5- Absolute Maxima and Minima on closed bounded Region:

We organize the search for the absolute extrema of a continuous function $f(x,y)$ on a closed and bounded region R into three steps:

1- List the interior points of R where f may have Local maxima and minima and evaluate f at these points. There are the critical points of f.

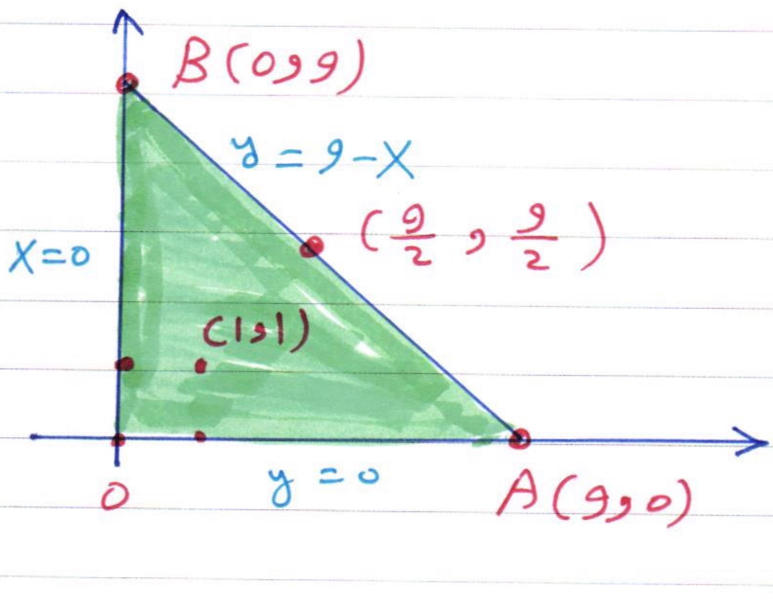
2- List the boundry points of R where f has local maxima and minima and evaluate f at these points. We show how to do this shortly.

3- Look the lists for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R.

Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists mad is step 1 and 2.

Example 1: Find the absolute maximum and minimum values of $f(x,y) = 2+2x+2y-x^2-y^2$ on the triangular region in the first qudrant bounded by the lines $x=0$, $y=0$, $y=9-x$.

Solution:



a- Interior Points:

$f_x = 2 - 2x = 0 \longrightarrow x = 1$

$f_y = 2 - 2y = 0 \longrightarrow y = 1$

yielding the single Point $(x,y) = (1,1)$

$f(1,1) = 2 + 2(1) + 2(1) - (1)^2 - (1)^2 = 4$

b- Boundary points:

(i) On the segment OA, $y = 0$

$f(x,y) = f(x,0) = 2 + 2x - x^2$

at $x = 0$ $\quad f(0,0) = 2 + 2(0) - (0)^2 = 2$

at $x = 9$ $\quad f(9,0) = 2 + 2*9 - 9^2 = -61$

$f'(x,0) = 2 - 2x = 0 \longrightarrow x = 1$

$f(x,0) = f(1,0) = 2 + 2(1) - (1)^2 = 3$

(ii) on the segment OB, $x=0$

$$f(x,y)=f(0,y)=2+2y-y^2$$

$$f(0,0)=2+2(0)-(0)^2=2$$

$$f(0,9)=2+2\times9-9^2=-61$$

$$f'(0,y)=2-2y=0 \longrightarrow y=1$$

$$f(0,1)=2+2(1)-(1)^2=3$$

(iii) We have alrady accounted for the values of f at end points of AB, so we need only look at the interior points of AB. With $y=9-x$, we have

$$f(x,y)=f(x,(9-x))=2+2x+2(9-x)$$
$$-x^2-(9-x)^2$$

$$f(x,(9-x))=-61+18x-2x^2$$

$$f'(x,(9-x)) = 18-4x = 0 \longrightarrow x = \frac{18}{4} = \frac{9}{2}$$

$$y = 9-x = 9-\frac{9}{2} = \frac{9}{2}$$

$$f(x,y) = f\left(\frac{9}{2},\frac{9}{2}\right) = 2+2\left(\frac{9}{2}\right)+2\left(\frac{9}{2}\right)$$

$$-\left(\frac{9}{2}\right)^2-\left(\frac{9}{2}\right)^2 = -\frac{41}{2}$$

Summary

We list all the conditions:

$4, \; 2, \; -61, \; 3, \; -\frac{41}{2}$

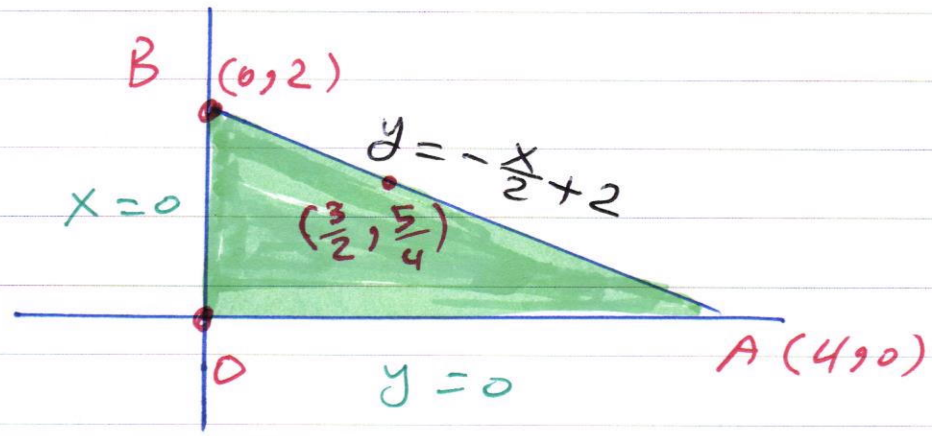
maximum (1,1)

minimum (9,0) (0,9)

Example 2: Find the absolute maximum and minimum values of the function on D, where D is the enclosed triangular region with vertices (0,0), (0,2), and (4,0).

$$f(x,y) = x + y - xy$$

Solution:



$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$

$$\frac{0-2}{4-0} = \frac{y-2}{x-0}$$

$$y = -\frac{1}{2}x + 2$$

a- interior points

$$f_x = 1 - y = 0 \longrightarrow y = 1$$

$$f_y = 1 - x = 0 \longrightarrow x = 1$$

The only critical point on D is (1,1)

$$f(1,1) = 1 + 1 - 1*1 = 1$$

b - boundry points:

(i) OA

$$f(x,y) = f(x,0) = x$$

at $x=0$. $\quad f(0,0) = 0$

at $x=4$ $\quad f(4,0) = 4$

(ii) OB

$$f(x,y) = f(0,y) = y$$

at $\cancel{x}y=0$ $\quad f(0,0) = 0$

at $y=2$ $\quad f(0,2)=2$

(iii) AB

$$f\left(x,\left(-\frac{x}{2}+2\right)\right)=x-\frac{x}{2}+2+\frac{x^2}{2}-2x$$

$$f\left(x,\left(-\frac{x}{2}+2\right)\right)=\frac{x^2}{2}-\frac{3}{2}x+2$$

$$f_x=x-\frac{3}{2}=0 \longrightarrow x=\frac{3}{2}$$

$$y=-\frac{x}{2}+2=-\frac{3}{4}+2=\frac{5}{4}$$

the critical point is then $\left(\frac{3}{2},\frac{5}{4}\right)$

$$f\left(\frac{3}{2},\frac{5}{4}\right)=\frac{3}{2}+\frac{5}{4}-\frac{3}{2}*\frac{5}{4}=\frac{7}{8}$$

$f(1,1)=1$

$f(0,2)=2$

$f(0,0)=0$ minimum

$f(4,0)=4$ maximum

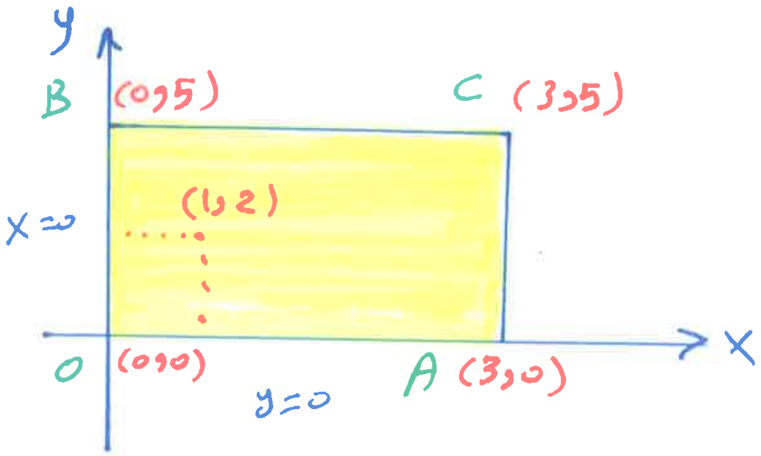
$f\left(\frac{3}{2},\frac{5}{4}\right)=\frac{7}{8}$

Example 3: Find the absolute maximum and absolute minimum values of:

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed region R with vertices (0,0), (3,0), (3,5), and (0,5).

solution:



a- interior points:

$$f_x = 3y - 6 - 0 + 0 = 0$$

$$3y - 6 = 0 \longrightarrow y = 2$$

$$f_y = 3x - 0 - 3 + 0 = 0$$

$3x - 3 = 0 \longrightarrow x = 1$

The interior critical point is (1,2)

$f(1,2) = 3(1)(2) - 6(1) - 3(2) + 7 = 1$

b- boundry points:

i) oA

$f(x,y) = f(x,0) = -6x + 7$

at $x = 0$ $\quad f(0,0) = 7$

at $x = 3$ $\quad f(3,0) = -11$

$f'(x,0) = -6 \neq 0$ ther is no interior points along oA

ii) oB

$f(x,y) = f(0,y) = -3y + 7$

at $y = 5$ $\quad f(0,5) = -15 + 7 = -8$

$f'(0,y) = -3 \neq 0$ ther is no interior point along OB

iii) BC

$$f(x,y) = f(x,5) = 15x - 6x - 15 + 7$$
$$= 9x - 8$$

at $(3,5)$ $f(3,5) = 19$

$f'(x,5) = 9 \neq 0$ ther is no interior point along BC

iv) AC

$$f(x,y) = f(3,y) = 3(3)y - 6(3) - 3y + 7$$
$$= 6y - 11$$

$f'(3,y) = 6 \neq 0$ ther is no inter point along AC

$f(3,0) = -11$ minimum

$f(3,5) = 19$ maximum

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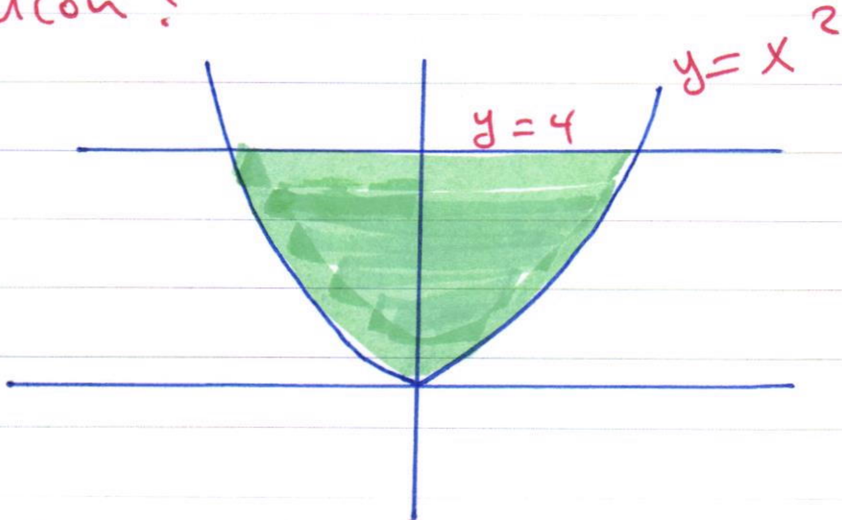
Example 4: Find the absolute extrema of the function $z = f(x,y) = 4x^2 + y^2 - 4xy$ in the region bounded by the curves $y = x^2$ and $y = 4$.

solution:



a- interior points:

$$\left.\begin{aligned} z_x &= 8x - 4y = 0 \\ z_y &= 2y - 4x = 0 \end{aligned}\right\} \quad y = 2x$$

كل النقاط التي تحقق $y=2x$ هي حرجة

2- bounded points

e) prabola $y=x^2$

for the curve $y=x^2$

$z=f(x,y)=f(x,x^2)=4x^2+x^4-4x^3$

$z_x(x,x^2)=8x+4x^3-12x^2=0$

$4x(2+x^2-3x)=0$

either $x=0 \longrightarrow y=(0)^2=0$ $(0,0)$

or $(x^2-3x+2)=0 \longrightarrow (x-2)(x-1)=0$

$x=2 \longrightarrow y=4$ $(2,4)$

$x=1 \longrightarrow y=1$ $(1,1)$

$f(0,0)=z(0,0)=0$

$f(1,1)=z(1,1)=4(1)^2+(1)^4-4(1)^3=1$

$f(2,4)=z(2,4)=4(2)^2+(2)^4-4(2)^3=0$

$$f(-2,4) = z(-2,4) = 4(-2)^2 + (-2)^4 - 4(-2)^3 = 64$$

(ii) line $y = 4$

for the line $y = 4$

$$z = f(x,4) = 4x^2 + 4^2 - 4 \times 4x$$

$$z = 4x^2 - 16x + 16$$

$$z'_x = 8x - 16 = 0 \longrightarrow x = 2 , y = 4$$

$(2,4)$ موجودة

$z(0,0) = 0$ minimum

$z(1,1) = 1$

$z(2,4) = 0$ minimum

$z(-2,4) = 64$ maximum

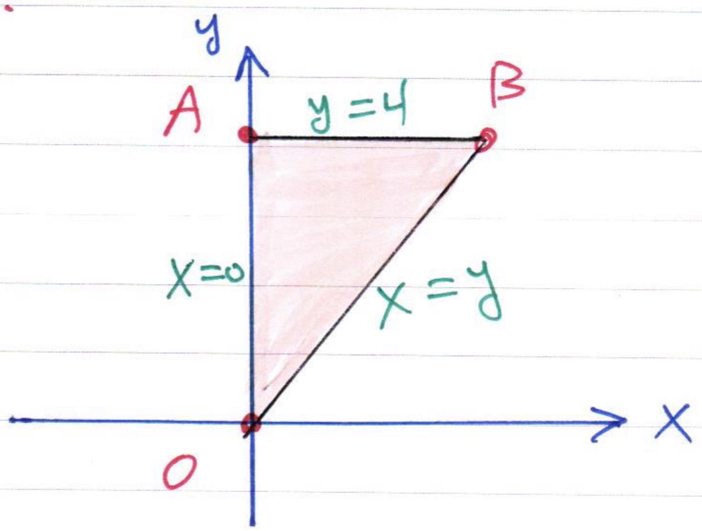
Example 5: Find the absolute maxima and minima of the function:

$$f(x,y) = x^2 - xy + y^2 + 1$$

on the closed triangular plate in the first qudrant bounded by the lines:

$x = 0$, $y = 4$ and $x = y$.

solution:



a- interior points:

$$f_x = 2x - y = 0 \quad (1)$$

$$f_y = -x + 2y = 0 \quad (2)$$

from ① and ② $x = 0$, $y = 0$

$(0,0)$ is not intror Point in the solid body -

b - boundry Points:

i) OA

$f(x,y) = f(0,y) = y^2 + 1$

$f'(0,y) = 2y - 0 = 0 \longrightarrow y = 0 , x = 0$

at $(0,0) \rightarrow f(0,0) = 1$

at $(0,4) \rightarrow f(0,4) = 17$

ii) AB

$f(x,y) = f(x,4) = x^2 - 4x + 7$

$f'(x,4) = 2x - 4 = 0 \longrightarrow x = 2$ and $y = 4$

The intrior point on AB is $(2,4)$

at $(2,4) \longrightarrow f(2,4) = 13$

at $(4,4) \longrightarrow f(4,4) = 17$

(iii) OB

$f(x,y) = f(x,x) = x^2 + 1$

$f'(x,x) = 2x - 0 = 0 \longrightarrow x = 0, y = 0$

which is not an interior point on OB, the end point values have been founded above.

$f(0,0) = 1$ minimum.

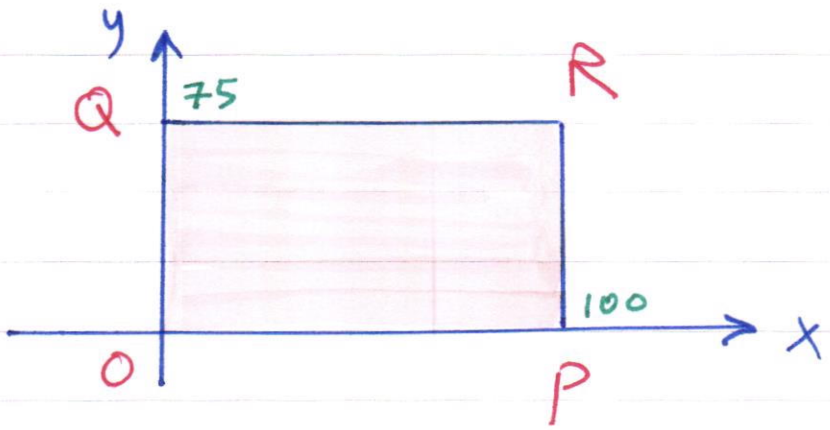
$f(0,4) = 17 \longrightarrow$ maximum

$f(2,4) = 13$

$f(4,4) = 17 \longrightarrow$ maximum

Example 6: Find the absolute maxima and minima of the function on D, where $f(x,y) = 10x + 20y - 0.5xy$ and $D = \{100 \geq x \geq 0 \,,\, 75 \geq y \geq 0\}$

solution:



a- interior points:

$f_x = 10 - 0.5y = 0 \quad (1) \longrightarrow y = 20$

$f_y = 20 - 0.5x = 0 \quad (2) \longrightarrow x = 40$

$(40, 20)$ is the interior point in the solid body.

$f(40,20) = 400$

b- boundry points :

(i) OP

$f(x,y) = f(x,0) = 10x$

$f'(x,0) = 10 \neq 0$ ∴ ther is no interior points along op

at $(0,0) \longrightarrow f(0,0) = 0$

at $(100,0) \longrightarrow f(100,0) = 1000$

(ii) OQ

$f(x,y) = f(0,y) = 20y$

$f'(0,y) = 20 \neq 0$ ∴ ther is no interior points along OQ

at $(0,75) \longrightarrow f(0,75) = 1500$

(iii) QR

$f(x,y) = f(x,75) = -27.5x + 1500$

$f'(x,75) = -27.5 \neq 0$ ∴ ther is no intrior points along QR.

at $(100,75) \longrightarrow f(100,75) = -1250$

(iv) PR

$f(x,y) = f(100,y) = 1000 - 30y$

$f'(x,y) = -30 \neq 0$ ∴ ther is no intror points along PR

$f(40,20) = 400$

$f(0,0) = 0$

$f(100,0) = 1000$

$f(0,75) = 1500$ maximum

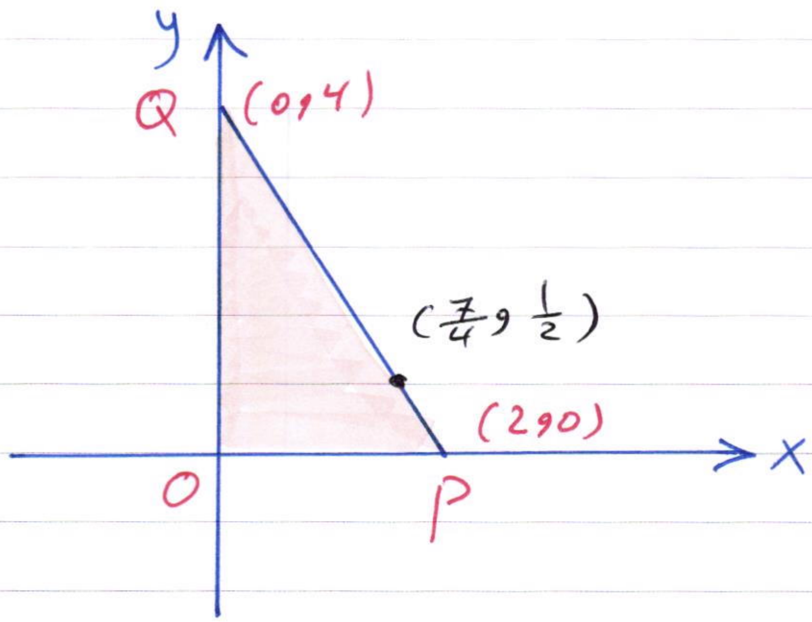
$f(100,75) = -1250$ minimum

Example 7: Find the maximum and minimum value of $f(x,y) = 8 + xy - x - 2y$ on the triangular region R with vertices $(0,0)$, $(2,0)$, and $(0,4)$.

Solution:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$

$$\frac{4-0}{0-2} = \frac{y-0}{x-2} \longrightarrow y = 4 - 2x$$



a- Interior points:

$f_x = y - 1 = 0 \longrightarrow y = 1$

$f_y = x - 2 = 0 \longrightarrow x = 2$

$(2,1)$ is the critical point which is not interior point on the domain of f

تقع خارج المنطقة لذلك تهمل .

b- bounded points:

i) OP

$f(x,y) = f(x,0) = 8 - x$

$f'(x,0) = -1 \neq 0 \quad \therefore$ there is no any interior point along OP.

at $(0,0) \longrightarrow f(0,0) = 8$

at $(2,0) \longrightarrow f(2,0) = 6$

ii) OQ

$f(x,y) = f(0,y) = 8-2y$

$f'(0,y) = -2 \neq 0$ ∴ there is no any interior point along OQ.

at $(0,4) \longrightarrow f(0,4) = 0$

iii) PQ

$f(x,y) = f(x,(4-2x)) = 8 + x(4-2x) - x - 2(4-2x)$

$f(x,(4-2x)) = -2x^2 + 7x$

$f'(x,(4-2x)) = -4x + 7 = 0 \longrightarrow x = \frac{7}{4}$

$y = 4 - 2x = 4 - 2\left(\frac{7}{4}\right) = 4 - \frac{14}{4} = \frac{1}{2}$

$\left(\frac{7}{4}, \frac{1}{2}\right)$ is the critical interior point along $y = 4 - 2x$

at $\left(\frac{7}{4}, \frac{1}{2}\right) \longrightarrow f\left(\frac{7}{4}, \frac{1}{2}\right) = 6.125$

$f(0,0) = 8 \longrightarrow$ maximum

$f(2,0) = 6$

$f(0,4) = 0$

$f\left(\frac{7}{4}, \frac{1}{2}\right) = 6.125 \longrightarrow$ minimum.

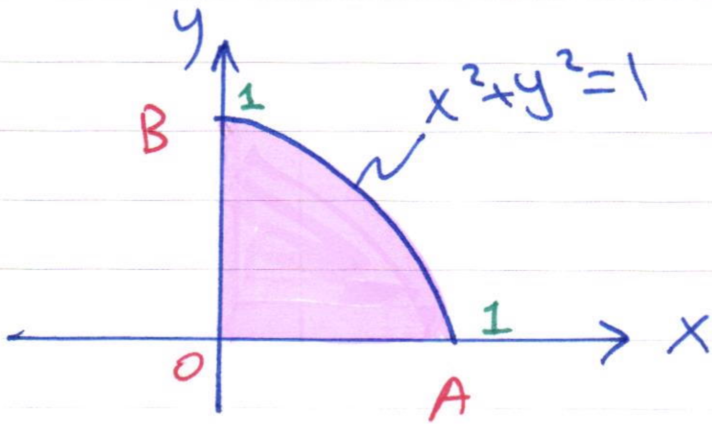
Example 8: Find the maximum and minimum value of $f(x,y) = x^2 + y^2 + y + 1$

subjected to: $x^2 + y^2 \leq 1$

$x \geq 0$

$y \geq 0$

Solution:



$x^2 + y^2 \leq 1 \;;\; x \geq 0 \;;\; y \geq 0$

a-interior point:

$f_x = 2x = 0 \longrightarrow x = 0$

$f_y = 2y + 1 = 0 \longrightarrow y = -\frac{1}{2}$

$(0, -\frac{1}{2})$ is the critical point, which is not the interior point of the domain.

Therfore there are no interior points.

b- bounded Points

(i) OA

$f(x,y) = f(x,0) = x^2 + 1$

$f'(x,0) = 2x = 0 \longrightarrow (0,0)$ is not interior point.

at $(0,0) \longrightarrow f(0,0) = 1$

at $(1,0) \longrightarrow f(1,0) = 2$

(ii) OB

$f(x,y) = f(0,y) = y^2 + y + 1$

$f'(0,y) = 2y + 1 \longrightarrow y = -\frac{1}{2}$

$(0, -\frac{1}{2})$ is not interior point along OB

at $(0,1) \longrightarrow f(0,1) = 3$

(iii) AB

$y = \sqrt{1-x^2}$

$f(x,(\sqrt{1-x^2})) = x^2 + (1-x^2) + \sqrt{1-x^2} + 1$

$f(x,(\sqrt{1-x^2})) = 2 + \sqrt{1-x^2}$

$f'(x,(\sqrt{1-x^2})) = 0 + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot 2x = 0$

$0 = \frac{x}{\sqrt{1-x^2}} \longrightarrow x = 0, \quad y = 1$

$(0,1)$ is not intrior point along AB,

it is vertex point in (iii).

$f(0,0) = 1$ minimum

$f(0,1) = 3$ maximum

$f(1,0) = 2$