Deflection of Beams

In previous lecture it was stated that lateral loads applied to a beam not only give rise to internal bending and shearing stresses in the bar, but
also cause the bar to deflect in a direction perpendicular to its longitudinal axis.

The deformation of a beam is expressed in terms of the deflection of the beam from its original unloaded position. The deflection is measured from the original neutral surface to the neutral surface of the deformed
beam. The configuration assumed by the deformed neutral surface is known as the ***elastic curve*** of the beam.

Figure 1 represents the beam in its original undeformed state and Fig.2 represents the beam in the deformed configuration it has assumed under the action of the load.

Fig.1

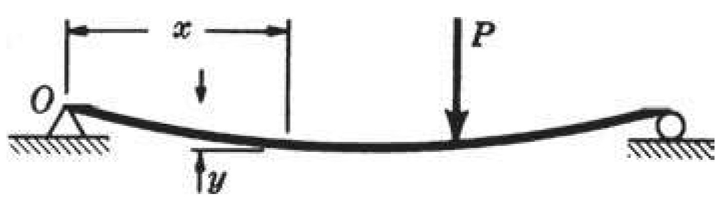


Fig.2

The displacement y is defined as the *deflection* of the beam. Often it will be necessary to determine the deflection y for every value of x along the beam. This relation is the *elastic curve* or *deflection curve* of the beam.

Specifications for the design of beams frequently impose limitations upon the deflections as well as the stresses.

For example, in many building codes the maximum allowable deflection of a beam is not to exceed 1/300 of the length of the beam.

Components of aircraft usually are designed so that deflections do not exceed some preassigned value, so that the aerodynamic characteristics are not altered.

Thus, a well-designed beam must not only be able to carry the loads to which it will be subjected but it must also not undergo undesirably large deflections.

Differential Equation of the Elastic Curve

Let us derive the differential equation of the deflection curve of a beam loaded by lateral forces.

In previous lecture the relationship

$$M = EI / \rho \quad \text{-----------(1)}$$

In this expression M denotes the bending moment acting at a particular cross section of the beam, ρ the radius of curvature of the neutral surface of the beam at this same section, E the modulus of elasticity, and I the moment of the cross sectional area about the neutral axis passing through the centroid of the cross section.

Let the heavy line in Fig. 2 represent the deformed neutral surface of the bent beam.

An expression for the *curvature* at any point along the curve representing the deformed beam is readily available from differential calculus. It is

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \text{ ------------(2)}$$

In this expression, *dy*/*dx* represents the slope of the curve at any point; and for small beam deflections this quantity and in particular its square are small in comparison to unity and may reasonably be neglected.
This assumption of small deflections simplifies the expression for curvature into

$$\frac{1}{\rho} \approx \frac{d^2y}{dx^2} \text{ ------------(3)}$$

Hence for small deflections, Eq. (8.1) becomes

$$EI\frac{d^2y}{dx^2} = M \text{ -----------(4)}$$

This is the differential equation of the deflection curve of a beam loaded by lateral forces. It is called the **Euler-Bernoulli equation** of bending of a beam.
In any problem it is necessary to integrate this equation to obtain an algebraic relationship between the deflection *y* and the coordinate *x* along the length of the beam.

Deflection by Double-Integration Method

The double-integration method for calculating deflections of beams merely consists of integrating Eq. (4).
The first integration yields the slope *dy*/*dx* at any point in the beam and the second integration gives the deflection *y* for any value of *x*. The bending moment *M* must, of course, be expressed as a function of the coordinate *x* before the equation can be integrated. For the cases to be studied here the integrations are straightforward.

Deflection curve equation is :

$$EI\frac{d^2y}{dx^2} = M \quad N.m \text{----------------(4)}$$

From first integration we obtained Slop equation (dy/dx):

$$EI\frac{dy}{dx} = \int M\,dx + C_1 \quad N.m^2$$

Deflection equation (y) is :

$$EIy = \int\int M\,dx\,dx + C_1x + C_2 \quad N.m^3$$

C_1 and C_2 are constant of integration which must be evaluated from the given conditions of beam and its loading.

The conditions of **simple support beam** are:

At X = 0 , y = 0

And

At X= L , y= 0

y max.(**maximum deflection**) at dy/dx= 0(**slop = 0**)

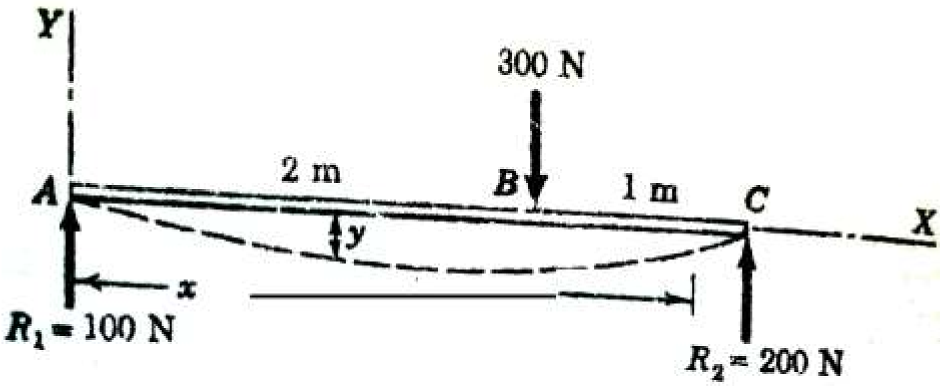
while, the conditions of **cantilever beam** are:

at X = 0 , y = 0 and dy/dx =0

y max.(**maximum deflection**) at X= L

Ex 1/

601. A concentrated load of 300 N is supported as shown in Fig. 6-4. Determine the equations of the elastic curve between each change of load point, and the maximum deflection in the beam.



$$EI\frac{d^2y}{dx^2} = M = (100x - 300\langle x-2\rangle)\ \text{N}\cdot\text{m}$$

$$EI\frac{dy}{dx} = (50x^2 - 150\langle x-2\rangle^2 + C_1)\ \text{N}\cdot\text{m}^2$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right)\ \text{N}\cdot\text{m}^3$$

1. At A where $x = 0$, the deflection $y = 0$. Substituting these values in Eq. (c), we find that $C_2 = 0$. Remember that $\langle x-2\rangle^3$ is to be ignored for negative values.

2. At the other support where $x = 3$, the deflection y is also zero. Knowing that $C_2 = 0$ and substituting these values in the deflection equation (c), we obtain

$$0 = \frac{50}{3}(3)^3 - 50(3-2)^3 + 3C_1 \quad \text{or} \quad C_1 = -133\ \text{N}\cdot\text{m}^2$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 - 133x\right)\ \text{N}\cdot\text{m}^3 \quad \text{-------(1)}$$

Assume the Maximum deflection occure in segment AB , $X \le 2$,and

$dy/dx = 0$,

$EI\ dy/dx = 50/2\ X^2 - 133$

$133 = 25\ X^2$

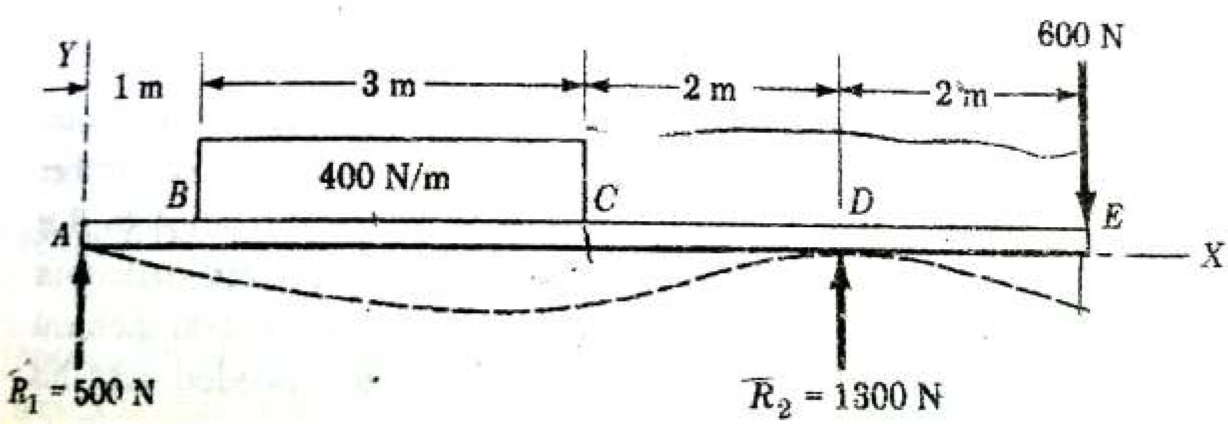
$X = 1.63\ m$

We substitute in eq.(1)

Max. $EIy = 50/3\ (1.63)^3 - 133(1.63) = -\ 145 N.m^3$

Ex.2/

602. Find the value of EIy at the position midway between the supports and at the overhanging end for the beam shown in Fig. 6–5.



$$EI\frac{d^2y}{dx^2} = M = \left(500x - \frac{400}{2}\langle x-1\rangle^2 + \frac{400}{2}\langle x-4\rangle^2 + 1300\langle x-6\rangle\right) \text{N·m}$$

$$EI\frac{dy}{dx} = \left(250x^2 - \frac{200}{3}\langle x-1\rangle^3 + \frac{200}{3}\langle x-4\rangle^3 + 650\langle x-6\rangle^2 + C_1\right) \text{N·m}^2$$

$$EIy = \left(\frac{250}{3}x^3 - \frac{50}{3}\langle x-1\rangle^4 + \frac{50}{3}\langle x-4\rangle^4 + \frac{650}{3}\langle x-6\rangle^3 + C_1x + C_2\right) \text{N·m}^3$$

To determine C_2, we note that $EIy = 0$ at $x = 0$, which gives $C_2 = 0$. Note that we ignore the negative terms in the pointed brackets. Next we use the condition that $EIy = 0$ at the right support where $x = 6$. This gives

Finally, to obtain the midspan deflection, we substitute $x = 3$

$$EIy = \frac{250}{3}(3)^3 - \frac{50}{3}(2)^4 - 1308(3) = -1941\ \text{N}\cdot\text{m}^3$$

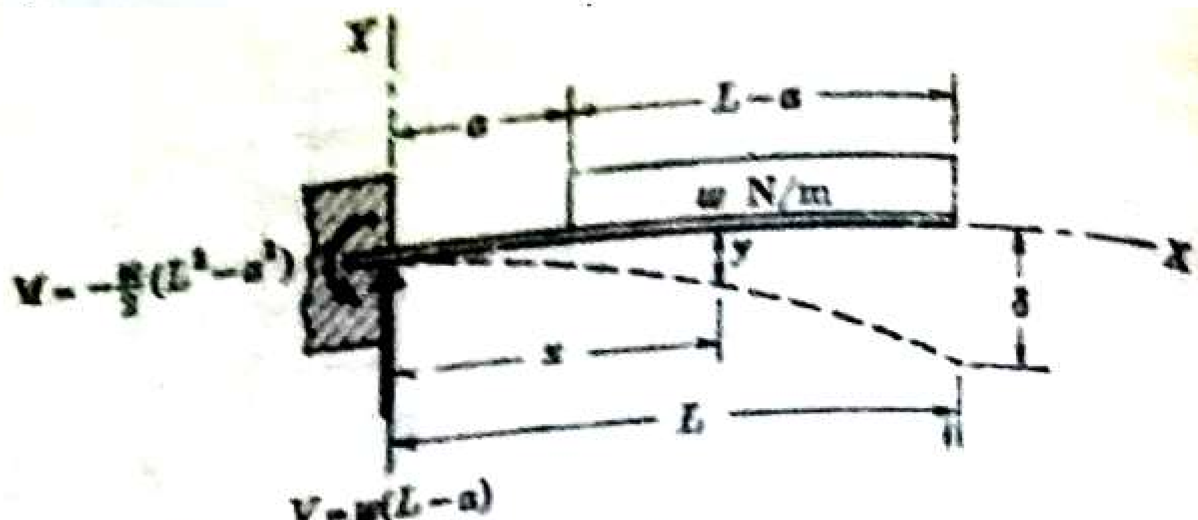
Also, at the overhanging end where $x = 8$, we have

$$EIy = \frac{250}{3}(8)^3 - \frac{50}{3}(7)^4 + \frac{50}{3}(4)^4 + \frac{650}{3}(2)^3 - 1308(8)$$

$$= -1814\ \text{N}\cdot\text{m}^3 \qquad \textit{Ans.}$$

Ex3/

604. Determine the equation of the elastic curve of a cantilever beam supporting a uniformly distributed load of w N/m over part of its length as shown in Fig. 6-7.



$$V = w(L - a)$$

$$M = -w(L - a)\left(a + \frac{L - a}{2}\right)$$

= - w (L- a)((2a+L-a)/2) = - w (L-a) ((L+a)/2)

$$= -\frac{w}{2}(L^2 - a^2)$$

In the cantilever beam Σ M is taken about free end

$$EI\frac{d^2y}{dx^2} = w(L-a)x - \frac{w}{2}(L^2 - a^2) - \frac{w}{2}\langle x - a\rangle^2$$

$$EI\frac{dy}{dx} = w(L-a)\frac{x^2}{2} - \frac{w}{2}(L^2 - a^2)x - \frac{w}{6}\langle x - a\rangle^3 + C_1$$

However, the slope dy/dx is zero at $x = 0$, so $C_1 = 0$. We may now integrate the slope equation (with $C_1 = 0$) and obtain the deflection equation:

$$EIy = w(L-a)\frac{x^3}{6} - \frac{w}{4}(L^2 - a^2)x^2 - \frac{w}{24}\langle x - a\rangle^4 + C_2$$

Since $y = 0$ at $x = 0$, we find that $C_2 = 0$

The value of the maximum deflection, which occurs at the free end, is denoted by δ. Evidently $\delta = -y$; so on substituting $x = L$ and simplifying, we obtain

$$EI\delta = \frac{w(L-a)}{8}\left(L^3 + L^2a + La^2 - \frac{a^3}{3}\right)$$