

changing Cartesian Integrals into polar Integrals :

The procedure for changing Cartesian Integral $\iint_R f(x,y) dA$ into a polar integral has two steps :

step 1 : substitute

$$x = r \cos \theta, \quad y = r \sin \theta$$

and replace $dx dy$ by

$$dx dy = r dr d\theta$$

in the Cartesian integral.

step 2 : supply polar limits of integration for boundary of R .

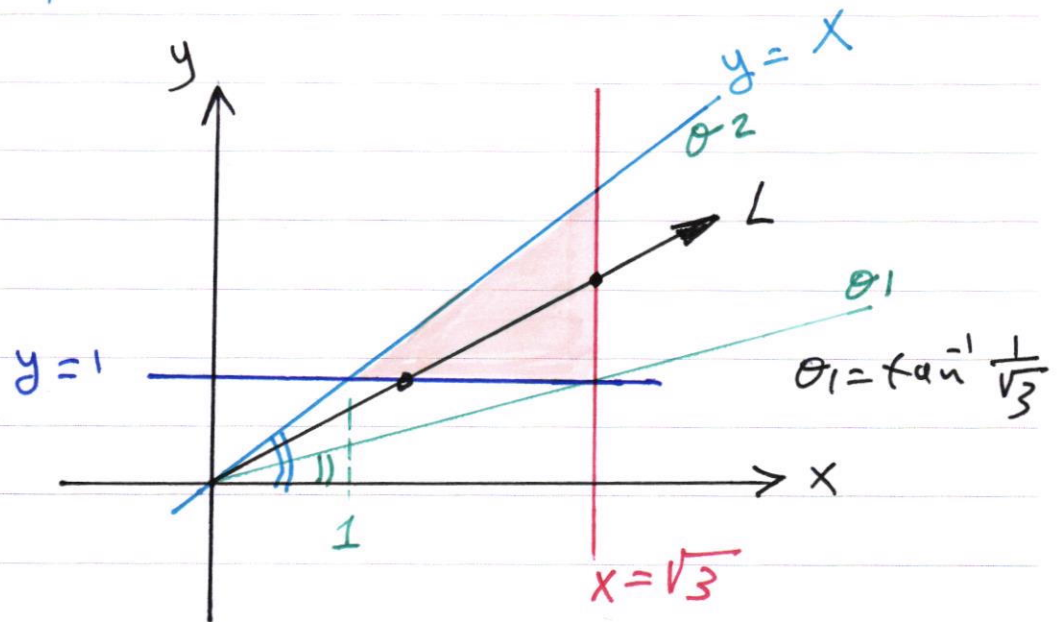
The Cartesian integral becomes :

$$\iint_R f(x,y) dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

Ex1: Change the Cartesian integral into an equivalent polar integral, then solve it.

$$\int_1^{\sqrt{3}} \int_1^x dy dx$$

solution:



$$\int_{x=1}^{x=\sqrt{3}} \int_{y=1}^{y=x} dy dx = \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{4}} \int_{r=\csc\theta}^{r=\sqrt{3}\sec\theta} r dr d\theta$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{\csc\theta}^{\sqrt{3}\sec\theta} r dr d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{3}{2} \sec^2\theta - \frac{1}{2} \csc^2\theta \right) d\theta$$

$$= \left[\frac{3}{2} \tan\theta + \frac{1}{2} \cot\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = 2 - \sqrt{3}$$

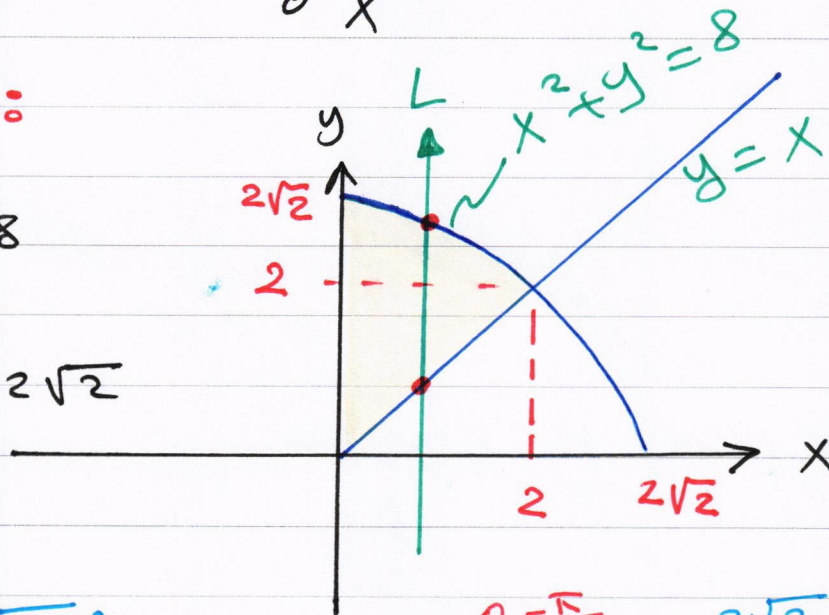
Ex 2 : Use polar coordinates to evaluate the double integral $\int_0^2 \int_{x^2}^{\sqrt{8-x^2}} \cos(x^2+y^2) dy dx$

Solution :

$$x^2 + y^2 = 8$$

$$r^2 = 8$$

$$r = \sqrt{8} = 2\sqrt{2}$$



$$x=2 \quad y=\sqrt{8-x^2}$$

$$\theta = \frac{\pi}{2} \quad r = 2\sqrt{2}$$

$$\iint \cos(x^2+y^2) dy dx = \iint \cos(r^2) r dr d\theta$$

$$x=0 \quad y=x$$

$$\theta = \frac{\pi}{4} \quad r=0$$

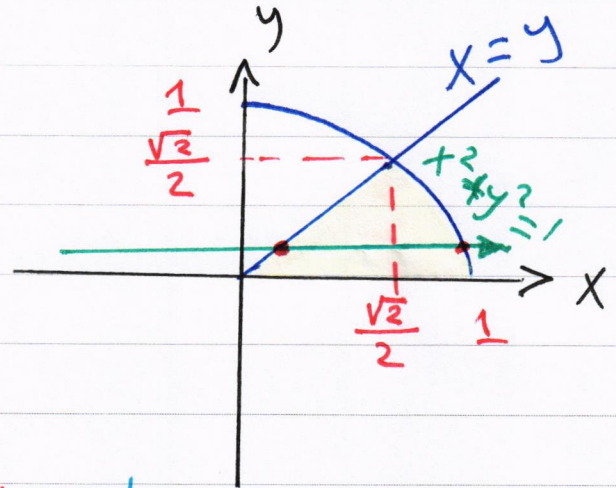
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} \int_0^{2\sqrt{2}} 2r \cos(r^2) dr d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} \sin r^2 \Big|_0^{2\sqrt{2}} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} [\sin 8 - \cancel{\sin 0}] d\theta = \frac{1}{2} \sin 8 \theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{8} \sin 8$$

Ex 3: Evaluate the integral

$$\int_0^{\frac{\sqrt{2}}{2}} \int_y^{\sqrt{1-y^2}} \frac{y^2}{\sqrt{x^2+y^2}} dx dy$$

Solution:



$$\iint_{y=0, x=y}^{y=\frac{\sqrt{2}}{2}, x=\sqrt{1-y^2}} \frac{y^2}{\sqrt{x^2+y^2}} dx dy = \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r=0}^{r=1} \frac{r^2 \sin^2 \theta}{r} r dr d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^1 r^2 \sin^2 \theta dr d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{r^3 \sin^2 \theta}{3} \right]_0^1 d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{4}} \sin^2 \theta d\theta = \frac{1}{3} \int_0^{\frac{\pi}{4}} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

$$= \frac{1}{3} \left[\frac{1}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \right]_0^{\frac{\pi}{4}} = \frac{\pi - 2}{24}$$

EX 4: Using polar coordinates, evaluate the integral $\iint_R (x^2 + y^2) dx dy$

where $R: x^2 + y^2 - 2y \leq 0$

Solution: $y = r \sin \theta, x = r \cos \theta$

$$x^2 + y^2 - 2y = 0$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r \sin \theta = 0$$

$$r^2 (\cancel{\cos^2 \theta} + \sin^2 \theta) = 2r \sin \theta$$

$$r^2 = 2r \sin \theta$$

$$r = 2 \sin \theta$$



$$\iint_R (x^2 + y^2) dx dy = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} r^2 \cdot r dr d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} r^3 dr d\theta = \int_0^{\pi} \left[\frac{r^4}{4} \right]_0^{2\sin\theta} d\theta$$

$$= \int_0^{\pi} \frac{(2\sin\theta)^4}{4} d\theta = 4 \int_0^{\pi} \sin^4\theta d\theta$$

$$= 4 \int_0^{\pi} \frac{1}{2} (1 - \cos 2\theta) \cdot \frac{1}{2} (1 - \cos 2\theta) d\theta$$

$$= \int_0^{\pi} [1 - 2\cos 2\theta + \cos^2 2\theta] d\theta$$

$$= \int_0^{\pi} [1 - 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)] d\theta$$

$$= \int_0^{\pi} [1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2} \cos 4\theta] d\theta$$

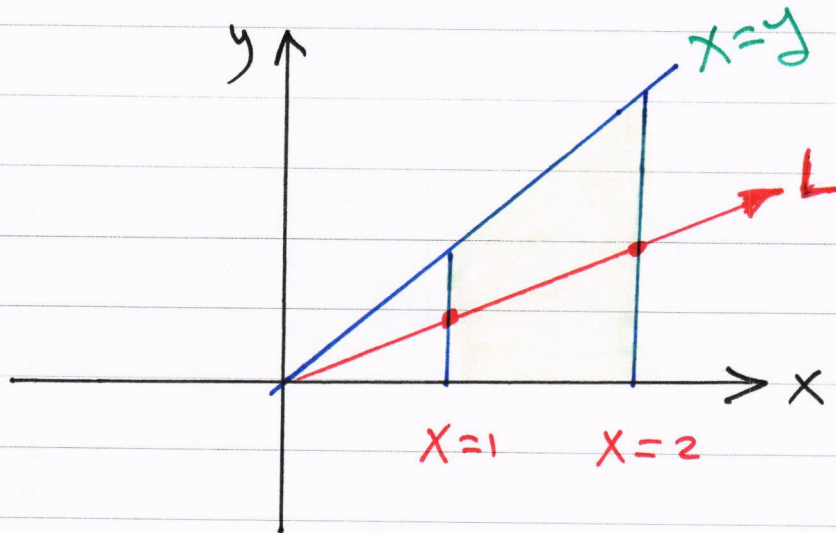
$$= \int_0^{\pi} \left[\frac{3}{2} - 2\cos 2\theta + \frac{1}{2} \cos 4\theta \right] d\theta$$

$$= \left[\frac{3}{2} \theta - \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_0^{\pi}$$

$$= \frac{3\pi}{2}$$

Ex 5 : Compute $\int_1^2 \int_0^x \frac{1}{(x^2+y^2)^{3/2}} dy dx$

by switching to polar coordinates.



$$x=1 \rightarrow r \cos \theta = 1 \rightarrow r_1 = \sec \theta$$

$$x=2 \rightarrow r \cos \theta = 2 \rightarrow r_2 = 2 \sec \theta$$

$$\int_{x=0}^{x=2} \int_{y=0}^{y=x} \frac{1}{(x^2+y^2)^{3/2}} dy dx = \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r_1=\sec \theta}^{r_2=2 \sec \theta} \frac{1}{r^3} r dr d\theta$$

$$= \int_0^{\pi/4} \int_{\sec \theta}^{2\sec \theta} \frac{1}{r^2} dr d\theta$$

$$= \int_0^{\pi/4} \left[-\frac{1}{r} \right]_{\sec \theta}^{2\sec \theta} d\theta$$

$$= \int_0^{\pi/4} \left[-\frac{1}{2\sec \theta} + \frac{1}{\sec \theta} \right] d\theta$$

$$= \int_0^{\pi/4} \left[\frac{1}{2} \frac{1}{\sec \theta} \right] d\theta$$

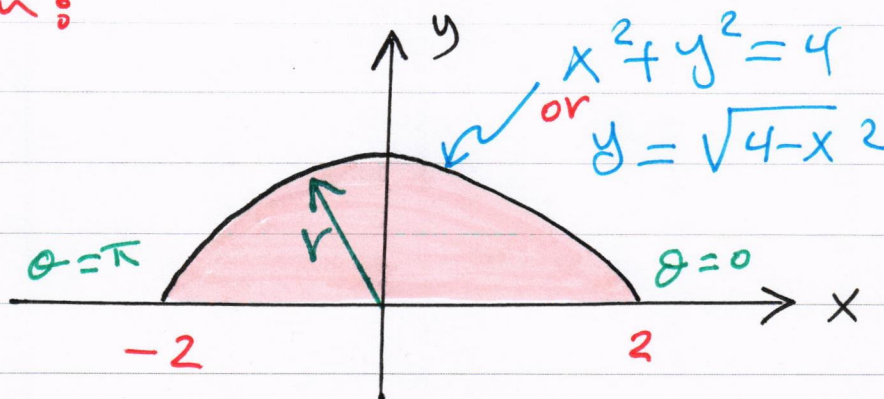
$$= \int_0^{\pi/4} \frac{1}{2} \cos \theta d\theta$$

$$= \frac{1}{2} \sin \theta \Big|_0^{\pi/4} = \frac{\sqrt{2}}{4}$$

Ex 6 : Evaluate $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} e^{-(x^2+y^2)} dy dx$

by converting to an equivalent integral in polar coordinates.

solution :



$$\int_{x=-2}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} e^{-(x^2+y^2)} dy dx = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2} e^{-r^2} \cdot r dr d\theta$$

$$= \int_0^\pi \left[-\frac{1}{2} \right]_{r=0}^2 -2r e^{-r^2} \cdot dr d\theta = \int_0^\pi -\frac{1}{2} [e^{-r^2}]_0^2 d\theta$$

$$= \int_0^\pi -\frac{1}{2} [e^{-4} - (e^0)] d\theta$$

$$= \int_0^{\pi} \frac{e^0 - e^{-4}}{2} d\theta$$

$$= \int_0^{\pi} \frac{1 - e^{-4}}{2} d\theta = \left(\frac{1 - e^{-4}}{2} \right) \theta \Big|_0^{\pi}$$

$$= \left(\frac{1 - e^{-4}}{2} \right) \pi$$

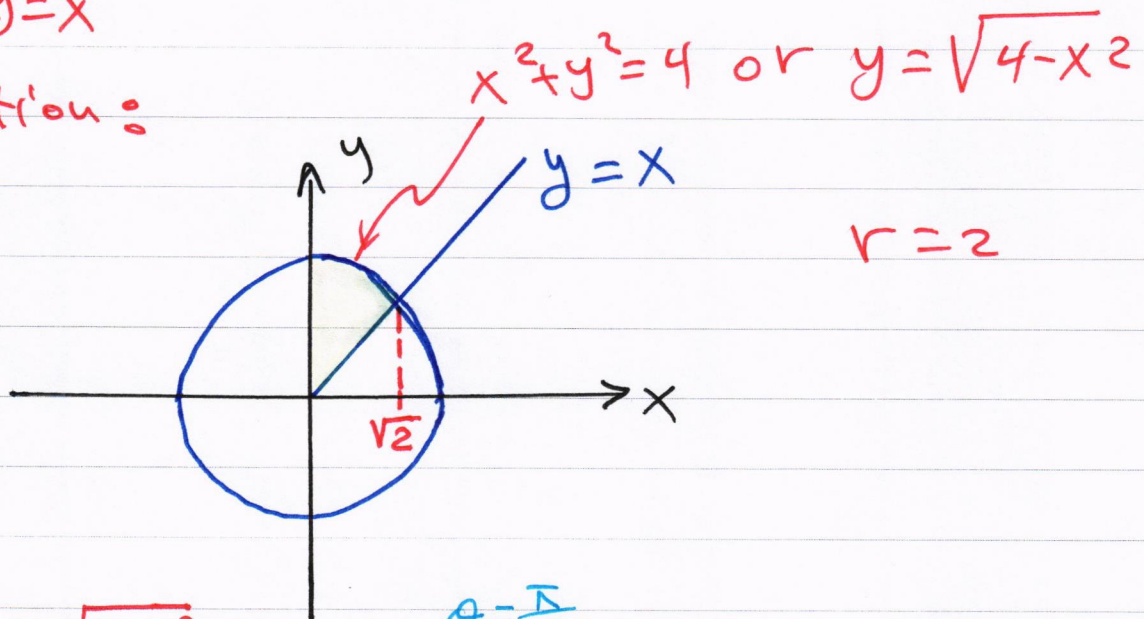
Ex 7: Convert the integral to an integral in polar coordinates and then compute the integral.

$$x = \sqrt{2} \quad y = \sqrt{4-x^2}$$

$$\iint xy \, dy \, dx$$

$$x=0 \quad y=x$$

Solution:



$$x = \sqrt{2} \quad y = \sqrt{4-x^2}$$

$$\theta = \frac{\pi}{2} \quad r = 2$$

$$\iint xy \, dy \, dx = \int \int r \cos \theta \cdot r \sin \theta \cdot r \, dr \, d\theta$$

$$x=0 \quad y=x$$

$$\theta = \frac{\pi}{4} \quad r = 0$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 r^3 \sin \theta \cos \theta \, dr \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 r^3 \frac{\sin(2\alpha)}{2} dr d\alpha$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^4}{8} \right]_0^2 \sin(2\alpha) d\alpha$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2 \sin(2\alpha) d\alpha = -\cos 2\alpha \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

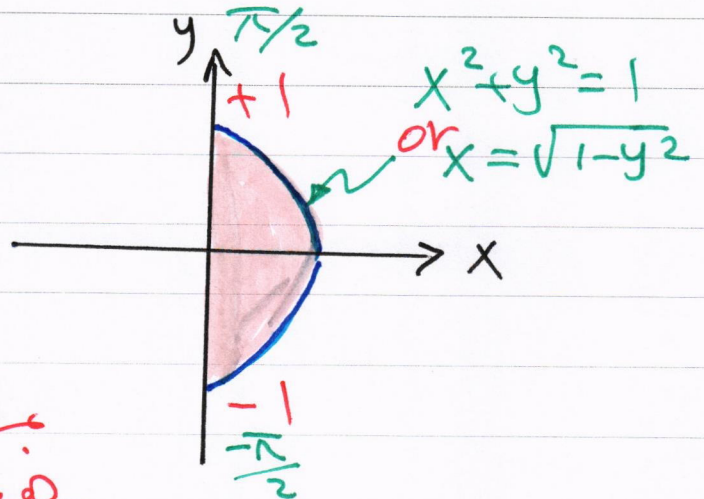
$$= [-\cos(180) - (-\cos 90)] = 1$$

Ex 8: Evaluate $\iint_R xy^2 dA$, where R is enclosed by $x=0$ and $x=\sqrt{1-y^2}$

solution:

$$\text{at } x=0 \text{ to } x=\sqrt{1-y^2}$$

$$y = -1 \text{ to } 1$$



بما أنه لم يحدد الطريقة في هذا السؤال، فممكن حله بطريقتين:

Cartesian coordinates:

$$y = -1 \text{ to } 1, \quad x = 0 \text{ to } \sqrt{1-y^2}$$

$$\int_{y=-1}^1 \int_{x=0}^{\sqrt{1-y^2}} xy^2 dx dy = \int_{-1}^1 \left[\frac{x^2 y^2}{2} \right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \int_{-1}^1 \frac{1}{2} (y^2 - y^4) dy = \frac{1}{2} \left(\frac{1}{3} y^3 - \frac{1}{5} y^5 \right) \Big|_{-1}^1$$

$$= \frac{1}{6} (1^3 - (-1)^3) - \frac{1}{10} (1^5 - (-1)^5)$$

$$= \frac{1}{3} - \frac{1}{5} = \frac{2}{5}$$

polar coordinates : $\theta = \frac{\pi}{2}$ $r = 1$
 $y = +1$ $x = \sqrt{1-y^2}$

$$\iint xy^2 dx dy = \iint r \cos \theta \cdot r^2 \sin^2 \theta \cdot r dr d\theta$$

$y = -1$ $x = 0$ $\theta = -\frac{\pi}{2}$ $r = 0$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^4 \cos \theta \sin^2 \theta dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^5}{5} \right]_0^1 \cos \theta \sin^2 \theta d\theta$$

$$= \frac{1}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta \sin^2 \theta d\theta$$

$$= \frac{1}{5} \left[\frac{\sin^3 \theta}{3} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

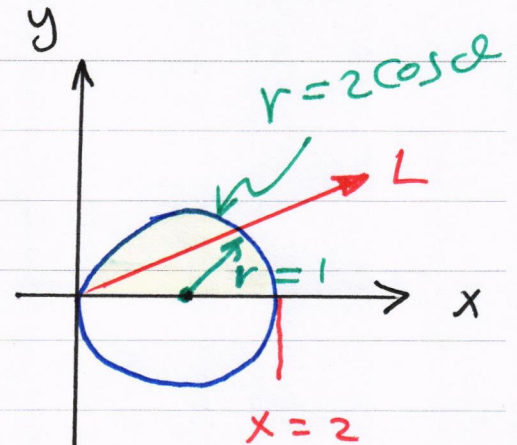
$$= \frac{1}{5} \left[\frac{\sin^3(\frac{\pi}{2})}{3} - \frac{\sin^3(-\frac{\pi}{2})}{3} \right]$$

$$= \frac{1}{5} \left[\frac{1}{3} - \left(-\frac{1}{3} \right) \right] = \frac{2}{15}$$

EX 9: Convert the integral to an integral in polar coordinates and then compute the integral.

Solution:

$$\int_{x=0}^x \int_{y=0}^{y=\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx$$



$$y = \sqrt{2x-x^2} \rightarrow x^2+y^2=2x$$

$$r^2 = 2r \cos \theta \rightarrow r = 2 \cos \theta$$

$$\int_{x=0}^x \int_{y=0}^{y=\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2 \cos \theta} r \cdot r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^{2 \cos \theta} d\theta = \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^3 \theta \, d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^2 \theta) \cos \theta \, d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} [\cos \alpha - \sin^2 \alpha \cos \alpha] d\alpha$$

$$= \frac{8}{3} \left[\sin \alpha - \frac{\sin^3 \alpha}{3} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{3} \left(1 - \frac{1}{3} \right) = \frac{16}{3}$$