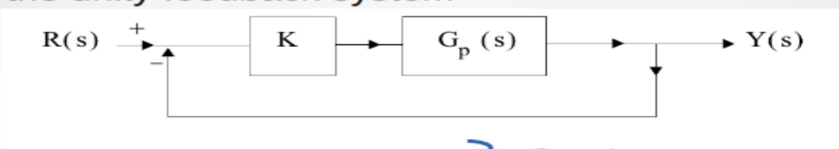
Draw the root locus plot

matlab code

```
figure;
num = [1];
denum = [1 4 3];
rlocus(num,denum);
```



Given the unity feedback system



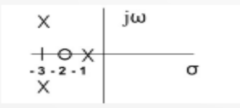
with

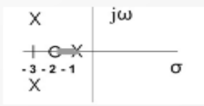
$$G_p(s) = \frac{s+2}{(s+1)(s^2+6s+18)}$$

Open loop zeros $s = -2$

Open loop poles $s = -1, -3 \pm 3j$

Pole Zero plot

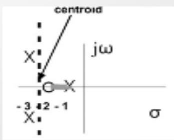




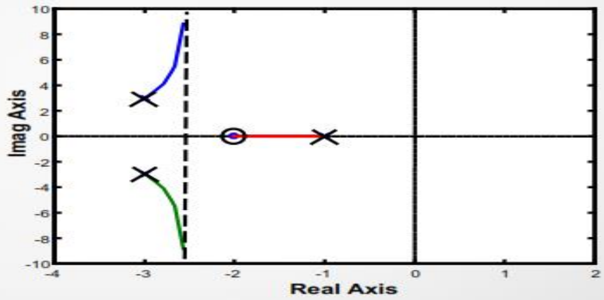
Centroid and Asymptotes

centroid = $\frac{-3-3-1-(-2)}{2} = -\frac{5}{2}$

RD = 2



Draw the root locus obeying the rules defined



matlab code

```
figure;
num = [1 2];
denum = [1 7 24 18];
rlocus(num,denum);
```

Another method (matlab code) to plot the root locus in matlab,
matlab code
num = [1 2];
den = conv([1 1],[1 6 18]);
sys = tf(num,den);
rlocus (sys);

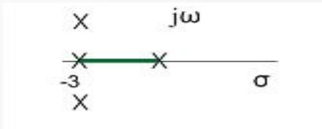
Same block diagram with

$$G_p(s) = \frac{1}{s(s+3)(s+3-j7.4)(s+3+j7.4)}$$

Characteristic polynomial

$$\Delta(s) = 1 + K\frac{1}{s(s+3)(s+3-j7.4)(s+3+j7.4)}$$

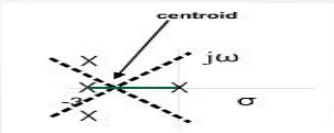
Pole Zero Plot



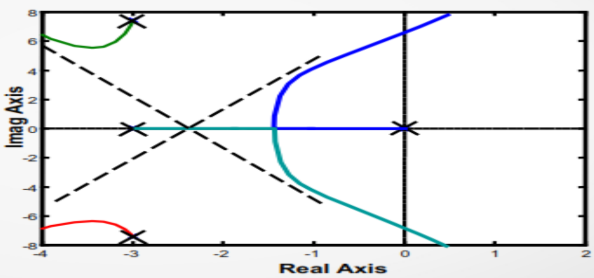
Centroid and Asymptotes

$$\text{centroid} = \frac{-3-3-3-(0)}{4} = \frac{-9}{4} = -2.25$$

RD = 4



Draw the root locus obeying the rules defined



matlab code

```
figure;
num = [1];
denum = [1 9 82 192 0];
rlocus(num,denum);
```

<u>Exercise :</u> plot the root locus in matlab for the following system.

$$\frac{1}{s(s+1)(s+5)(s+10)}$$

```
a=conv([1 0], [1 1])
b=conv([1 5], [1 10])
D=conv(a,b)
N=1
sys=tf(N, D)
rlocus(sys)
```