Lecture No. 8

-Struts-

8-1 Introduction: -

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions. Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded. Long, slender columns or struts, however, fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one or more of the following reasons:

(a) The strut may not be perfectly straight initially;

(b) The load may not be applied exactly along the axis of the strut;

(c) one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties throughout the strut.

8-2 Euler's theory: -

a- <u>Strut with pinned ends: -</u>

Consider the axially loaded strut shown in Fig. 8.1 subjected to the crippling load P_e, producing a deflection y at a distance x from one end. Assume that the ends are either pin-jointed or rounded so that there is no moment at either end.

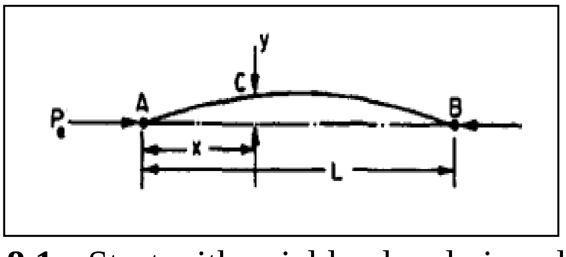


Figure 8.1:- Strut with axial load and pinned ends.

B.M. at C $= EI\frac{d^2y}{dx^2} = -P_e y$

$$EI\frac{d^2y}{dx^2} + P_e y = 0$$

$$\frac{d^2y}{dx^2} + \frac{P_e}{EI}y = 0$$

i.e. in operator form, with D = d / d x ,

$$(D^2 + n^2)y = 0 \;\; where\; n^2 = \frac{P_e}{EI}$$

This is a second-order differential equation which has a solution of the form

$$y = Acosnx + Bsinnx$$

$$y = Acos\sqrt{\frac{P_e}{EI}}x + Bsin\sqrt{\frac{P_e}{EI}}x \;,$$

B.C.

At x=0, y=0 $\quad \therefore A = 0$

And at x=L, y=0 $\therefore BsinL\sqrt{\frac{P_e}{EI}} = 0$

If B = 0 then y = 0 and the strut has not yet buckled. Thus the solution required is,

$$\therefore sinL\sqrt{\frac{P_e}{EI}} = 0 \;, \therefore L\sqrt{\frac{P_e}{EI}} = \pi$$

$$\boxed{\therefore P_e = \frac{\pi^2 EI}{L^2}} \qquad \text{....8.1}$$

It should be noted that other solutions exist for the equation,

$$sinL\sqrt{\frac{P_e}{EI}} = 0 \; i.e.\, sinnL = 0$$

The solution chosen of $n\,L = \pi$ is just one particular solution; the solutions $nL = 2\pi$, $3\pi, 5\pi$ etc. are equally as valid mathematically and they do.

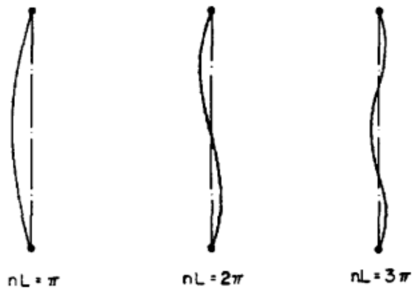


Figure 8.2:- Strut failure modes.

b- <u>One end fixed, the other free:</u> -

Consider now the strut of Fig. 8.3 with the origin at the fixed end.

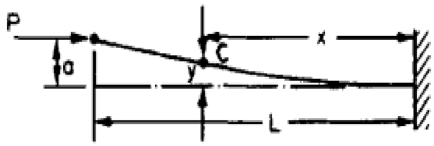


Figure 8.3: - Fixed-free strut.

B.M. at C $= EI\frac{d^2y}{dx^2} = +P(a-y)$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI} \qquad ....8.2$$

$$(D^2+n^2)y = n^2a$$

$$y = A\cos nx + B\sin nx + (particular\ solution)$$

The particular solution is a particular value of y which satisfies eqn. (8.2), and in this case can be shown to be y = a.

$$y = A\cos nx + B\sin nx + a$$

Now at x= 0, y=0 $\qquad \therefore A = -a$

and at x=0, $\frac{dy}{dx} = 0$ $\qquad \therefore$ B =0

$$\therefore y = -a\cos nx + a$$

But when x=L, y= a

$$a=-a \cos nL + a$$

$$0 = \cos nL$$

The fundamental mode of buckling in this case therefore is given when $n\,L = \frac{\pi}{2}$.

$$L\sqrt{\left(\frac{P}{EI}\right)} = \frac{\pi}{2}$$

$$\therefore P_e = \frac{\pi^2 EI}{4L^2} \qquad \text{....8.3}$$

c- Fixed ends: -

Consider the strut of Fig. 8.4 with the origin at the center.

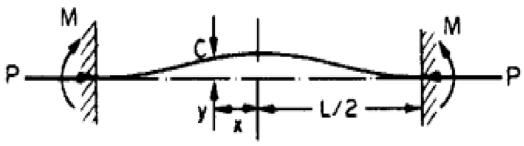


Figure 8.4: - Strut with fixed ends.

In this case the B.M. at C is given by,

B.M. at C $= EI\frac{d^2y}{dx^2} = M - Py$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{M}{EI}$$

$$(D^2+n^2)y=\frac{M}{EI}$$

Here the particular solution is

$$y=\frac{M}{n^2EI}=\frac{M}{P}$$

$$y=A\cos nx+B\sin nx+\frac{M}{P}$$

Now when $x=0$, $\frac{dy}{dx} = 0 \;\; \therefore B = 0$

And when $x=\frac{1}{2}L, y = 0 \;\; \therefore A = -\frac{M}{P}\sec\frac{nL}{2}$

$$y = -\frac{M}{P}\sec\frac{nL}{2}\cos nx + \frac{M}{P}$$

But when x=$\frac{1}{2}L, \frac{dy}{dx} = 0$

$$0 = \frac{nM}{P}\sec\frac{nL}{2}\sin\frac{nL}{2}$$

$$0 = \frac{nM}{P}\tan\frac{nL}{2}$$

The fundamental buckling mode is then given when $\frac{nL}{2} = \pi$

$$\frac{L}{2}\sqrt{\frac{P}{EI}} = \pi$$

$$\therefore P_e = \frac{4\pi^2 EI}{L^2} \quad \text{....8.4}$$

8-3 Comparison of Euler theory with experimental results (see Fig. 8.5)

Between L / k = 40 and L / k = 100 neither the Euler results nor the yield stress are close to the experimental values, each suggesting a critical load which is in excess of that which is actually required for failure - a very unsafe situation! Other formulae have therefore been derived to attempt to obtain closer agreement between the actual failing load and the predicted value in this particular range of slenderness ratio.

(a) Straight-line formula,

$$\mathbf{P = \sigma_y A [1 - n (L / k)]}$$

the value of (**n**) depending on the material used and the end condition.

(b) Johnson parabolic formula,

$$\mathbf{P = \sigma_y A [l - b (L/k)^2]}$$

the value of (**b**) depending also on the end condition.

Neither of the above formulae proved to be very successful, and they were replaced by:

(c) Rankine-Gordon formula,

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

where $\mathbf{P_e}$ is the Euler buckling load and $\mathbf{P_c}$ is the crushing (compressive yield) load = $\boldsymbol{\sigma_y}$ **A**. This formula has been widely used.

8-4 Euler "validity limit"

From the graph of Fig. 8.5 and the comments above, it is evident that the Euler theory is unsafe for small L/k ratios. It is useful, therefore, to determine the limiting value of L / k below which the Euler theory should not be applied; this is termed the validity limit.

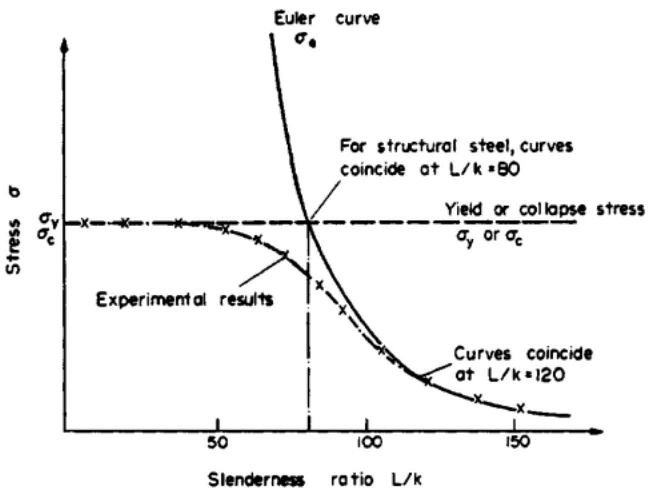


Figure 8.5: - Comparison of Experimental results with Euler curve.

The validity limit is taken to be the point where the Euler σ_e equals the yield or crushing stress σ_y, i.e. the point where the strut load

$$P = \sigma_y A$$

Now the Euler load can be written in the form,

$$P_e = C\frac{\pi^2 EI}{L^2} = C\frac{\pi^2 EAk^2}{L^2}$$

where C is a constant depending on the end condition of the strut.

Therefore in the limiting condition

$$\sigma_y A = C\,\frac{\pi^2 E A k^2}{L^2}$$

$$\frac{L}{k} = \sqrt{\left(\frac{C\pi^2 E}{\sigma_y}\right)}$$

The value of this expression will vary with the type of end condition; as an example, low carbon steel struts with pinned ends give L / k = 80.

8-5 Rankine or Rankine-Gordon formula,

As stated above, the Rankine formula is a combination of the Euler and crushing loads for a strut

$$\frac{\mathbf{1}}{\boldsymbol{P_R}} = \frac{\mathbf{1}}{\boldsymbol{P_e}} + \frac{\mathbf{1}}{\boldsymbol{P_c}}$$

*For very short struts P_e is very large; $1/P_e$ can therefore be neglected and $P_R = P_c$.

*For very long struts P_e is very small and $1/P_e$ is very large so that $1/P_c$ can be neglected. Thus The Rankine formula is therefore valid for extreme values of L/k. It is also found to be fairly accurate for the intermediate values in the range under consideration. Thus, re-writing the formula in terms of stresses,

$$\frac{\mathbf{1}}{\boldsymbol{\sigma A}} = \frac{\mathbf{1}}{\boldsymbol{\sigma_e A}} + \frac{\mathbf{1}}{\boldsymbol{\sigma_y A}}$$

$$\frac{\mathbf{1}}{\boldsymbol{\sigma}} = \frac{\mathbf{1}}{\boldsymbol{\sigma_e}} + \frac{\mathbf{1}}{\boldsymbol{\sigma_y}} = \frac{\boldsymbol{\sigma_e + \sigma_y}}{\boldsymbol{\sigma_e \sigma_y}}$$

$$\boldsymbol{\sigma} = \frac{\boldsymbol{\sigma_e \sigma_y}}{\boldsymbol{\sigma_e + \sigma_y}} = \frac{\boldsymbol{\sigma_y}}{\left[\mathbf{1} + \left(\frac{\boldsymbol{\sigma_y}}{\boldsymbol{\sigma_e}}\right)\right]}$$

For a strut with both **ends pinned**,

$$\boldsymbol{\sigma_e} = \frac{\boldsymbol{\pi^2 E}}{\left(\boldsymbol{L}/\boldsymbol{k}\right)^{\mathbf{2}}}$$

$$\sigma = \frac{\sigma_y}{1 + \frac{\sigma_y}{\pi^2 E}\left(L/_k\right)^2}$$

Rankine stress $\sigma_R = \frac{\sigma_y}{1+a\left(L/_k\right)^2}$

where a = $\frac{\sigma_y}{\pi^2 E}$, theoretically, but having a value normally found by experiment for various materials. This will take into account other types of end condition.

Therefore Rankine load $\boxed{P_R = \frac{\sigma_y A}{1+a\left(L/_k\right)^2}}$

Example 8-1:- A circular shaft of diameter 60 mm and its length is 1.5m. Given a factor of safety of 3, a compressive yield stress of 300 MN/m^2 and a constant (a) of 1/7500, determine the allowable load which can be carried by shaft according to the Rankine-Gordon formulae.

Sol.

$$I = \frac{\pi D^4}{64} = \frac{\pi(0.060)^4}{64} = 6.361 * 10^{-7} m^4$$

$$A = \frac{\pi D^2}{4} = \frac{\pi(0.060)^2}{4} = 2.827 * 10^{-3} m^2$$

$$\therefore\ k^2 = \frac{I}{A} = \frac{6.361 * 10^{-7}}{2.827 * 10^{-3}} = 2.2497 * 10^{-4} m^2$$

According to Rankine –Gordon formula

$$P_R = \frac{\sigma_y A}{1+a\left(L/_k\right)^2} = \frac{300*10^6*2.827*10^{-3}}{1+\frac{1}{7500}\left(\frac{1.5^2}{2.2497*10^{-4}}\right)} = 363.443 KN$$

With a factor of safety of 3 the maximum permissible load therefore becomes,

$$P_{max} = \frac{363.443}{3} = 121.147 KN.$$

Example 8-2:- In an experiment an alloy rod 1 m long and of 6 mm diameter, when tested as a simply supported beam over a length of 750 mm, was found to have a maximum deflection of 5.8 mm under the action of a central load of 5 N.

a- Find the Euler buckling load when this rod is tested as a strut, pin-jointed and guided at both ends.
b- What will be the central deflection of this strut when the material reaches a yield stress of 240 MN/m^2?

Note: - Take maximum stress $= \frac{P}{A} \pm \frac{My}{I}$ $where\ M = P\delta_{max}$

Sol.

$$I = \frac{\pi D^4}{64} = \frac{\pi(0.006)^4}{64} = 6.361 * 10^{-11} m^4$$

For simply supported beam with a concentrated load W at center,

$$\delta = \frac{WL^3}{48\,EI} = \frac{5(0.75)^3}{48(6.361 * 10^{-11})E} = 0.0058 \rightarrow E = 119.0994\ G\,N/m^2$$

$$But\ P_e = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(119.0994 * 10^9)(6.361 * 10^{-11})}{1^2} = 74.77N$$

Take maximum stress$= 240 * 10^6 = \frac{P}{A} + \frac{My}{I} = \frac{74.77}{\pi(0.003)^2} + \frac{74.77(\delta)(0.003)}{6.361*10^{-11}}$

$$240 * 10^6 = 2644447.8 + 3.526 * 10^9 \delta$$

$$\therefore\ \delta = 0.0673\ m$$

Example 8-3: - In tests it was found that a tube 2 m long, 50 mm outside diameter and 2 mm thick when used as a pin-jointed strut failed at a load of 43 kN. In a compression test on a short length of this tube failure occurred at a load of 115 kN.

a- Determine whether the value of the critical load obtained agrees with that given by the Euler theory.
b- Find from the test results the value of the constant a in the Rankine-Gordon formula. Assume E = 200 GN/m^2.

Sol.

$$A = \frac{\pi(D^2 - d^2)}{4} = \frac{\pi(0.050^2 - 0.046^2)}{4} = 3.016 * 10^{-4} m^2$$

$$I = \frac{\pi(D^4 - d^4)}{64} = \frac{\pi(0.050^4 - 0.046^4)}{64} = 8.7 * 10^{-8} m^4$$

$$\sigma_y = \frac{P}{A} = \frac{115 * 10^3}{3.016 * 10^{-4}} = 381.308 MPa.$$

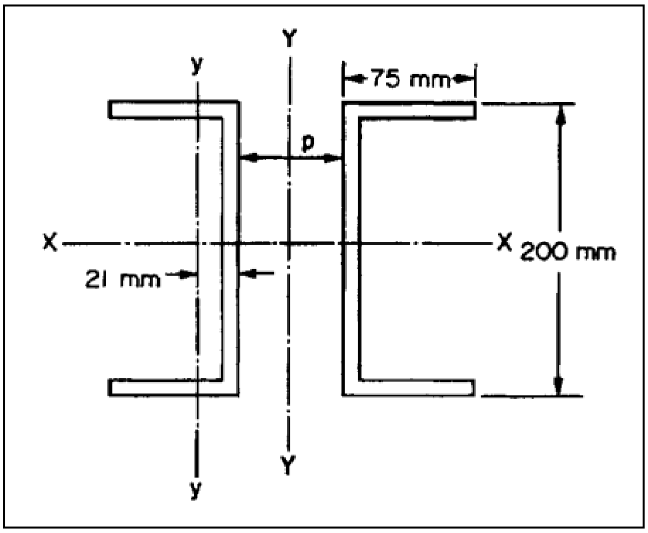
$$P_e = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (200*10^9)(8.7*10^{-8})}{2^2} = 42.937 kN$$ So, yes

a = $\frac{\boldsymbol{\sigma_y}}{\pi^2 E} = \frac{381.308*10^6}{\pi^2(200*10^9)} = \frac{1}{5176.7}$

Example 8-4: - A stanchion is made from two 200 mm x 75 mm channels placed back to back, as shown in figure below, with suitable diagonal bracing across the flanges. For each channel I_{xx} = 20*10^{-6} m^4, I_{yy} = 1.5*10^{-6} m^4, the cross-sectional area is 3.5*10^{-3} m^2 and the centroid is 21 mm from the back of the web.

At what value of p will the radius of gyration of the whole cross-section be the same about the X and Y axes?

The strut is 6 m long and is pin-ended. Find the Euler load for the strut and discuss briefly the factors which cause the actual failure load of such a strut to be less than the Euler load. E = 210 GN/m^2.



Sol.

For stanchion,

I_{xx}=2(20*10^{-6}) = 40*10^{-6} m^2

$I_{yy}=I_{yy}+Ad^2=2[1.5*10^{-6}+3.5*10^{-3}*(0.021+0.5p)^2]$

$I=Ak^2 \rightarrow k=\sqrt{\frac{I}{A}}$

$k_{xx}=k_{yy}$ and $A_{xx}=A_{yy}=A \quad \therefore I_{xx}=I_{yy}$

$\therefore 40*10^{-6}=2[1.5*10^{-6}+3.5*10^{-3}*(0.021+0.5p)^2]$

$\therefore$ p=0.1034 m

$$P_e=\frac{\pi^2 EI}{L^2}=\frac{\pi^2(210*10^9)(40*10^{-6})}{6^2}=2.303\ MN$$

……………………………………..End………………………………………