

3-2 Big- M Method to Solve LPP

In the previous article of the simplex algorithm we have seen that the method must start with a basic feasible solution. In previous examples so far, we have looked at problems that, when put into standard LP form, conveniently have an all slack starting solution. An all slack solution is only a possibility when all of the constraints in the problem have inequalities. Today, we are going to look at methods for dealing with LPs having other constraint types.

Remember that simplex needs a place to start. It must start from a basic feasible solution then move to another basic feasible solution to improve the objective value. With these assumptions, we can obtain an initial basic feasible solution/ dictionary by letting all slack variables be basic, all original variables be non- basic. Obviously, these assumptions do not hold for every LP. What do we do when they don't?

When a basic feasible solution is not readily apparent, the Big M method or the two- phase simplex method may be used to solve the problem.

Big- M method:

Step 1- express the problem in the standard form.

Step 2- add non- negative artificial variable to the left side of each of the equations corresponding to constraints of the type ' \geq ' or '='.

When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that variables will be zero in the final solution(provided the solution of the problem exists).

On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very large price (per unit penalty) to these variables in the objective function. Such large will be designated by $-M$ for maximization problems ($+M$ for minimizing problem), where $M > 0$.

Step 3- in the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

Ex : find the optimal solution for the following LPP.

$$\text{Max } z = 5x_1 + 12x_2 + 4x_3$$

$$x_1 + 2x_2 + x_3 \leq 5$$

$$2x_1 + x_2 + 3x_3 = 2$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

Convert to the standard form:

$$Z = 5x_1 + 12x_2 + 4x_3 - MR$$

$$x_1 + 2x_2 + x_3 + s_1 = 5$$

$$2x_1 - x_2 + 3x_3 + R = 2$$

$$R = 2 - 2x_1 + x_2 - 3x_3$$

Substitute in Z

$$Z = 5x_1 + 12x_2 + 4x_3 - M(2 - 2x_1 + x_2 - 3x_3)$$

$$= (5 + 2M)x_1 + (12 - M)x_2 + (4 + 3M)x_3 - 2M$$

$$Z-(5+2M)X_1-(12-M)X_2-(4+3M)X_3=-2M$$

$$X_1+2X_2+X_3+S_1=5$$

$$2X_1-X_2+3X_3+R=2$$

Basic var.	X1	X2	X3	S1	R	SOLU
Z	-5-2M	-12+M	-4-3M	0	0	-2M
S1	1	2	1	1	0	5
R	2	-1	3	0	1	2
Z	-7/3	-40/3	0	0	4/3+M	8/3
S1	1/3	7/3	0	1	-1/3	13/3
X3	2/3	-1/3	1	0	1/3	2/3
Z	-3/7	0	0	40/7	-4/7+M	192/7
X2	1/7	1	0	3/7	-1/7	13/7
X3	5/7	0	1	1/7	2/7	9/7
Z	0	0	3/5	29/5	-2/5+M	141/5
X2	0	1	-1/5	2/5	-1/5	3/5
X1	1	0	7/5	1/5	2/5	9/5

EX: Find the optimal solution for the following LPP.

$$\text{MIN } z=4x_1+x_2$$

S.T

$$3x_1+x_2=3$$

$$4x_1+3x_2\geq 6$$

$$x_1+2x_2\leq 4$$

$$x_1\geq 0, x_2\geq 0$$

Convert to the standard form

$$\text{Min } z=4x_1+x_2+MR_1+MR_2$$

Sto:

$$3x_1+x_2+R_1=3$$

$$4x_1+3x_2-S_1+R_2=6$$

$$x_1+2x_2+S_2=4$$

$$x_1, x_2, S_1, S_2, R_1, R_2 \geq 0$$

$$R_1=3-3x_1-x_2$$

$$R_2=6-4x_1-3x_2+S_1$$

$$Z= 4x_1+x_2+M(3-3x_1-x_2)+M(6-4x_1-3x_2+S_1)$$

$$=(4-7M)x_1+(1-4M)x_2+MS_1+9M$$

$$Z-(4-7M)x_1-(1-4M)x_2-MS_1=9M$$

$$3x_1+x_2+R_1=3$$

$$4x_1+3x_2-S_1+R_2=6$$

$$x_1+2x_2+S_2=4$$

Basic var	X1	X2	S1	S2	R1	R2	Solu.
Z	4-7M	-1+4M	-M	0	0	0	9M
R1	3	1	0	0	1	0	3
R2	4	3	-1	0	0	1	6
S2	1	2	0	1	0	0	4
Z	0	1/3+5/3M	-M	0	4/3-7/3M	0	4+2M
X1	1	1/3	0	0	1/3	0	1
R2	0	5/3	-1	0	-4/3	1	2
S2	0	5/3	0	1	-1/3	0	3
Z	0	0	1/5	0	8/5-M	-1/5-m	18/5
X1	1	0	1/5	0	3/5	-1/5	3/5
X2	0	1	-3/5	0	-4/5	3/5	6/5
S2	0	0	1	1	1	-1	1
Z	0	0	0	-1/5	7/5-M	-M	17/5
X1	1	0	0	-1/5	2/5	0	2/5
X2	0	1	0	3/5	-1/5	0	9/5
S1	0	0	1	1	1	-1	1

EX:

$$\text{Max } z = 3x_1 + 2x_2 + x_3$$

$$\text{S.T. } 2x_1 + x_2 + x_3 = 12$$

$$3x_1 + 4x_3 = 11$$

$x_2 \geq 0$, $x_3 \geq 0$ and x_1 is unrestricted

Solution

SLPP

$$\text{Max } z = 3(x_1' - x_1'') + 2x_2 + x_3 - M R_1 - M R_2 \text{ subject to}$$

$$2(x_1' - x_1'') + x_2 + x_3 + R_1 = 12$$

$$3(x_1' - x_1'') + 4x_3 + R_2 = 11$$

$$X1', x1'', x2, x3, R1, R2 \geq 0$$

$$\text{Max } z = 3X1' - 3X1'' + 2X2 + X3 - M R1 - M R2$$

$$2x1' - 2x1'' + x2 + x3 + R1 = 12$$

$$3x1' - 3x1'' + 4x2 + R2 = 11$$

$$X1', x2'', X2, X3, R1, R2 \geq 0.$$

Cj	3	-3	2	1	-M	-M	
Basic var	X1'	X1''	X2	X3	R1	R2	SOL
R1	2	-2	1	1	1	0	12
R2	3	-3	4	0	0	1	11
Z	-5m-3	5m+3	-5m-2	-m-1	0	0	-23m
R1	0	0	-5/3	1	1	x	-14/3
X1'	1	-1	4/3	0	0	x	-11/3
Z	0	0	5/3 M+2	-M-1	0	x	-14M/3-11
X3	0	0	-5/3	1	x	x	14/3
X1'	1	-1	4/3	0	x	x	11/3
Z	0	0	1/3	0	x	x	47/3

Since all $Z \geq 0$, optimal basic feasible solution is obtained

$$X1' = 11/3, x1'' = 0$$

$$X1 = x1' - x1'' = 11/3 - 0 = 11/3$$

Therefore the solution is $\text{max } z = 47/3, x1 = 11/3, x2 = 0, x3 = 14/3$

3-3 Two – Phase Method

The process of eliminating artificial variables is performed in phase- I of the solution and phase-II is used to get an optimal solution. Since the solution of LPP is computed in two phases, it is called as Two-Phase Simplex Method.

Phase I – in this phase, the simplex method is applied to a specially constructed auxiliary linear programming problem leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1- assign a cost-1 to each artificial variable and a cost 0 to all other variables in the objective function.

Step 2- construct the auxiliary LPP in which the new objective function z^* is to be maximized subject to the given set of constraints.

Step 3- solve the auxiliary problem by simplex method until either of the following three possibilities do arise.

i. $\text{Max } z^* < 0$ and at least one artificial vector appear in the optimum basic at a positive level ($\Delta_j \geq 0$). In this case, given problem does not possess any feasible solution.

ii. $\text{Max } z^* = 0$ and at least one artificial vector appears in the optimum basis at a zero level. In this case proceed to phase- II.

iii. $\text{Max } z^* = 0$ and one artificial vector appears in the optimum basis. In this case also proceed to phase- II.

Phase II- now assign the actual cost to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints.

Simplex method is applied to the modified simplex table obtained at the end of phase- I, until an optimum basic feasible solution has been attained. The artificial variables which are non- basic at the end of phase- I are removed.

Ex

$$\text{Max } z = 3x_1 - x_2$$

S.T

$$2x_1 + x_2 \geq 2$$

$$x_1 + 3x_2 \leq 2$$

$$x_2 \leq 4 \text{ and } x_1 \geq 0, x_2 \geq 0$$

$$\text{Max } Z = 3x_1 - x_2$$

Subject to

$$2x_1 + x_2 - s_1 + R_1 = 2$$

$$x_1 + 3x_2 + s_2 = 2$$

$$x_2 + s_3 = 4$$

$$x_1, x_2, s_1, s_2, s_3, R_1 \geq 0$$

Phase I

Cj	0	0	0	0	0	-1	
Basic var	X1	X2	s1	s2	s3	R1	SOL
R1	2	1	-1	0	0	1	2
S2	1	3	0	1	0	0	2
S3	0	1	0	0	1	0	4
Z	-2	-1	1	0	0	0	-2
x1	1	1/2	-1/2	0	0	x	1
S2	0	5/2	1/2	1	0	x	1
S3	0	1	0	0	1	x	4
Z	0	0	0	0	0	x	0

Since all $Z \geq 0$, $\max z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

Cj	3	-1	0	0	0	
Basic var	X1	X2	s1	s2	s3	SOL
X1	1	1/2	-1/2	0	0	1
S2	0	5/2	1/2	1	0	1
S3	0	1	0	0	1	4
Z	0	5/2	-3/2	0	0	3
X1	1	3	0	0	0	2
S1	0	5	1	2	0	2
S3	0	1	0	0	1	4
Z	0	10	0	3	0	6

Since all $Z \geq 0$, optimal basis feasible solution is obtained

therefore the solution is:

Max $z=6$, $x_1=2$, $x_2=0$

Ex

MAX $z=5x_1+8x_2$

S.T

$$3x_1+2x_2 \geq 3$$

$$x_1+4x_2 \geq 4$$

$$x_1+x_2 \leq 5$$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Standard LPP

Max $Z=5X_1+8X_2$ subject to

$$3X_1+2X_2-S_1+R_1=3$$

$$X_1+4X_2-S_2+R_2=4$$

$$X_1+X_2+S_3=5$$

$X_1, X_2, S_1, S_2, S_3, R_1, R_2 \geq 0$

Auxiliary LPP

Max $Z^*=0X_1+0X_2+0S_1+0S_2+0S_3-1R_1-1R_2$ subject to

$$3X_1+2X_2-S_1+R_1=3$$

$$X_1+4X_2-S_2+R_2=4$$

$$X_1+X_2+S_3=5 \quad X_1, X_2, S_1, S_2, S_3, R_1, R_2 \geq 0$$

Phase I

Cj	0	0	0	0	0	-1	-1	
Basic var	X1	X2	S1	S2	S3	R1	R2	SOL
R1	3	2	-1	0	0	1	0	3
R2	1	4	0	-1	0	0	1	4
S3	1	1	0	0	1	0	0	5
Z	-4	-6	1	1	0	0	0	-7
R1	5/2	0	-1	1/2	0	1	X	1
X2	1/4	1	0	-1/4	0	0	X	1
S3	3/4	0	0	1/4	1	0	X	4
Z	-5/2	0	1	-1/2	0	0	X	-1
X1	1	0	-2/5	1/5	0	X	X	2/5
X2	0	1	1/10	-3/10	0	X	X	9/10
S3	0	0	3/10	1/10	1	X	X	37/10
								0
Z*=0	0	0	0	0	0	0	X	←

Since all $Z \geq 0$ max $z^*=0$ and no artificial vector appears in the basis, we proceed to phase II.

Cj	5	8	0	0	0	
Basic var	X1	X2	s1	s2	s3	SOL
x1	1	0	-2/5	1/5	0	2/5
x2	0	1	1/10	-3/10	0	9/10
S3	0	0	3/10	1/10	1	37/10
Z	0	0	-6/5	-7/5	0	46/5
S2	5	0	-2	1	0	2
x2	3/2	1	-1/2	0	0	3/2
s3	-1/2	0	1/2	0	1	7/2
Z	7	0	-4	0	0	12
S3	3	0	0	1	2	16
X2	1	1	0	0	1/2	5
S1	-1	0	1	0	2	7
Z	3	0	0	0	4	40

X1=9/5 X2=8/5 Z=141/5