Queuing Theory

What is Queuing?

- Any obstruction of traffic flow results in a queue
- Traffic queues in congested periods is a source of considerable delay and loss of performance
- Under extreme conditions queuing delay can account for 90% or more of a motorist's total trip travel time

Queuing theory

1. **Queuing theory is a broad field of study of situations that involve lines or queues**
    - retail stores
    - manufacturing plants
    - transportation
        - traffic lights
        - toll booths
        - stop signs
        - etc.
2. **Processes by which queues form and dissipate**

Queuing Theory – acronyms

• FIFO - a family of models that use the principle of "first in first out"

• LIFO - "last in first out"

• a/d/N notation

- a - arrival type (either D- deterministic, or Mmechanistic (i.e. exponential distribution or similar))
- d - departure type (either D- deterministic, or Mmechanistic)
- N - number of "channels"

Notation Example

• D/D/1

- Deterministic arrivals
- Deterministic departures
- One departure channel

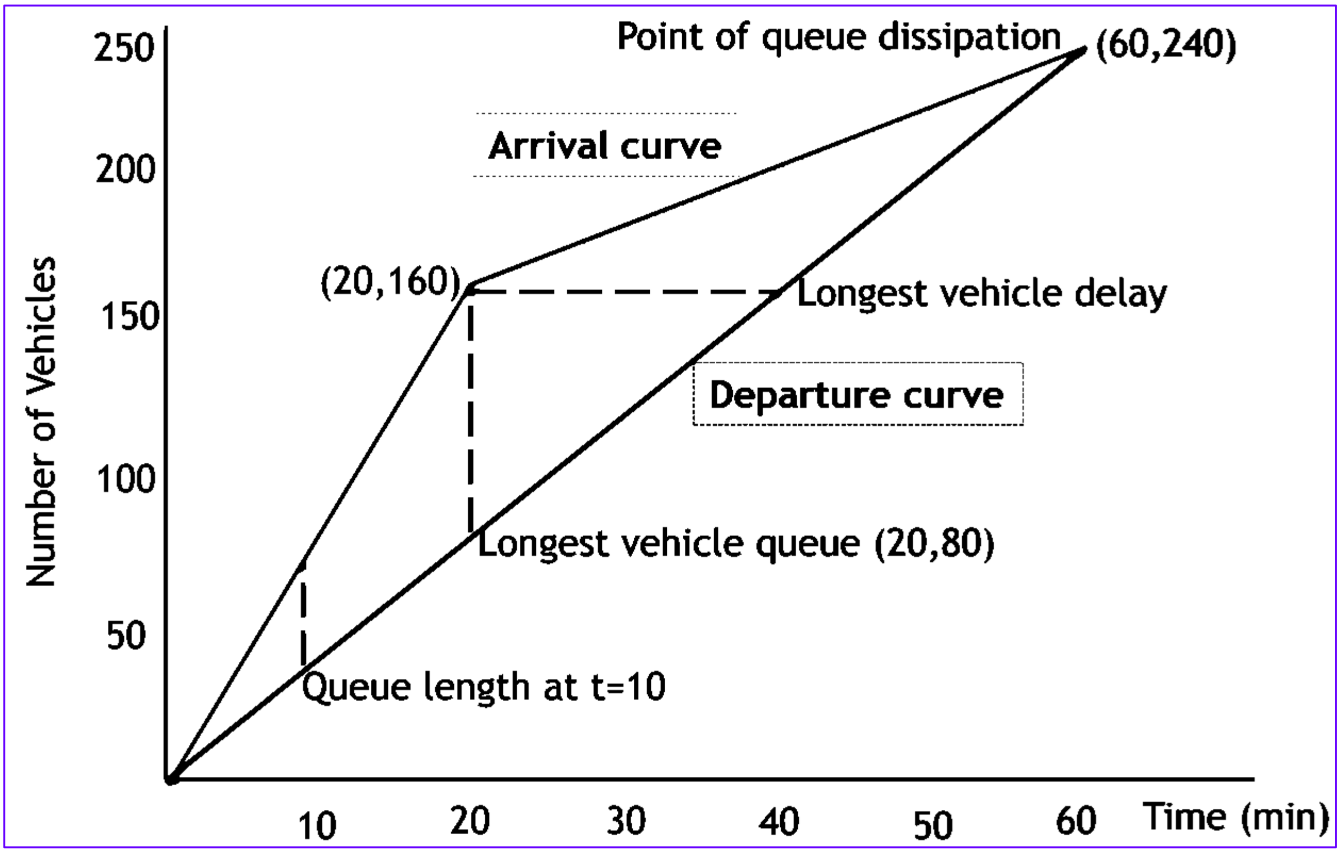
• M/D/1

- Exponential arrivals
- Deterministic departures
- One departure channel

D/D/1 Queuing Example

- Entrance gate to National Park
- Deterministic arrivals and departures, one fee booth, first in first out
- At the opening of the booth (8:00am), there is no queue, cars arrive at a rate of 480veh/hr for 20 minutes and then changes to 120veh/hr
- The fee booth attendant spends 15seconds with each car
- Determine the following
    - What is the longest queue? When does it occur?
    - When will the queue dissipate?
    - What is the total time of delay by all vehicles?
    - What is the average delay, longest delay?
    - What delay is experienced by the 200th car to arrive?
- Arrival rate (denoted as λ)
    - $\lambda = \frac{480 \text{ veh/h}}{60 \text{ min/h}} = 8\ veh/\min\ for\ t \leq 20\ min$
    - $\lambda = \frac{120 \text{ veh/h}}{60 \text{ min/h}} = 2\ veh/\min\ for\ t > 20\ min$
- Departure rate (denoted as μ)
    - $\mu = \frac{60 \text{ sec/min}}{15 \text{ sec/veh}} = 4\ veh/\min$

- Let t-> number of minutes after start of queue
- Vehicle arrival can be written as
  - $8t \; for \; t \leq 20 \; min$
  - $160 + 2 \; t - 20 \; for \; t > 20 \; min$
- Vehicle departure can be written as
  - 4



- What is the longest vehicle queue? When does it occur?
  - Occurs at 20th minute
  - Vehicle queue = 80
- When will the queue dissipate?
  - 160 + 2 t – 20 = 4t – t = 60 min
  - Since queue started at 8am, 240 vehicles would have arrived, and 240 vehicles would have departed
- What is the total time of delay by all vehicles?
  - Area between the arrival and departure curves
  - 0.5(80*20) + 0.5 (80*40) = 2400 veh-min
- What is the average delay per vehicle?

- 2400 veh-min / 240 vehicles = 10 min / veh

- What is the average queue length?
  - 2400 veh-min / 60 min = 40 vehicles

D/D/1 queuing

- Easy graphical interpretation
- Mathematical construct is also easy

M/D/1 queuing

- Arrival pattern is not often deterministic – Often random (unless peak periods)
- Graphical solution is sometime difficult
- However, mathematical construct is straightforward
- Define a new term traffic density (ρ)
  - $\rho = \lambda\ \mu$
  - λ: average vehicle arrival rate (vehicle per unit time)
  - μ: average vehicle departure rate (vehicle per unit time)
  - ρ: traffic intensity (unitless)
- When ρ <1
  - D/D/1 process will not predict any queue information
  - However, M/D/1 is based on random arrivals, will predict queue formations.

M/D/1 queuing performance

- Average length of queue

$$\bar{Q} = \rho^2 / 2(1-\rho)$$

- Average waiting time in queue

$$\bar{w} = \rho / 2\mu(1-\rho)$$

- Average time spent in the system

$$t = 2-\rho / 2\mu(1-\rho)$$

M/D/1 queuing example

- Let us use the same example as D/D/1, but the vehicle arrival rate is
    - 180 veh/hr and poisson distributed
- Compute the following
    - Average length of queue
    - Average waiting time
    - Average time spent in the system

- Arrival rate

$$\lambda = \frac{180 \text{ veh/h}}{60 \text{ min/h}} = 3\ veh/\min\ for\ all\ t$$

- Departure rate

$$\mu = \frac{60 \text{ sec/min}}{15 \text{ sec/veh}} = 4\ veh/\min$$

- Traffic intensity

$$\rho = {}^{\lambda}/_{\mu} = \frac{3 \text{ veh/min}}{4 \text{ veh/ min}} = 0.75$$

- Average length of queue

$$\bar{Q} = \frac{0.75^2}{2(1-0.75)} = 1.125 \text{ veh}$$

- Average waiting time in queue

$$\bar{W} = \frac{0.75^2}{2*4(1-0.75)} = 0.375 \text{ min/veh}$$

- Average time spent in the system

$$\bar{t} = \frac{2-0.75}{2*4\ (1-0.75)} = 0.625 \text{ min/veh}$$

M/M/1 queuing

- Exponentially distributed arrival and departure times
- One departure channel
- Example-toll booth

- Average length of queue

$$\bar{Q} = \frac{\rho}{(1-\rho)}^2$$

- Average waiting time in queue

$$\bar{w} = \frac{\lambda}{\mu(\mu-\lambda)}$$

- Average time spent in the system

$$\bar{t} = \frac{1}{\mu(\mu-\lambda)} = \bar{w} + 1$$

M/M/1 queuing example

Assume the park attendant takes an average of 15 sec to distribute brochures but the distribution time varies depending on whether park patrons have questions relating to park operating policies. Given average arrival rate of 180 veh/h, compute

- ✓ Average length of queue
- ✓ Average waiting time
- ✓ Average time spent in the system

- Average length of queue

$$\bar{Q} = \frac{\rho}{(1-\rho)}^2 = 2.25\ veh$$

- Average waiting time in queue

$$\bar{w} = \frac{\lambda}{\mu(\mu-\lambda)} = 0.75 \text{ min/veh}$$

- Average time spent in the system

$$\bar{t} = \frac{1}{\mu(\mu-\lambda)} = \bar{w} + 1 = 1 \text{ min/veh}$$