Transfer function from State Variable Representation

Assume a linear control system with an input vector u(t) and described by the state equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Taking Laplace transform for both sides, then

$$sX(s) - x(0) = AX(s) + B\,U(s)$$

$$X(s)[sI - A] = x(0) + B\,U(s)$$

In order to obtain the transfer function, the initial values $x(0)$ must be zero, then

$$X(s) = [sI - A]^{-1}B\,U(s) \qquad (4)$$

The output equation is

$$y(t) = Cx(t) + Du(t)$$

Taking Laplace transform

$$Y(S) = CX(S) + DU(S) \qquad (5)$$

Substituting by the value of X(S) given by (4) into the output equation (5), then

$$Y(S) = C\{[sI - A]^{-1}B\,U(s)\} + DU(S)$$

$$Y(S) = \{C\,[sI - A]^{-1}B + D\}\,U(S)$$

$$T.F.G(S) = \frac{Y(S)}{U(S)} = C\,[sI - A]^{-1}B + D \qquad (6)$$

Example 1

Consider the dynamic equation for the system with output y(t) and input u(t) shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\,u$$

Determine the system transfer function

First, we must calculate $[sI - A]^{-1}$

$$[sI - A]^{-1} = \begin{bmatrix} \frac{S+3}{(S+2)(S+1)} & \frac{1}{(S+2)(S+1)} \\ \frac{-2}{(S+2)(S+1)} & \frac{S}{(S+2)(S+1)} \end{bmatrix}$$

$$\frac{Y(S)}{U(S)} = C\,[sI - A]^{-1}B + D$$

$$\frac{Y(S)}{U(S)} = [0 \quad 1] \begin{bmatrix} \frac{S+3}{(S+2)(S+1)} & \frac{1}{(S+2)(S+1)} \\ \frac{-2}{(S+2)(S+1)} & \frac{S}{(S+2)(S+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [0]$$

$$\frac{Y(S)}{U(S)} = \begin{bmatrix} \frac{-2}{(S+2)(S+1)} \\ \frac{S}{(S+2)(S+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{S}{(S+2)(S+1)}$$

Example: Detail Algebra

$$T(s) = C(sI - A)^{-1} B + D$$

$$= [1 \quad 0]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [0]$$

$$= [1 \quad 0]\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 0]\left(\frac{1}{s(s+3)+2}\right)\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2}[1 \quad 0]\begin{bmatrix} 1 \\ s \end{bmatrix} = \frac{1}{s^2+3s+2}$$

Revue For Inverse Matrix

For a square matrix A, the inverse is written A^{-1}. When A is multiplied by A^{-1} the result is the identity matrix I. Non-square matrices do not have inverses.

Example: For matrix $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$, its inverse is $A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$ since

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $$A^{-1}A = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here are three ways to find the inverse of a matrix:

1. Shortcut for 2x2 matrices

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse can be found using this formula:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

2. Augmented matrix method

Use Gauss-Jordan elimination to transform [A | I] into [I | A^{-1}].

Example: The following steps result in $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

so we see that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$.

3. Adjoint method

$A^{-1} = \frac{1}{\det A}$(adjoint of A) or $A^{-1} = \frac{1}{\det A}$(cofactor matrix of A)^T

Example: The following steps result in A^{-1} for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$.

The cofactor matrix for A is $\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$, so the adjoint is $\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$. Since det A = 22, we get

$$A^{-1} = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

Example: Find the cofactor matrix of $\mathbf{A}$ given that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$.

Solution: First find the cofactor of each element.

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

The cofactor matrix is thus $\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$.

Example 2

Consider the dynamic equation for the system with output y(t) and input u(t) shown below:

The state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

The output equation:

$$\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

Calculate the transfer matrix.

$$SI - A = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -5 & -5 \end{bmatrix} = \begin{bmatrix} S & -1 & 0 \\ 0 & S+4 & -3 \\ 1 & 5 & S+5 \end{bmatrix}$$

$$adj(SI-A)=\begin{bmatrix}(S+4)(S+5)+15 & -3 & -(S+4)\\ --(S+5) & S(S+5) & -(5S+1)\\ 3 & --3S & S(S+4)\end{bmatrix}^T$$

$$\Delta = \det(SI-A)=S\{(S+4)(S+5)+15\}+3=S^3+9S^2+20S+15$$

$$[SI-A]^{-1}=\frac{1}{\Delta}\begin{bmatrix}S^2+9S+35 & S+5 & 3\\ 3 & S(S+5) & 3S\\ -(S+4) & -(5S+1) & S(S+4)\end{bmatrix}$$

$$\frac{C(S)}{R(S)}= C\,[sI-A]^{-1}B+D$$

$$\frac{C(S)}{R(S)}=\frac{1}{\Delta}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}S^2+9S+35 & S+5 & 3\\ 3 & S(S+5) & 3S\\ -(S+4) & -(5S+1) & S(S+4)\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\frac{C(S)}{R(S)}=\frac{1}{\Delta}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}S+5 & 3\\ S(S+5) & 3S\\ -(5S+1) & S(S+4)\end{bmatrix}$$

$$\frac{C(S)}{R(S)}=\frac{1}{\Delta}\begin{bmatrix}S+5 & 3\\ -(5S+1) & S(S+4)\end{bmatrix}$$

This means

$$\frac{C_1(S)}{R_1(S)}=\frac{S+5}{S^3+9S^2+20S+15}$$

$$\frac{C_1(S)}{R_2(S)}=\frac{3}{S^3+9S^2+20S+15}$$

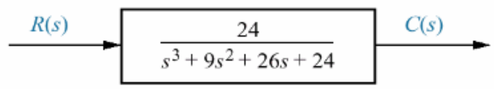
$$\frac{C_2(S)}{R_1(S)}=\frac{-(5S+1)}{S^3+9S^2+20S+15}$$

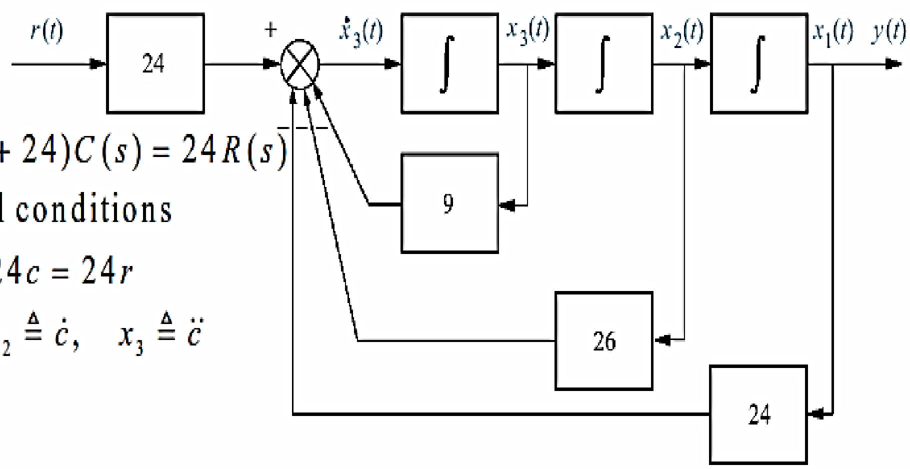
$$\frac{C_2(S)}{R_2(S)}=\frac{S(S+4)}{S^3+9S^2+20S+15}$$

Representation the system in block diagram

Example:
(Constant in the Numerator)

3 integrators





$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

with zero initial conditions

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Let $x_1 \triangleq c, \quad x_2 \triangleq \dot{c}, \quad x_3 \triangleq \ddot{c}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$

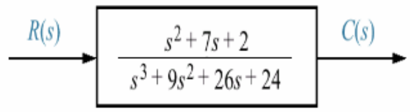
The dynamic equations in state-space representation are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example:
(Polynomial in the Numerator)

For the block containing denominator

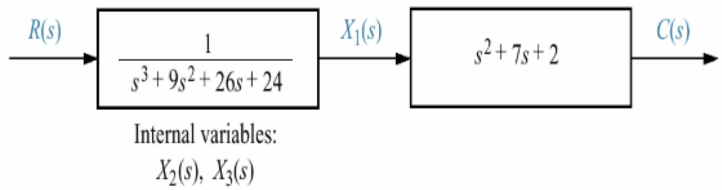


$$(s^3 + 9s^2 + 26s + 24)X_1(s) = 24R(s)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + r$$



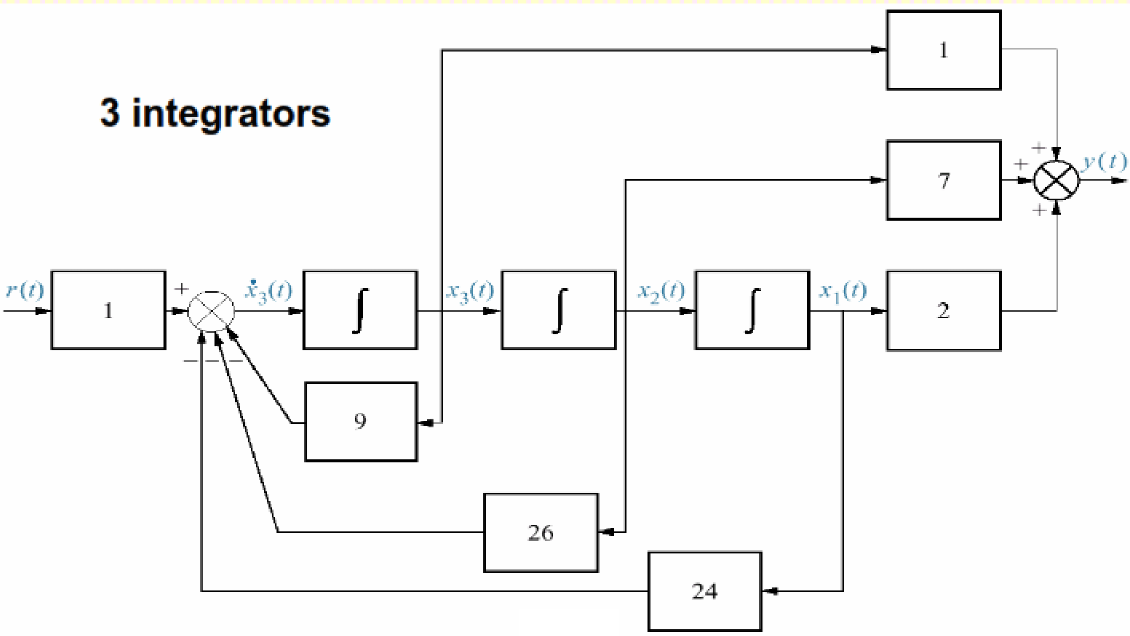
For Numerator Block

$$C(s) = (s^2 + 7s + 2)X_1(s)$$

Taking inverse Laplace transform

$$y = x_3 + 7x_2 + 2x_1$$

3 integrators



Exercise

Write the dynamic equations for the above system

Characteristic Equation

The characteristic equation plays an important role in the study of linear systems. It can be defined with respect to the differential equation, the transfer function, or state equations.

Characteristic Equation from Differential Equation

In T.F. all initial values must be zero, therefore, if we have certain D.E.

$$\dddot{c}(t) + 9\ddot{c}(t) + 24\dot{c}(t) + 20c(t) = 10\dot{r}(t) + 10r(t)$$

Then

$$S^3C(s) + 9S^2C(s) + 24SC(s) + 20C(s) = 10SR(s) + 10R(s)$$

$$C(s)\{ S^3 + 9S^2 + 24S + 20\} = R(s)\{10S + 10\}$$

Then the characteristic equation is obtained by setting the homogeneous part of the above equation to zero. This mean the characteristic equation is:

$$S^3 + 9S^2 + 24S + 20 = 0$$

Characteristic Equation from Transfer Function

If the transfer function of a control system is given, the characteristic equation is obtained by equating the denominator polynomial of the T.F. to zero.

Suppose the transfer function of control system is given as:

$$\frac{C(s)}{R(s)} = \frac{10(S+1)}{(S+2)^2(S+5)}$$

The T.F. must be in polynomial format as:

$$\frac{C(s)}{R(s)} = \frac{10S + 10}{S^3 + 9S^2 + 24S + 20}$$

Then, the characteristic equation is:

$$S^3 + 9S^2 + 24S + 20 = 0$$

Characteristic Equation from State Equation

The T.F. obtained from the matrices A, B, C, and D as:

$$\frac{Y(S)}{U(S)} = C\,[sI - A]^{-1}B + D$$

$$\frac{Y(S)}{U(S)} = \frac{C\,adj\,[sI - A]\;\;B}{|sI - A|} + D$$

$$\frac{Y(S)}{U(S)} = \frac{C\,adj\,[sI - A]\;\;B + D|sI - A|}{|sI - A|}$$

As we said before, the characteristic equation is obtained by equating the denominator polynomial of the T.F. to zero, then it can be obtained as:

$$|sI - A| = 0$$

Suppose that the matrix A is:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 2 \\ -1 & -5 & -3 \end{bmatrix}$$

Then

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 2 \\ -1 & -5 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s+4 & -2 \\ 1 & 5 & s+3 \end{bmatrix}$$

the characteristic equation is:

$$|sI - A| = s\{(s+4)(s+3) + 10\} + 2 = s^3 + 7s^2 + 22s + 2 = 0$$

Eigen Values

The eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ of n×n matrix A are the roots of the characteristic equation. The eigen values are sometimes called the dynamic roots.

Consider the matrix A,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Find the eigen values?

First we calculate (sI-A)

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}$$

Replace each S by λ

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

Solving this cubic equation for λ, we get the three eigen values as:

$$(\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

The eigen values of matrix A are

$$\lambda_1 = -1 \text{ , } \lambda_2 = -2 \text{ and } \lambda_3 = -3$$

Problems for homework

1.

Given the state equations of a linear system as $\dot{x}(t) = A\,x(t) + B\,u(t)$

where,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Find the system eigen values

2.

Consider the system described by $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

Find a) the dynamic equations in vector-matrix form

c) Draw the block diagram represent this system.