Double Integrals in Polar Coordinates

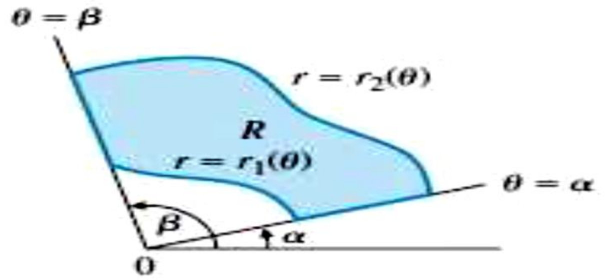
Such integrals are important for two reasons: first, they arise naturally in many applications, and second, many double integrals in rectangular coordinates can be evaluated more easily if they are converted to polar coordinates.

Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates.

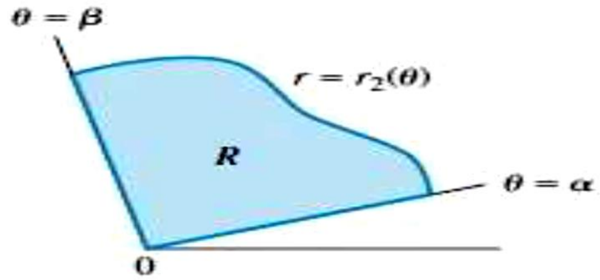
A ***simple polar region R*** in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves,

$r = r1(\theta)$ and $r = r2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \leq \beta$ **(ii)** $\beta - \alpha \leq 2\pi$ **(iii)** $0 \leq r1(\theta) \leq r2(\theta)$

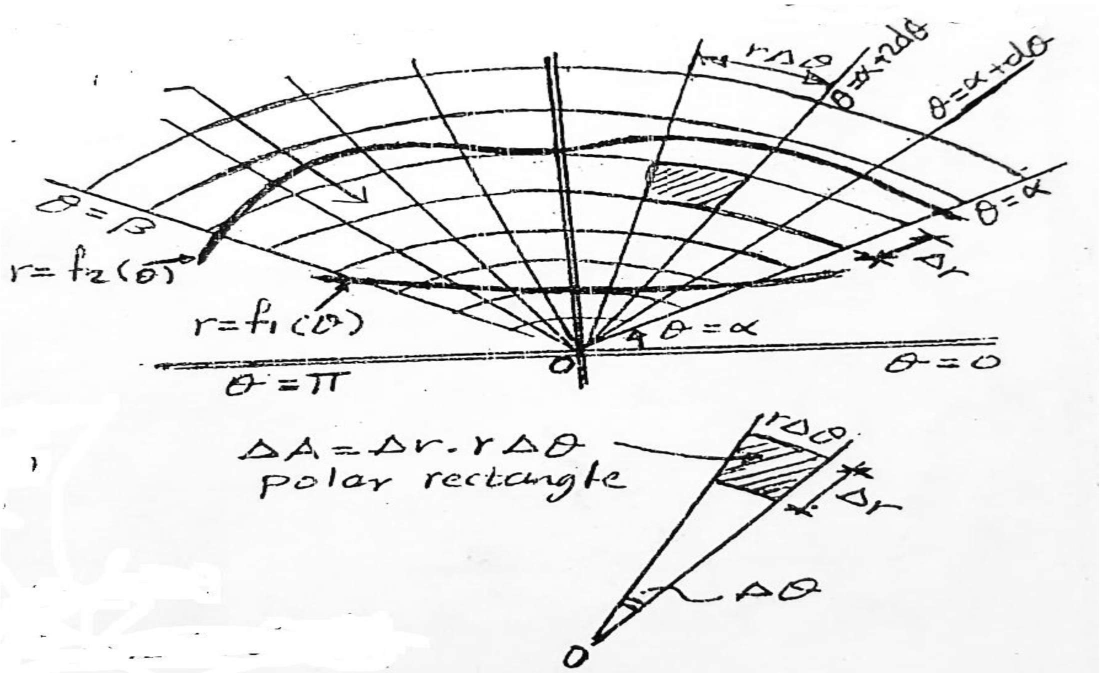


(a)



(b)

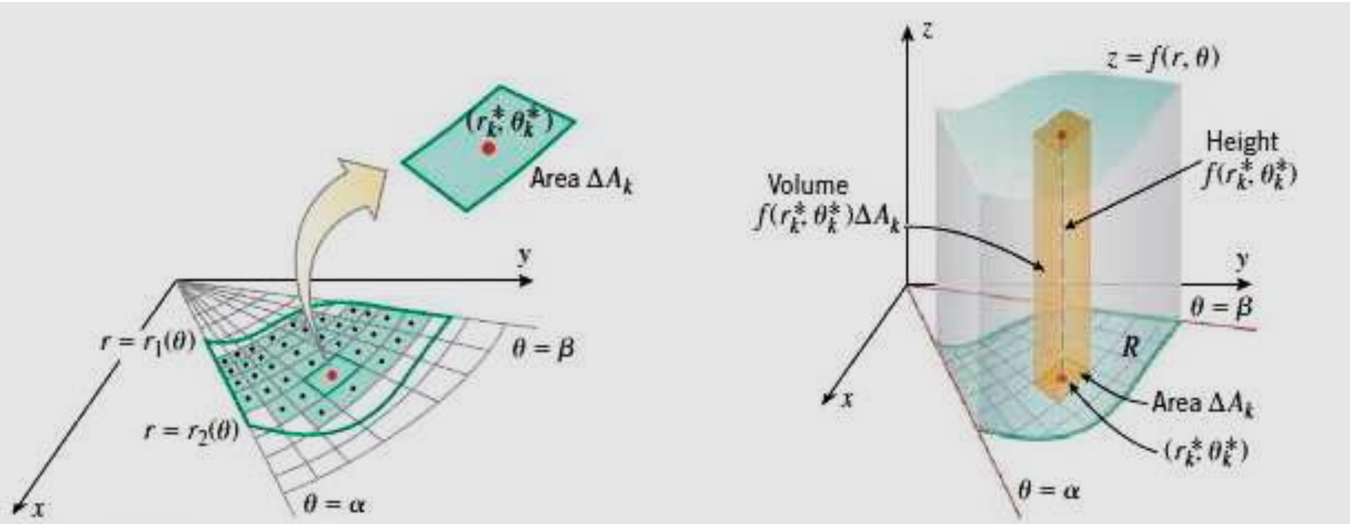
To find the integral of a function f(x,y) over a region R, the region is divided with rectangles. When we work with polar coordinates, (r and θ) It is the natural divided into " polar rectangular"



The volume problem in polar coordinates Given a function *f(r, θ)* that is continuous and nonnegative on a simple polar region *R*, find the volume of the solid that is enclosed between the region *R* and the surface whose equation in cylindrical coordinates is ***z = f(r, θ)***.

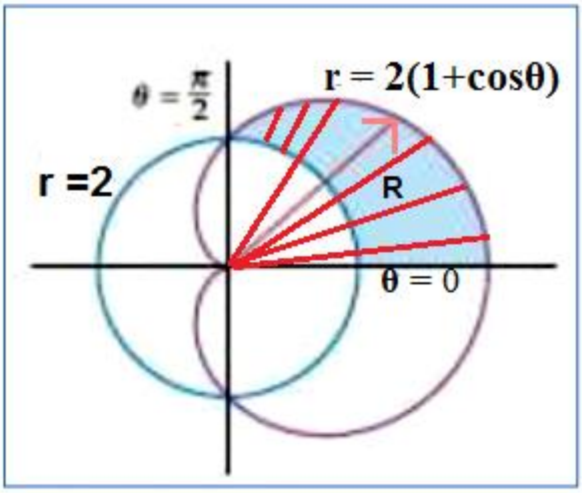
The volume V can be expressed as the iterated integral

$$V = \iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)r\,dr\,d\theta$$



Example1: Evaluate

$$\iint_R \sin\theta \, dA$$



Where R is the region in the first quadrant that is outside the circle **r = 2** and inside the cardioid **r = 2(1+cosθ).**

Solution:

Step 1: hold **θ** fixed (constant), and let r increase to trace a ray out from the origin.

Step 2: integrate from r- value where the ray enters R to the r – value where its leaves R.

Step 3: chooses θ limits to include all the rays from the origin that intersect R.

2= 2(1+cosθ)

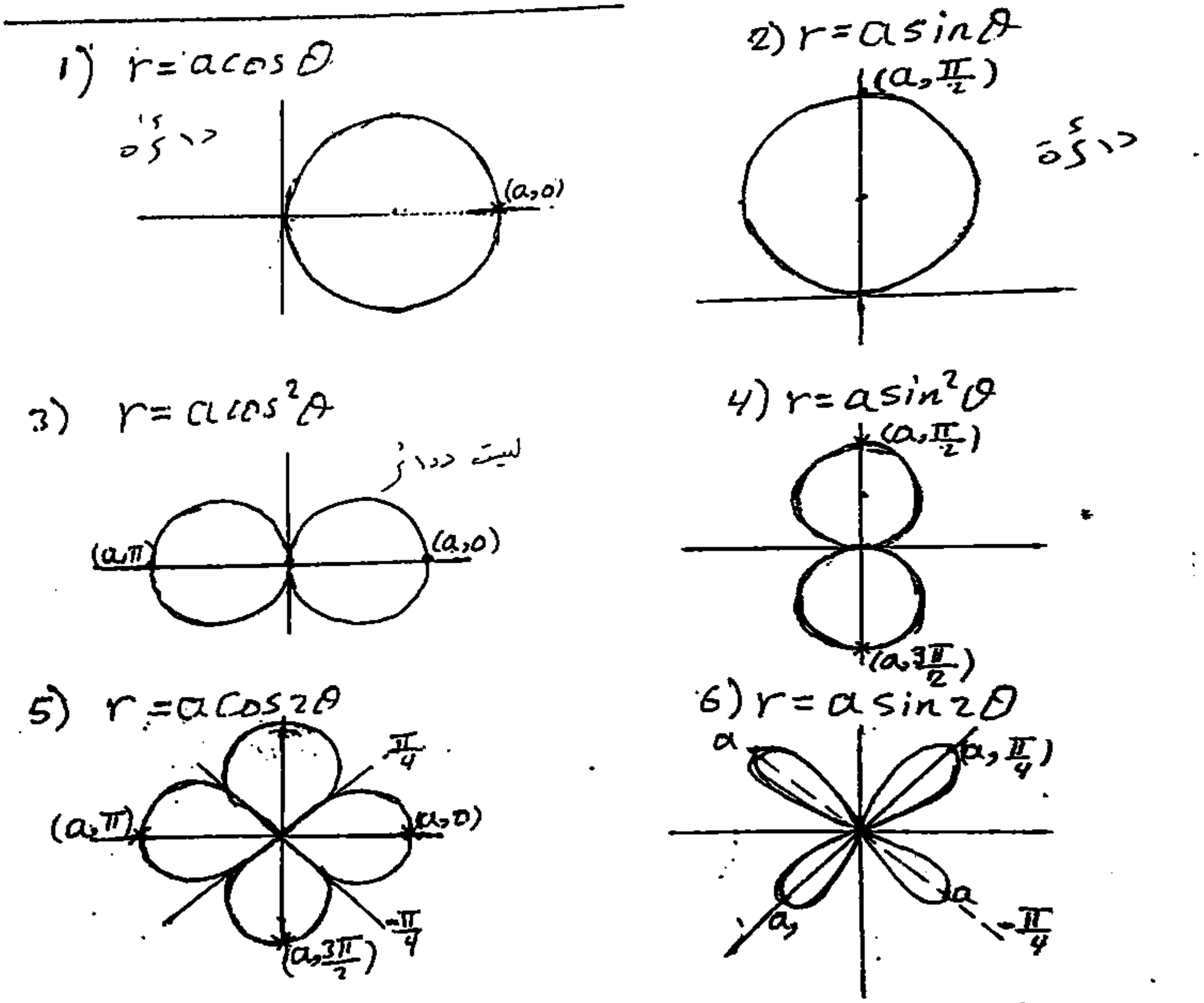
Cos θ = 1- 1 = 0 $\rightarrow$ **θ** = $\frac{\pi}{2}$ or **θ** = $\frac{3\pi}{2}$

$$\iint_R \sin\theta\, dA = \int_0^{\frac{\pi}{2}}\int_2^{2(1+\cos\theta)} (\sin\theta) r\, dr\, d\theta = \int_0^{\pi/2}\left[\frac{1}{2}r^2 \sin\theta\right]_{r=2}^{2(1+\cos\theta)} d\theta$$

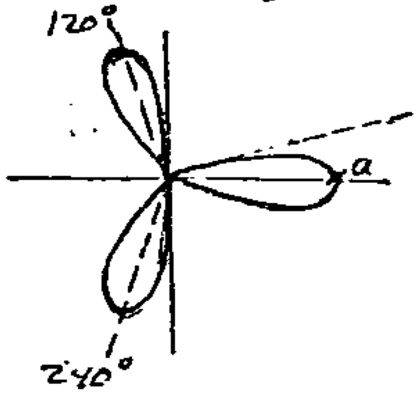
$$= 2\int_0^{\pi/2} [(1+\cos\theta)^2 \sin\theta - \sin\theta]\, d\theta = 2[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta]_0^{\frac{\pi}{2}}$$

$$= 2\left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3}$$

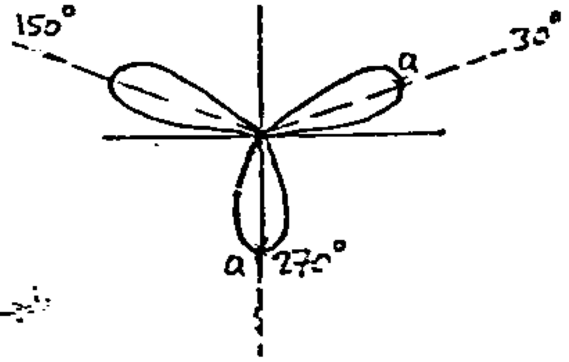
Standard polar curves



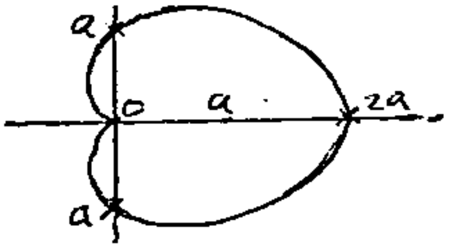
7) $r = a\cos 3\theta$



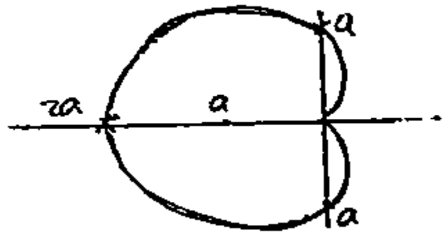
8) $r = a\sin 3\theta$



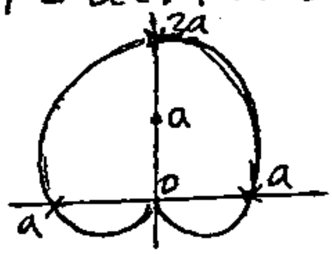
9) $r = a(1+\cos\theta)$



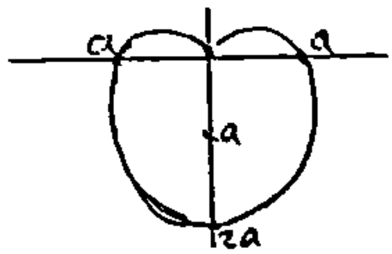
10) $r = a(1-\cos\theta)$



11) $r = a(1+\sin\theta)$

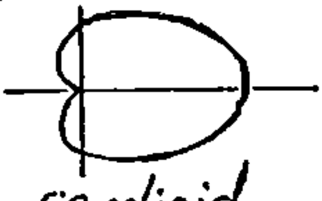


12) $r = a(1-\sin\theta)$



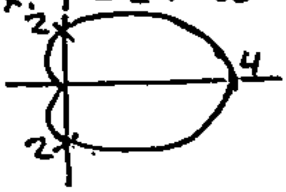
13) Graphs of $r = a + b\cos\theta$

i) if $a = b$, we get

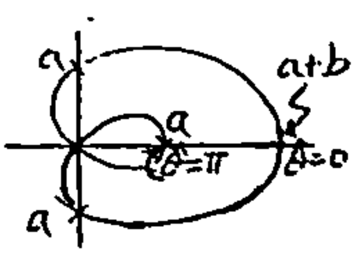


cardioid

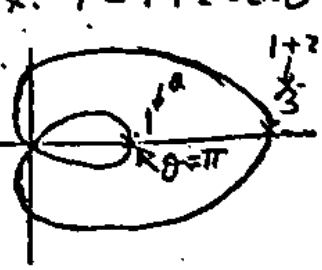
Ex: $r = 2 + 2\cos\theta$

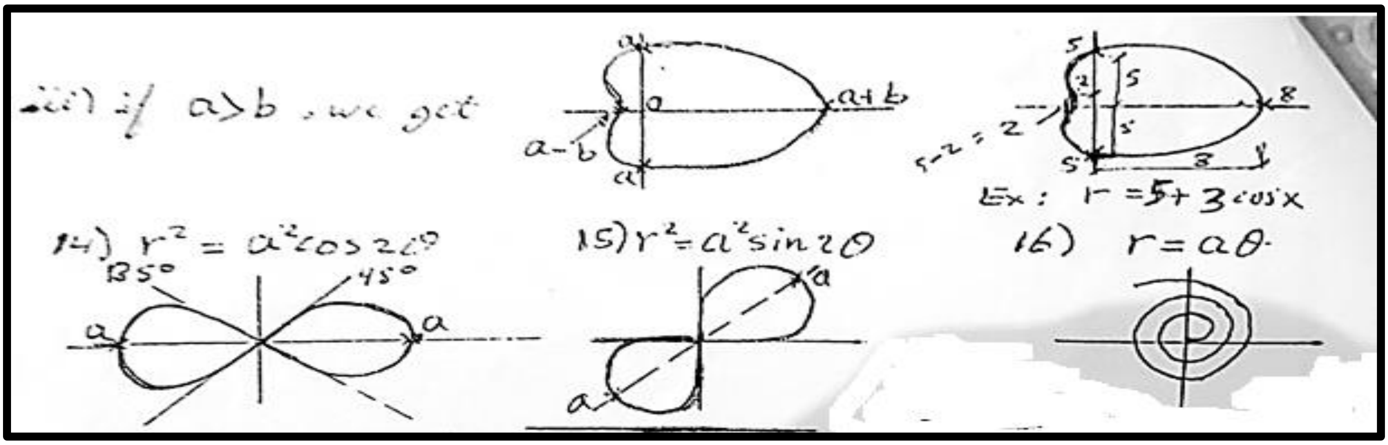


ii) if $a < b$, we get



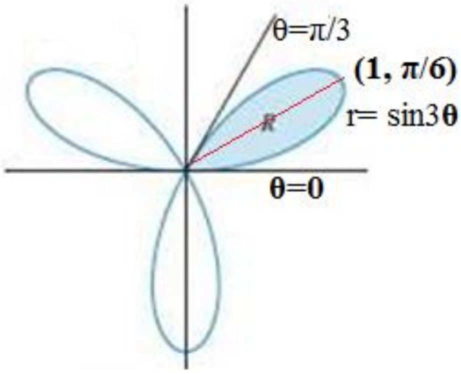
Ex: $r = 1 + 2\cos\theta$





Example 2: Use a polar double integral to find the area enclosed by the three leaves rose $r = \sin 3\theta$.

Solution: The rose is sketched in Figure. We will calculate the area of the petal R in the first quadrant and multiply by three.



$r = a \sin 3\theta$

$0 = a \sin 3\theta \rightarrow \quad \sin 3\theta = 0 \qquad (a=1)$

$\rightarrow 3\theta = \sin^{-1}(0) = \pi \rightarrow \theta = \frac{\pi}{3}$

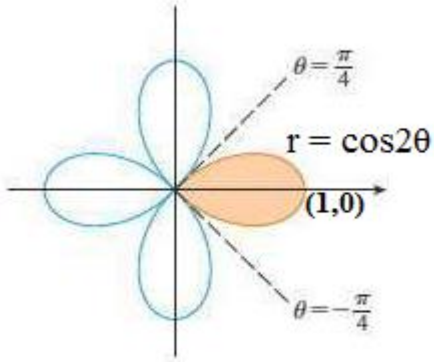
$$A = 3\iint_R dA = 3\int_0^{\frac{\pi}{3}}\int_0^{\sin 3\theta} r\,dr\,d\theta =$$

$$\frac{3}{2}\int_0^{\frac{\pi}{3}} \sin^2 3\theta\,d\theta =$$

$$\frac{3}{4}\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta = \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{1}{4}\pi$$

Example3: Use a double integral to find the area enclosed by one loop of the four leaved rose $r = \cos 2\theta$

Solution: From the sketch of the curve in Figure, we see that a loop is given by the region.



$r = a\cos 2\theta \qquad (a=1)$

$0 = a\cos 2\theta \rightarrow \quad \cos 2\theta = 0$

$\rightarrow 2\theta = \cos^{-1}(0) = \frac{\pi}{2} \rightarrow \theta = \frac{\pi}{4}$

$-\theta = -\frac{\pi}{4}$

$$R = \{(r,\theta) \mid -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, 0 \le r \le \cos 2\theta\}$$

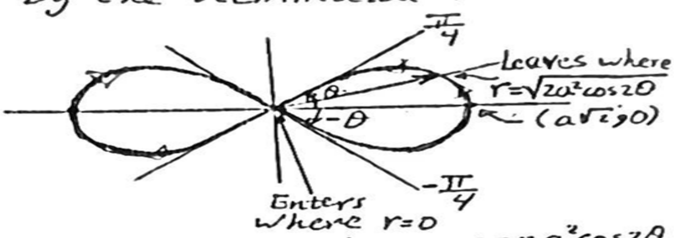
$$\text{Area of } R = \iint_R dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{1}{2}r^2\right]_0^{\cos 2\theta} d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta$$

$$= \frac{1}{4}\int_{-\pi/4}^{\pi/4} (1+\cos 4\theta)\,d\theta = \frac{1}{4}[\theta + \frac{1}{4}\sin 4\theta]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

Example 4

Find the area enclosed by the leminscate $r^2 = 2a^2\cos 2\theta$



solution: area of right-hand half $= \iint r\,dr\,d\theta$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{r=0}^{\sqrt{2a^2\cos 2\theta}} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{2a^2\cos 2\theta}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \frac{2a^2\cos 2\theta}{2}\,d\theta$$

$$= a^2\int_{-\pi/4}^{\frac{\pi}{4}} \cos 2\theta\,d\theta = a^2\frac{\sin 2\theta}{2}\Big]_{-\frac{\pi}{4}}^{\pi/4} = \frac{a^2}{2}[1-(-1)] = a^2$$

Total area $= 2a^2$

if $r=0$ $\quad 0 = 2a^2\cos 2\theta$

$0 = \cos 2\theta$

$2\theta = \cos^{-1}(0)$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$-\theta = -\frac{\pi}{4}$

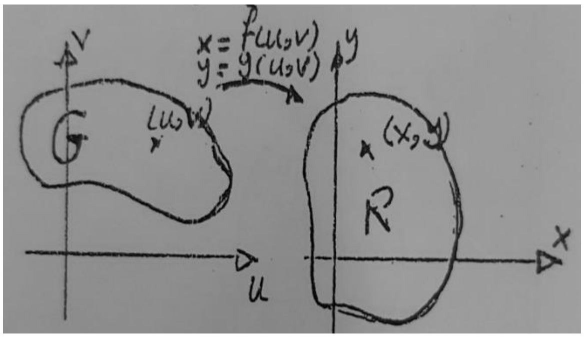
if $\theta = 0$ $\quad r^2 = 2a^2$

$r = a\sqrt{2}$

Change of Variable in a Double Integral

Some Cartesian double integral cannot be integrated unless it is transformed to another region of integration.If a region **G** in the **uv- plane** is transformed into the region **R** in the **xy- plane** by differentiable equation of the form **x= f(u,v)**, **y = g(u,v)** then a function **ϕ (x,y)**

Defined on R can be throughout of a function ϕ **[f(u,v), g(u,v)]** defined on G. The integral of **ϕ (x,y)** over **R** and the integral of **ϕ [f(u,v), g(u,v)] over G are related by the equation :**



$$\iint\limits_{\mathbf{R}} \phi(x,y)\,dx\,dy = \iint\limits_{\mathbf{G}} \phi[f(u,v), g(u,v)]\ |J|\,du\,dv \qquad \text{..... 1}$$

Where:

$J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the determinate of partial derivative and it is called Jacobian of the coordinate transformation.

In the case of polar coordinates, we have **r &θ** in place of **u &v**

Note

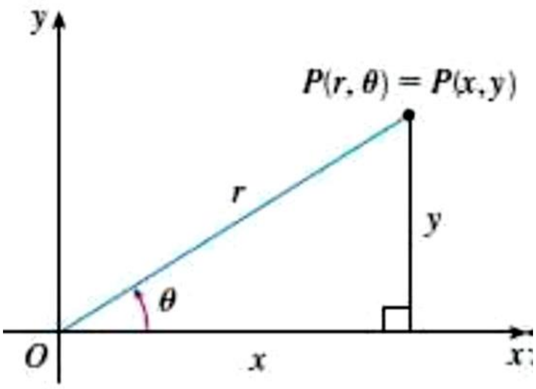
$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\frac{y}{x}$

$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Hence eq. 1becomes

$$\iint_R \Phi(x, y)\,dx\,dy = \iint_G \Phi(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

<u>Example 5:</u> Find the polar moment of inertia about the origin of a thin plate of density δ=1 bounded by the quarter circle $\mathbf{x^2+y^2=1}$ in the **first quadrant** using polar coordinates.

<u>Solution</u>

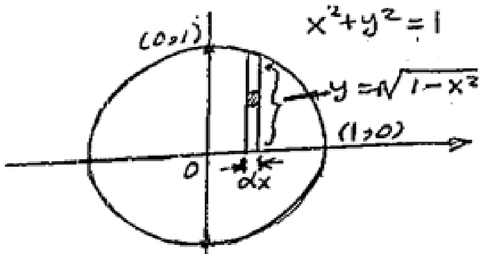
$\mathbf{x^2+y^2=1 \rightarrow y^2=1-x^2 \rightarrow y=\pm}\sqrt{1-x^2}$

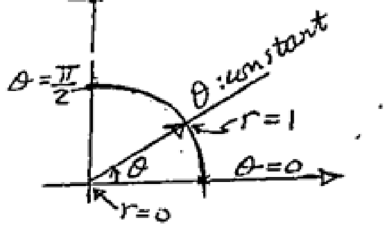
$y = \sqrt{1-x^2}$

$r^2\sin^2\theta = 1 - r^2\cos^2\theta$

$r^2(\sin^2\theta + \cos^2\theta) = 1$

$\rightarrow r^2 = 1 \rightarrow r = 1$





$r = x^2+y^2 = 1$

$r = 1$

$$I_o = \iint (x^2+y^2)\,\delta(x,y)\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1} (r^2\cos^2\theta + r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^2\overbrace{(\cos^2\theta+\sin^2\theta)}^{=1}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^2\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \int_0 \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{1}{4}\left[\theta\right]_0^{\pi/2} = \frac{\pi}{8}$$

Example 6: Change the double integral to an equivalent double integral in terms of polar coordinates and evaluate the resulting integral

$$\int_{x=0}^{x=2a}\int_{y=0}^{y=\sqrt{2ax-x^2}} x^2 dydx$$

$y = \sqrt{2ax - x^2}$

$y^2 = 2ax - x^2$

$y^2 + x^2 - 2ax = 0$ is the equation of a circle of radius =a & its center is point (a,0)

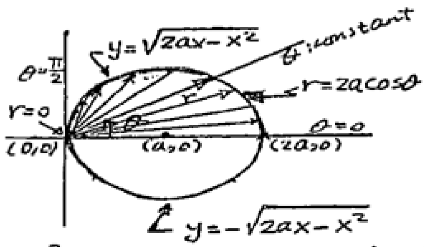
Find the limits

x= r cosθ = 2a → Cos θ = $\frac{2a}{r}$ but r = 2a cosθ

$\cos\theta = \frac{2a}{2a\,cos\theta} \rightarrow cos^2\theta = 1 \rightarrow cos\theta = 1 \rightarrow \theta = \cos^{-1} 1 = 0$

For x= r cosθ =0 → Cos θ = 0 $\rightarrow \theta = \cos^{-1} 0 = \frac{\pi}{2}$

$$= \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2a\cos\theta} \underbrace{x^2}_{r^2\cos^2\theta}\ \underbrace{dydx}_{rdrd\theta}$$



$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\cos^2\theta\, dr\, d\theta$$

$$\underbrace{y^2+x^2}_{r^2}-2ax=0$$
$$r^2-2ar\cos\theta=0$$
$$r(r-2a\cos\theta)=0$$
$$r=0 \text{ or } r=2a\cos\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta}\cos^2\theta\, d\theta$$

$$= \int_0^{\pi/2}\frac{16a^4\cos^6\theta}{4}d\theta = 4a^4\int_0^{\pi/2}\cos^6\theta\, d\theta$$

$$= 4a^4\int_0^{\pi/2}(\cos^2\theta)^3 d\theta = 4a^4\int_0^{\pi/2}\left(\frac{1+\cos2\theta}{2}\right)^3 d\theta$$

$$= \frac{a^4}{2}\int_0^{\pi/2}(1+3\cos2\theta+3\cos^2 2\theta+\cos^3 2\theta)\,d\theta$$

$$= \frac{a^4}{2}\int_0^{\pi/2}\left[1+3\cos2\theta+3\left[\frac{1+\cos4\theta}{2}\right]+\left[1-\sin^2 2\theta\right]\cos2\theta\right]d\theta$$

$$= \frac{a^4}{2}\int_0^{\pi/2}\left[1+3\cos2\theta+\frac{3}{2}+\frac{3\cos4\theta}{2}+\cos2\theta-\sin^2 2\theta\cos2\theta\right]d\theta$$

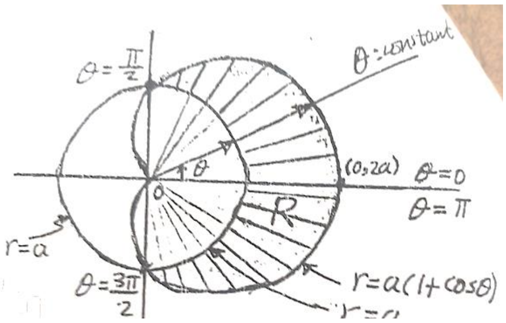
$$= \frac{a^4}{2}\int_0^{\pi/2}\left[\frac{5}{2}+4\cos2\theta+\frac{3}{2}\cos4\theta-\sin^2 2\theta\cos2\theta\right]d\theta$$

$$= \frac{a^4}{2}\left[\frac{5}{2}\theta+2\sin2\theta+\frac{3}{8}\sin4\theta-\frac{\sin^3 2\theta}{3\times2}\right]_0^{\pi/2}$$

$$= \frac{a^4}{2}\left[\left(\frac{5\pi}{4}+0+0-0\right)-(0+0+0-0)\right] = \frac{5a^4\pi}{8}$$

Example 7: find the centroid of the region that lies inside the cardioid **r = a (1+cosθ)**

And outside the circle **r=a**



Solution

$m = \iint \delta\,(r,\theta)dA \quad Since\ \delta\,(r,\theta) = 1$

$m = \iint dA$

$$A = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=a}^{r=a(1+\cos\theta)} r\,dr\,d\theta + \int_{\theta=\frac{3\pi}{2}}^{\theta=2\pi} \int_{r=a}^{r=a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \int_a^{a(1+\cos\theta)} r\,dr\,d\theta = 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta = \frac{2}{2}\int_0^{\frac{\pi}{2}} [a^2(1+\cos\theta)^2 - a^2]\,d\theta$$

$$= a^2 \int_0^{\pi/2} [1+2\cos\theta+\cos^2\theta-1]\,d\theta = a^2\int_0^{\pi/2}\left[2\cos\theta+\frac{1+\cos 2\theta}{2}\right]d\theta$$

$$= a^2\left[2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{a^2}{4}\left[8\sin\theta+2\theta+\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{4}\left[\left(8+\frac{2\pi}{2}+0\right)-(0+0+0)\right] = \frac{a^2}{4}(8+\pi)$$

$$M_x = \iint y\,\delta\,(x,y)dA = \iint y\,dA = \int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\sin\theta\, r\,dr\,d\theta + \int_{\frac{3\pi}{2}}^{2\pi}\int_a^{a(1+\cos\theta)} r\sin\theta\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_a^{a(1+\cos\theta)}\sin\theta\,d\theta + \int_{3\pi/2}^{2\pi}\left[\frac{r^3}{3}\right]_a^{a(1+\cos\theta)}\sin\theta\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}[(1+\cos\theta)^3-1]\sin\theta\,d\theta + \frac{a^3}{3}\int_{\frac{3\pi}{2}}[(1+ \cdots$$

$$= \frac{a^3}{3}\int_0^{\pi/2}[1+3\cos\theta+3\cos^2\theta+\cos^3\theta-1]\sin\theta\,d\theta + \frac{a^3}{3}\int_{\frac{3\pi}{2}}^{2\pi}[1+3\cos\theta+3\cos^2\theta+ \cdots$$

$$= \frac{a^3}{3}\left[\frac{-3\cos^2\theta}{2}-\frac{3\cos^3\theta}{3}-\frac{\cos^4\theta}{4}\right]_0^{\pi/2} + \frac{a^3}{3}\left[\frac{-3\cos^2\theta}{2}-\frac{3\cos^3\theta}{3}-\frac{\cos^4\theta}{4}\right]_{\frac{3\pi}{2}}^{2\pi}$$

$$= \frac{a^3}{3}\left[(0-0-0)-\left(-\frac{3}{2}-\frac{3}{3}-\frac{1}{4}\right)\right] + \frac{a^3}{3}\left[\left(-\frac{3}{2}-\frac{3}{3}-\frac{1}{4}\right)-(0-0-0)\right]$$

$$= \frac{a^3}{3}[2.75] + \frac{a^3}{3}[-2.75] = \frac{a^3}{3}[2.75-2.75] = 0$$

$$\bar{y} = \frac{M_x}{m} = \frac{0}{\frac{a^2}{4}(8+\pi)} = 0$$

$$M_y = \iint x\,dA = \int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\cos\theta\, r\,dr\,d\theta + \int_{3\pi/2}^{2\pi}\int_a^{a(1+\cos\theta)} r\cos\theta\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_a^{a(1+\cos\theta)}\cos\theta\,d\theta + \int_{3\pi/2}^{2\pi}\left[\frac{r^3}{3}\right]_a^{a(1+\cos\theta)}\cos\theta\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}\left[(1+\cos\theta)^3-1\right]\cos\theta\,d\theta + \frac{a^3}{3}\int_{3\pi/2}^{2\pi}\cdots$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left[1+3\cos\theta+3\cos^2\theta+\cos^3\theta-1\right]\cos\theta\,d\theta + \frac{a^3}{3}\int_{3\pi/2}^{2\pi}\left[1+3\cos\theta+3\cos^2\theta+\cdots\right.$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left[3\cos\theta+3\cos^2\theta+\cos^3\theta\right]\cos\theta\,d\theta + \frac{a^3}{3}\int_{3\pi/2}^{2\pi}\left[3\cos\theta+3\cos^2\theta+\cdots\right.$$

$$= \frac{a^3}{3}\int_0^{\pi/2}(3\cos^2\theta+3\cos^3\theta+\cos^4\theta)\,d\theta + \frac{a^3}{3}\int_{3\pi/2}^{2\pi}(3\cos^2\theta+3\cos^3\theta+\cos^4\theta)\,d\theta$$

$= \ldots\ldots$

$= \ldots$

$$M_y = \frac{a^3}{24}(32+15\pi)$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{a^3}{24}(32+15\pi)}{\frac{a^2}{4}(8+\pi)} = \frac{a(32+15\pi)}{6(8+\pi)} = \frac{a(32+15\pi)}{(48+6\pi)}$$