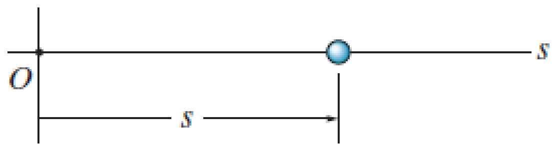
DYNAMICS

Dynamics: dynamics, which deals with the accelerated motion of a body. The subject of dynamics will be presented in two parts: kinematics, which treats only the geometric aspects of the motion, and kinetics, which is the analysis of the forces causing the motion. To develop these principles, the dynamics of a particle will be discussed first, followed by topics in rigid-body dynamics in two and then three dimensions.

kinematics of a particle

We will begin our study of dynamics by discussing the kinematics of a particle that moves along a rectilinear or straight-line path. Recall that a particle has a mass but negligible size and shape. Therefore we must limit application to those objects that have dimensions that are of no consequence in the analysis of the motion. In most problems, we will be interested in bodies of finite size, such as rockets, projectiles, or vehicles. Each of these objects can be considered as a particle, as long as the motion is characterized by the motion of its mass center and any rotation of the body is neglected.

Rectilinear Kinematics. The kinematics of a particle is characterized by specifying, at any given instant, the particle's position, velocity, and acceleration Position. The straight-line path of a particle will be defined using a single coordinate axis s, Fig. a. The origin O on the path is a fixed point, and from this point the position coordinate s is used to specify the location of the particle at any given instant. The magnitude of s is the distance from O to the particle, usually measured in meters (m) or feet (ft), and the sense of direction is defined by the algebraic sign on s. Although the choice is arbitrary, in this case s is positive since the coordinate axis is positive to the right of the origin. Likewise, it is negative if the particle is located to the left of O. Realize that position is a vector quantity since it has both magnitude and direction. Here, however, it is being represented by the algebraic scalar s, rather than in boldface s, since the direction always remains along the coordinate axis.

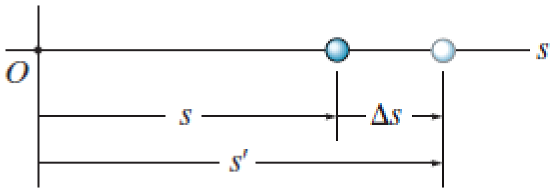


Position

(a)

Displacement. The displacement of the particle is defined as the change in its position. For example, if the particle moves from one point to another, Fig. b, the displacement is

$$\Delta s = s' - s$$



Displacement

(b)

In this case Δs is positive since the particle's final position is to the right of its initial position, i.e., s' > s. Likewise, if the final position were to the left of its initial position, Δs would be negative.

The displacement of a particle is also a vector quantity, and it should be distinguished from the distance the particle travels. Specifically, the distance traveled is a positive scalar that represents the total length of path over which the particle travels

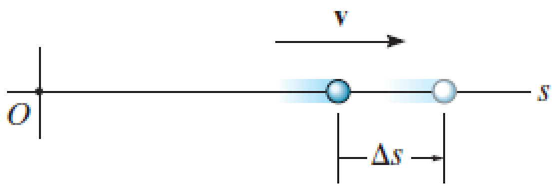
Velocity. If the particle moves through a displacement Δs during the time interval Δt, the average velocity of the particle during this time interval is

$$v_{avg} = \frac{\Delta s}{\Delta t}$$

If we take smaller and smaller values of t, the magnitude of s becomes smaller and smaller. Consequently, the instantaneous velocity is a vector defined as $v = \lim_{\Delta t \to 0} (\Delta s / \Delta t)$, or

$$\boxed{v = \frac{ds}{dt}}$$

Since t or dt is always positive, the sign used to define the sense of the velocity is the same as that of s or ds. For example, if the particle is moving to the right, Fig. c, the velocity is positive; whereas if it is moving to the left, the velocity is negative. The magnitude of the velocity is known as the speed, and it is generally expressed in units of m/s or ft/s. Occasionally, the term "average speed" is used. The average speed is always a positive scalar and is defined as the total distance traveled by a particle, sT , divided by the elapsed time Δt; i.e.,

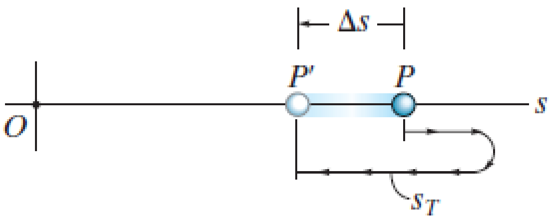


Velocity

(c)

$$(v_{sp})_{avg} = \frac{s_T}{\Delta t}$$

For example, the particle in Fig. d travels along the path of length s_T in time Δt, so its average speed is $(\upsilon sp)avg = S_T/\Delta t$, but its average velocity is $\upsilon_{avg} = -\Delta s/\Delta t$



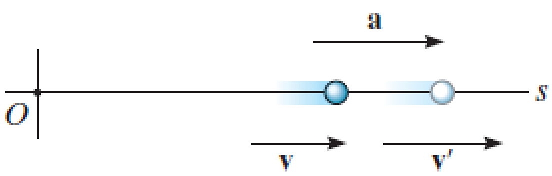
Average velocity and
Average speed

(d)

Acceleration. Provided the velocity of the particle is known at two points, the average acceleration of the particle during the time interval Δt is defined as

$$a_{\text{avg}} = \frac{\Delta v}{\Delta t}$$

Here Δυ represents the difference in the velocity during the time interval Δt, i.e., $\Delta\upsilon = \upsilon' - \upsilon$, Fig. e.



Acceleration

(e)

The instantaneous acceleration at time t is a vector that is found by taking smaller and smaller values of Δt and corresponding smaller and smaller values of Δυ, so that

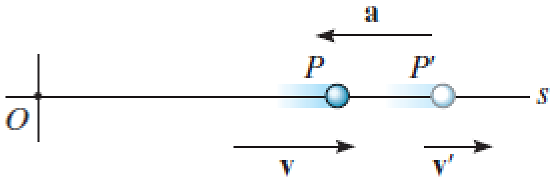
$a = \lim_{\Delta t \to 0} (\Delta v/\Delta t)$, or

$$\boxed{a = \frac{dv}{dt}}$$

Substituting Eq. 12–1 into this result, we can also write

$$a = \frac{d^2s}{dt^2}$$

Both the average and instantaneous acceleration can be either positive or negative. In particular, when the particle is slowing down, or its speed is decreasing, the particle is said to be decelerating. In this case, υ' in Fig. f is less than υ, and so Δυ = υ' - υ will be negative. Consequently, a will also be negative, and therefore it will act to the left, in the opposite sense to υ. Also, notice that if the particle is originally at rest, then it can have an acceleration if a moment later it has a velocity v'; and, if the velocity is constant, then the acceleration is zero since Δυ = υ - υ = 0. Units commonly used to express the magnitude of acceleration are m/s^2 or ft/s^2.



Deceleration

(f)

Finally, an important differential relation involving the displacement, velocity, and acceleration along the path may be obtained by eliminating the time differential dt between Eqs. 12–1 and 12–2. We have

$$dt = \frac{ds}{v} = \frac{dv}{a}$$

$$\boxed{a\,ds = v\,dv}$$

Although we have now produced three important kinematic equations, realize that the above equation is not independent of Eqs. 12–1 and 12–2.

Constant Acceleration, $a = a_c$. When the acceleration is constant, each of the three kinematic equations $a_c = dv/dt$, $\upsilon = ds/dt$,and $a_c\, ds = \upsilon\, d\upsilon$ can be integrated to obtain formulas that relate a_c , υ, s, and t.

Velocity as a Function of Time. Integrate $a_c = dv/dt$, assuming that initially $\upsilon = \upsilon_0$ when t = 0.

$$\int_{v_0}^{v} dv = \int_0^t a_c\, dt$$

$$\boxed{\begin{gathered} v = v_0 + a_c t \\ \text{Constant Acceleration} \end{gathered}} \qquad (12–4)$$

Position as a Function of Time. Integrate $\upsilon = ds/dt = \upsilon_0 + a_c t$, assuming that initially s = s_0 when t = 0.

$$\int_{s_0}^{s} ds = \int_{0}^{t} (v_0 + a_c t)\, dt$$

$$\boxed{\begin{array}{c} s = s_0 + v_0 t + \frac{1}{2} a_c t^2 \\ \text{Constant Acceleration} \end{array}} \qquad (12\text{–}5)$$

Velocity as a Function of Position. Either solve for t in Eq. 12–4 and substitute into Eq. 12–5, or integrate $\upsilon\, d\upsilon = a_c\, ds$, assuming that initially $\upsilon = \upsilon_0$ at $s = s_0$.

$$\int_{v_0}^{v} v\, dv = \int_{s_0}^{s} a_c\, ds$$

$$\boxed{\begin{array}{c} v^2 = v_0^2 + 2a_c(s - s_0) \\ \text{Constant Acceleration} \end{array}} \qquad (12\text{–}6)$$

The algebraic signs of s0 , v0 , and ac , used in the above three equations,are determined from the positive direction of the s axis as indicated by the arrow written at the left of each equation. Remember that these equations are useful only when the acceleration is constant and when t = 0, $s = s_0$, $\upsilon = \upsilon_0$. A typical example of constant accelerated motion occurs when a body falls freely toward the earth. If air resistance is neglected and the distance of fall is short, then the downward acceleration of the body when it is close to the earth is constant and approximately 9.81 m/s^2 or 32.2 ft/s^2.

Important Points

Dynamics is concerned with bodies that have accelerated motion.

- Kinematics is a study of the geometry of the motion.
- Kinetics is a study of the forces that cause the motion.
- Rectilinear kinematics refers to straight-line motion.
- Speed refers to the magnitude of velocity.

 Average speed is the total distance traveled divided by the total time. This is different from the average velocity, which is the displacement divided by the time.

 A particle that is slowing down is decelerating.

 A particle can have an acceleration and yet have zero velocity.

 The relationship a ds = υ dυ is derived from a = dυ/dt and υ = ds/dt, by eliminating dt.

Procedure for Analysis

Coordinate System.

- Establish a position coordinate s along the path and specify its *fixed origin* and positive direction.
- Since motion is along a straight line, the vector quantities position, velocity, and acceleration can be represented as algebraic scalars. For analytical work the sense of s, v, and a is then defined by their *algebraic signs*.
- The positive sense for each of these scalars can be indicated by an arrow shown alongside each kinematic equation as it is applied.

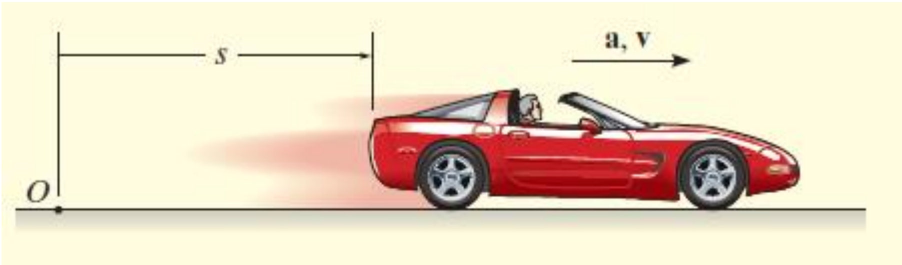
Kinematic Equations.

- If a relation is known between any *two* of the four variables a, v, s, and t, then a third variable can be obtained by using one of the kinematic equations, $a = dv/dt$, $v = ds/dt$ or $a\,ds = v\,dv$, since each equation relates all three variables.*
- Whenever integration is performed, it is important that the position and velocity be known at a given instant in order to evaluate either the constant of integration if an indefinite integral is used, or the limits of integration if a definite integral is used.
- Remember that Eqs. 12–4 through 12–6 have only limited use. These equations apply *only* when the *acceleration is constant* and the initial conditions are $s = s_0$ and $v = v_0$ when $t = 0$.

*Some standard differentiation and integration formulas are given in Appendix A.

PROBLEMS

1- The car on the left in the photo and in Fig. moves in a straight line such that for a short time its velocity is defined by $v = (3t^2 + 2t)$ ft/s, where t is in seconds. Determine its position and acceleration when t = 3 s. When t = 0, s = 0.



SOLUTION

Coordinate System. The position coordinate extends from the fixed origin O to the car, positive to the right.

Position. Since v = f(t), the car's position can be determined from v = ds>dt, since this equation relates υ, s, and t. Noting that s = 0 when t = 0, we have*

$$(\overset{+}{\rightarrow}) \qquad v = \frac{ds}{dt} = (3t^2 + 2t)$$

$$\int_0^s ds = \int_0^t (3t^2 + 2t)dt$$

$$s\Big|_0^s = t^3 + t^2\Big|_0^t$$

$$s = t^3 + t^2$$

When $t = 3$ s,

$$s = (3)^3 + (3)^2 = 36 \text{ ft}$$

Acceleration. Since $v = f(t)$, the acceleration is determined from $a = dv/dt$, since this equation relates a, v, and t.

$$(\overset{+}{\rightarrow}) \qquad a = \frac{dv}{dt} = \frac{d}{dt}(3t^2 + 2t)$$

$$= 6t + 2$$

When $t = 3$ s,

$$a = 6(3) + 2 = 20 \text{ ft/s}^2 \rightarrow \qquad \textit{Ans.}$$

NOTE: The formulas for constant acceleration *cannot* be used to solve this problem, because the acceleration is a function of time.

EXAMPLE 12.2

A small projectile is fired vertically *downward* into a fluid medium with an initial velocity of 60 m/s. Due to the drag resistance of the fluid the projectile experiences a deceleration of $a = (-0.4v^3)\ \text{m/s}^2$, where v is in m/s. Determine the projectile's velocity and position 4 s after it is fired.

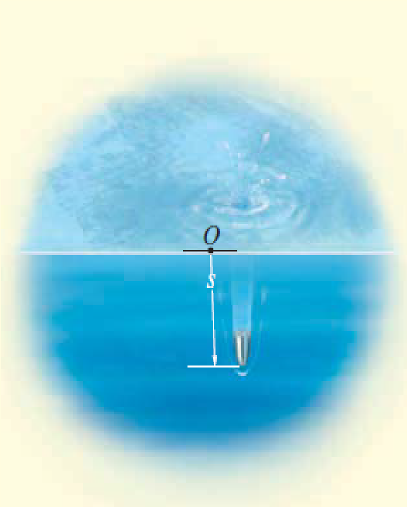


Fig. 12–3

SOLUTION

Coordinate System. Since the motion is downward, the position coordinate is positive downward, with origin located at O, Fig. 12–3.

Velocity. Here $a = f(v)$ and so we must determine the velocity as a function of time using $a = dv/dt$, since this equation relates v, a, and t. (Why not use $v = v_0 + a_c t$?) Separating the variables and integrating, with $v_0 = 60$ m/s when $t = 0$, yields

$$(+\downarrow) \qquad a = \frac{dv}{dt} = -0.4v^3$$

$$\int_{60\text{ m/s}}^{v} \frac{dv}{-0.4v^3} = \int_0^t dt$$

$$\frac{1}{-0.4}\left(\frac{1}{-2}\right)\frac{1}{v^2}\bigg|_{60}^{v} = t - 0$$

$$\frac{1}{0.8}\left[\frac{1}{v^2} - \frac{1}{(60)^2}\right] = t$$

$$v = \left\{\left[\frac{1}{(60)^2} + 0.8t\right]^{-1/2}\right\}\ \text{m/s}$$

Here the positive root is taken, since the projectile will continue to move downward. When $t = 4$ s,

$$v = 0.559\ \text{m/s}\downarrow \qquad \textit{Ans.}$$

Position. Knowing $v = f(t)$, we can obtain the projectile's position from $v = ds/dt$, since this equation relates s, v, and t. Using the initial condition $s = 0$, when $t = 0$, we have

$$(+\downarrow) \qquad v = \frac{ds}{dt} = \left[\frac{1}{(60)^2} + 0.8t\right]^{-1/2}$$

$$\int_0^s ds = \int_0^t \left[\frac{1}{(60)^2} + 0.8t\right]^{-1/2} dt$$

$$s = \frac{2}{0.8}\left[\frac{1}{(60)^2} + 0.8t\right]^{1/2}\bigg|_0^t$$

$$s = \frac{1}{0.4}\left\{\left[\frac{1}{(60)^2} + 0.8t\right]^{1/2} - \frac{1}{60}\right\}\ \text{m}$$

When $t = 4$ s,

$$s = 4.43\ \text{m} \qquad \textit{Ans.}$$

EXAMPLE 12.3

During a test a rocket travels upward at 75 m/s, and when it is 40 m from the ground its engine fails. Determine the maximum height s_B reached by the rocket and its speed just before it hits the ground. While in motion the rocket is subjected to a constant downward acceleration of 9.81 m/s^2 due to gravity. Neglect the effect of air resistance.

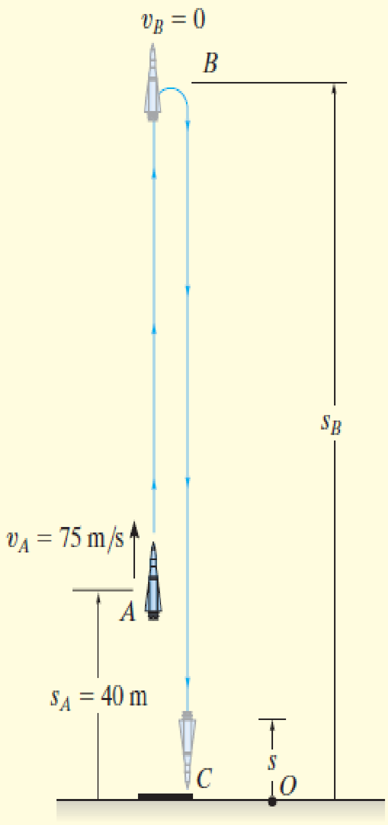


Fig. 12–4

SOLUTION

Coordinate System. The origin O for the position coordinate s is taken at ground level with positive upward, Fig. 12–4.

Maximum Height. Since the rocket is traveling *upward*, $v_A = +75$ m/s when $t = 0$. At the maximum height $s = s_B$ the velocity $v_B = 0$. For the entire motion, the acceleration is $a_c = -9.81$ m/s^2 (negative since it acts in the *opposite* sense to positive velocity or positive displacement). Since a_c is *constant* the rocket's position may be related to its velocity at the two points A and B on the path by using Eq. 12–6, namely,

$$(+\uparrow) \qquad v_B^2 = v_A^2 + 2a_c(s_B - s_A)$$

$$0 = (75 \text{ m/s})^2 + 2(-9.81 \text{ m/s}^2)(s_B - 40 \text{ m})$$

$$s_B = 327 \text{ m} \qquad \textit{Ans.}$$

Velocity. To obtain the velocity of the rocket just before it hits the ground, we can apply Eq. 12–6 between points B and C, Fig. 12–4.

$$(+\uparrow) \qquad v_C^2 = v_B^2 + 2a_c(s_C - s_B)$$

$$= 0 + 2(-9.81 \text{ m/s}^2)(0 - 327 \text{ m})$$

$$v_C = -80.1 \text{ m/s} = 80.1 \text{ m/s} \downarrow \qquad \textit{Ans.}$$

The negative root was chosen since the rocket is moving downward.

Similarly, Eq. 12–6 may also be applied between points A and C, i.e.,

$$(+\uparrow) \qquad v_C^2 = v_A^2 + 2a_c(s_C - s_A)$$

$$= (75 \text{ m/s})^2 + 2(-9.81 \text{ m/s}^2)(0 - 40 \text{ m})$$

$$v_C = -80.1 \text{ m/s} = 80.1 \text{ m/s} \downarrow \qquad \textit{Ans.}$$

NOTE: It should be realized that the rocket is subjected to a *deceleration* from A to B of 9.81 m/s^2, and then from B to C it is *accelerated* at this rate. Furthermore, even though the rocket momentarily comes to *rest* at B ($v_B = 0$) the acceleration at B is still 9.81 m/s^2 downward!

EXAMPLE 12.4

A metallic particle is subjected to the influence of a magnetic field as it travels downward through a fluid that extends from plate A to plate B, Fig. 12–5. If the particle is released from rest at the midpoint C, $s = 100$ mm, and the acceleration is $a = (4s)\text{ m/s}^2$, where s is in meters, determine the velocity of the particle when it reaches plate B, $s = 200$ mm, and the time it takes to travel from C to B.

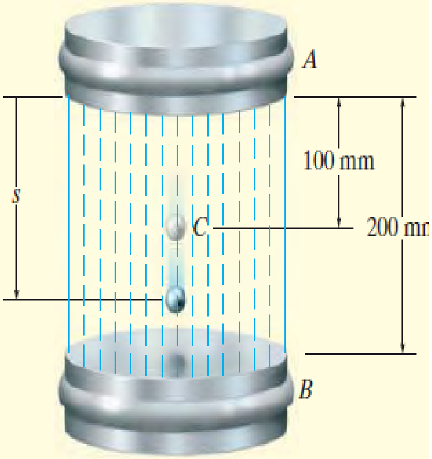


Fig. 12–5

SOLUTION

Coordinate System. As shown in Fig. 12–5, s is positive downward, measured from plate A.

Velocity. Since $a = f(s)$, the velocity as a function of position can be obtained by using $v\,dv = a\,ds$. Realizing that $v = 0$ at $s = 0.1$ m, we have

$$(+\downarrow) \qquad v\,dv = a\,ds$$

$$\int_0^v v\,dv = \int_{0.1\text{ m}}^s 4s\,ds$$

$$\frac{1}{2}v^2\Big|_0^v = \frac{4}{2}s^2\Big|_{0.1\text{ m}}^s$$

$$v = 2(s^2 - 0.01)^{1/2}\text{ m/s} \qquad (1)$$

At $s = 200\text{ mm} = 0.2$ m,

$$v_B = 0.346\text{ m/s} = 346\text{ mm/s}\downarrow \qquad \textit{Ans.}$$

The positive root is chosen since the particle is traveling downward, i.e., in the $+s$ direction.

Time. The time for the particle to travel from C to B can be obtained using $v = ds/dt$ and Eq. 1, where $s = 0.1$ m when $t = 0$. From Appendix A,

$$(+\downarrow) \qquad ds = v\,dt$$

$$= 2(s^2 - 0.01)^{1/2}dt$$

$$\int_{0.1}^s \frac{ds}{(s^2 - 0.01)^{1/2}} = \int_0^t 2\,dt$$

$$\ln\left(\sqrt{s^2 - 0.01} + s\right)\Big|_{0.1}^s = 2t\Big|_0^t$$

$$\ln\left(\sqrt{s^2 - 0.01} + s\right) + 2.303 = 2t$$

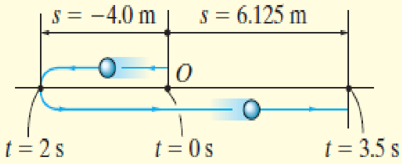
At $s = 0.2$ m,

$$t = \frac{\ln\left(\sqrt{(0.2)^2 - 0.01} + 0.2\right) + 2.303}{2} = 0.658\text{ s} \qquad \textit{Ans.}$$

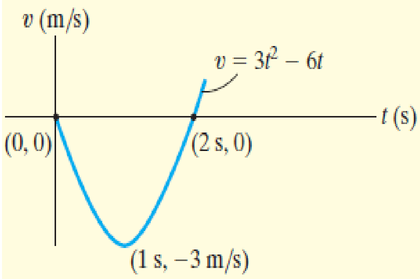
NOTE: The formulas for constant acceleration cannot be used here because the acceleration changes with position, i.e., $a = 4s$.

EXAMPLE 12.5

A particle moves along a horizontal path with a velocity of $v = (3t^2 - 6t)$ m/s, where t is the time in seconds. If it is initially located at the origin O, determine the distance traveled in 3.5 s, and the particle's average velocity and average speed during the time interval.



(a)



(b)

Fig. 12–6

SOLUTION

Coordinate System. Here positive motion is to the right, measured from the origin O, Fig. 12–6*a*.

Distance Traveled. Since $v = f(t)$, the position as a function of time may be found by integrating $v = ds/dt$ with $t = 0$, $s = 0$.

$(\overset{+}{\rightarrow})$

$$ds = v\,dt$$

$$= (3t^2 - 6t)\,dt$$

$$\int_0^s ds = \int_0^t (3t^2 - 6t)\,dt$$

$$s = (t^3 - 3t^2)\text{ m} \qquad (1)$$

In order to determine the distance traveled in 3.5 s, it is necessary to investigate the path of motion. If we consider a graph of the velocity function, Fig. 12–6*b*, then it reveals that for $0 < t < 2$ s the velocity is *negative*, which means the particle is traveling to the *left*, and for $t > 2$ s the velocity is *positive*, and hence the particle is traveling to the *right*. Also, note that $v = 0$ at $t = 2$ s. The particle's position when $t = 0$, $t = 2$ s, and $t = 3.5$ s can be determined from Eq. 1. This yields

$$s|_{t=0} = 0 \quad s|_{t=2\text{ s}} = -4.0\text{ m} \quad s|_{t=3.5\text{ s}} = 6.125\text{ m}$$

The path is shown in Fig. 12–6*a*. Hence, the distance traveled in 3.5 s is

$$s_T = 4.0 + 4.0 + 6.125 = 14.125\text{ m} = 14.1\text{ m} \qquad \textit{Ans.}$$

Velocity. The *displacement* from $t = 0$ to $t = 3.5$ s is

$$\Delta s = s|_{t=3.5\text{ s}} - s|_{t=0} = 6.125\text{ m} - 0 = 6.125\text{ m}$$

and so the average velocity is

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{6.125\text{ m}}{3.5\text{ s} - 0} = 1.75\text{ m/s} \rightarrow \qquad \textit{Ans.}$$

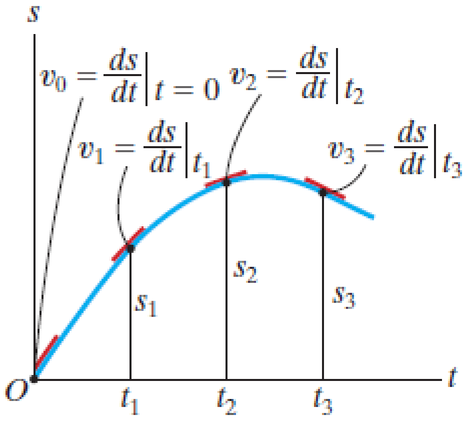
The average speed is defined in terms of the *distance traveled* s_T. This positive scalar is

$$(v_{sp})_{avg} = \frac{s_T}{\Delta t} = \frac{14.125\text{ m}}{3.5\text{ s} - 0} = 4.04\text{ m/s} \qquad \textit{Ans.}$$

NOTE: In this problem, the acceleration is $a = dv/dt = (6t - 6)$ m/s^2, which is not constant.

Rectilinear Kinematics: Erratic Motion

When a particle has erratic or changing motion then its position, velocity, and acceleration cannot be described by a single continuous mathematical function along the entire path. Instead, a series of functions will be required to specify the motion at different intervals. For this reason, it is convenient to represent the motion as a graph. If a graph of the motion that relates any two of the variables s,υ, a, t can be drawn, then this graph can be used to construct subsequent graphs relating two other variables since the variables are related by the differential relationships υ = ds/dt, a = dv/dt, or a ds = v dv. Several situations occur frequently.



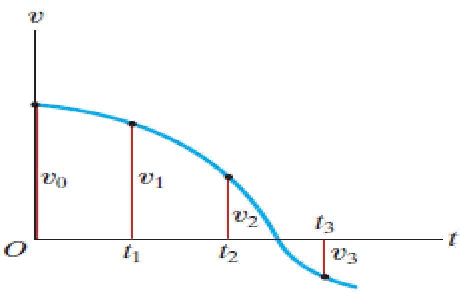
(a)

The s–t, v–t, and a–t Graphs. To construct the v-t graph given the s–t graph, Fig. a, the equation υ = ds/dt should be used, since it relates the variables s and t to v. This equation states that

$$\frac{ds}{dt} = v$$

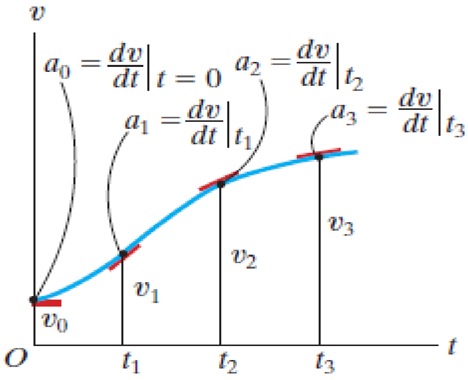
$$\text{slope of } s\text{–}t \text{ graph} = \text{velocity}$$

For example, by measuring the slope on the s–t graph when t = t_1, the velocity is v_1, which is plotted in Fig. b. The v/t graph can be constructed by plotting this and other values at each instant.



(b)

The a–t graph can be constructed from the v/t graph in a similar manner, Fig. a, since

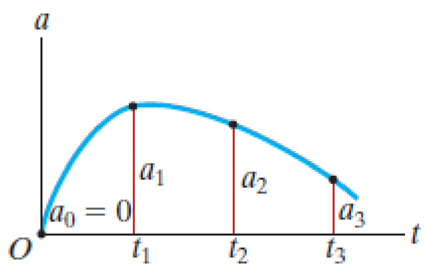


(a)

$$\frac{dv}{dt} = a$$

slope of v–t graph = acceleration

Examples of various measurements are shown in Fig. a and plotted in Fig. b.



(b)

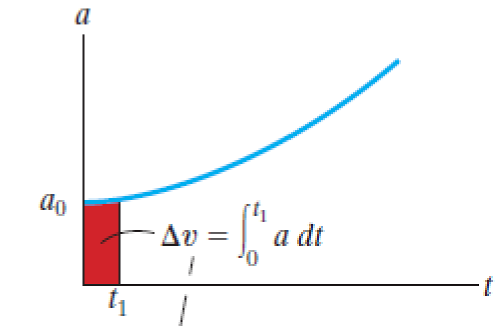
If the s–t curve for each interval of motion can be expressed by amathematical function s = s(t), then the equation of the v9t graph for the same interval can be obtained by differentiating this function with respect to time since v = ds/dt. Likewise, the equation of the a–t graph for the same interval can be determined by differentiating v = v(t) since a = dv/dt. Since differentiation reduces a polynomial of degree n to that of degree n – 1, then if the s–t graph is parabolic (a second-degree curve), the v/t graph will be a sloping line (a first-degree curve), and the a–t graph will be a constant or a horizontal line (a zero-degree curve).

If the a–t graph is given, Fig.a, the v-t graph may be constructed using a = dv/dt, written as

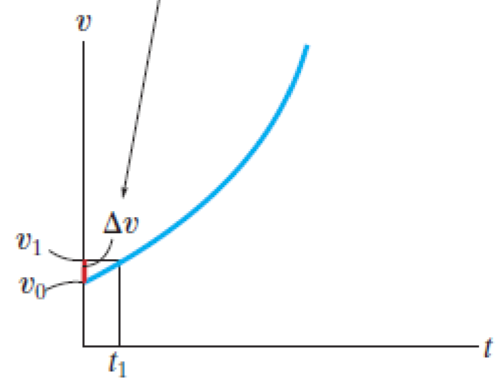
$$\Delta v = \int a\,dt$$

change in velocity = area under a–t graph

Hence, to construct the v-t graph, we begin with the particle's initial velocity v_0 and then add to this small increments of area (v) determined from the a–t graph. In this manner successive points, $\upsilon_1 = \upsilon_0 + \Delta\upsilon$, etc., for the v-t graph are determined, Fig. b. Notice that an algebraic addition of the area increments of the a–t graph is necessary, since areas lying above the t axis correspond to an increase in v ("positive" area), whereas those lying below the axis indicate a decrease in v ("negative" area).



(a)



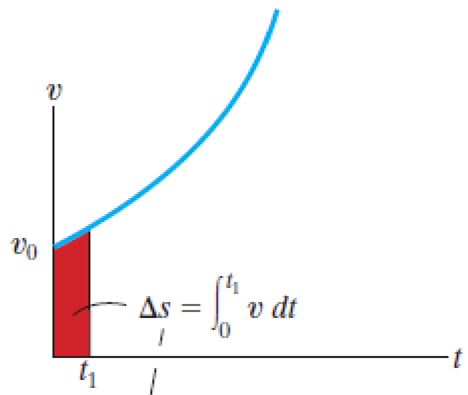
(b)

Similarly, if the v-t graph is given, Fig. a1, it is possible to determine the s–t graph using v = ds/dt, written as

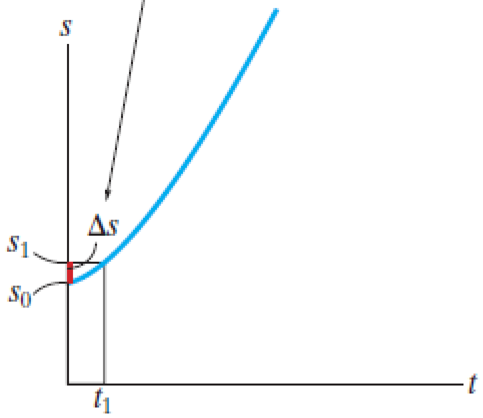
$$\Delta s = \int v\,dt$$

$$\text{displacement} = \begin{array}{c}\text{area under}\\ v\text{–}t\text{ graph}\end{array}$$

If segments of the a–t graph can be described by a series of equations, then each of these equations can be integrated to yield equations describing the corresponding segments of the v-t graph. In a similar manner, the s–t graph can be obtained by integrating the equations which describe the segments of the v-t graph. As a result, if the a–t graph is linear (a first-degree curve), integration will yield a v-t graph that is parabolic (a second-degree curve) and an s–t graph that is cubic (third-degree curve).



(a)



(b)

The v–s and a–s Graphs. If the a–s graph can be constructed, then points on the v-s graph can be determined by using υ dυ = a ds. Integrating this equation between the limits υ = υ_0 at s = s_0 and υ = υ1 at s = s_1 , we have,

$$\tfrac{1}{2}(v_1^2 - v_0^2) = \int_{s_0}^{s_1} a\,ds$$

area under
a–s graph

Therefore, if the red area in Fig. 12–11a is determined, and the initial velocity v_0 at s_0 = 0 is known, then $v_1 = (2\int_0^{s_1} a\,ds + v_0^2)^{1/2}$, , Fig. b. Successive points on the v–s graph can be constructed in this manner.

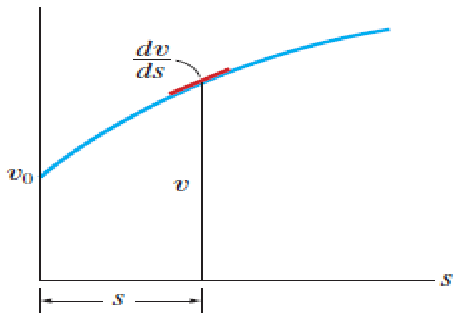
If the v–s graph is known, the acceleration a at any position s can be determined using a ds = v dv, written as

$$a = v\left(\frac{dv}{ds}\right)$$

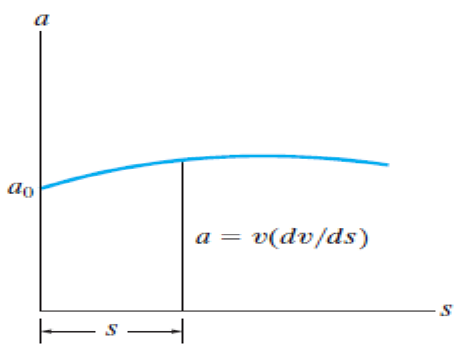
$$\text{acceleration} = \begin{array}{l}\text{velocity times}\\ \text{slope of}\\ v\text{–}s\text{ graph}\end{array}$$

Thus, at any point (s, v) in Fig. a, the slope dv>ds of the v–s graph is measured. Then with v and dv>ds known, the value of a can be calculated, Fig. b.

The v–s graph can also be constructed from the a–s graph, or vice versa, by approximating the known graph in various intervals with mathematical functions, v = f(s) or a = g(s), and then using a ds = v dv to obtain the other graph.



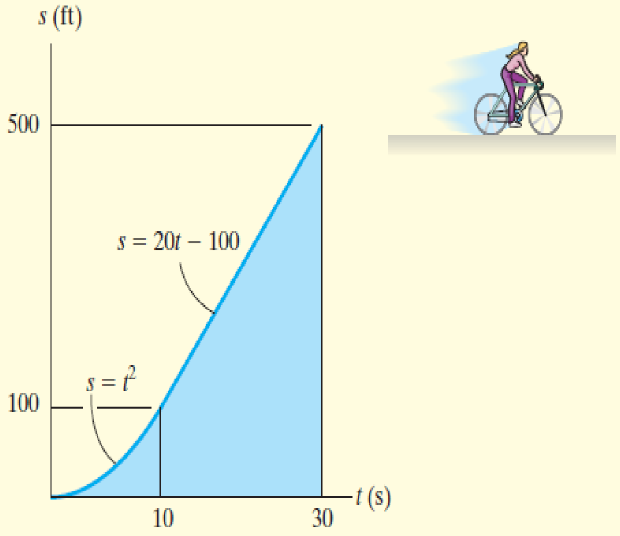
(a)



(b)

EXAMPLE 12.6

A bicycle moves along a straight road such that its position is described by the graph shown in Fig. 12–13*a*. Construct the v–t and a–t graphs for $0 \leq t \leq 30$ s.



(a)

SOLUTION

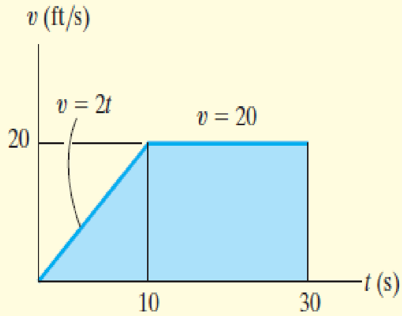
v–t Graph. Since $v = ds/dt$, the v–t graph can be determined by differentiating the equations defining the s–t graph, Fig. 12–13*a*. We have

$$0 \leq t < 10 \text{ s}; \qquad s = (t^2) \text{ ft} \qquad v = \frac{ds}{dt} = (2t) \text{ ft/s}$$

$$10 \text{ s} < t \leq 30 \text{ s}; \qquad s = (20t - 100) \text{ ft} \qquad v = \frac{ds}{dt} = 20 \text{ ft/s}$$

The results are plotted in Fig. 12–13*b*. We can also obtain specific values of v by measuring the *slope* of the s–t graph at a given instant. For example, at $t = 20$ s, the slope of the s–t graph is determined from the straight line from 10 s to 30 s, i.e.,

$$t = 20 \text{ s}; \qquad v = \frac{\Delta s}{\Delta t} = \frac{500 \text{ ft} - 100 \text{ ft}}{30 \text{ s} - 10 \text{ s}} = 20 \text{ ft/s}$$



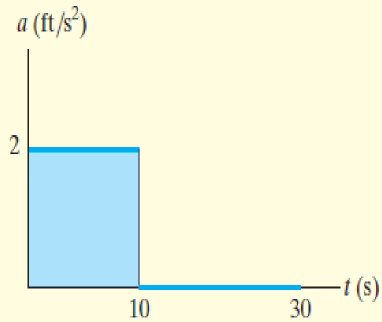
(b)

a–t Graph. Since $a = dv/dt$, the a–t graph can be determined by differentiating the equations defining the lines of the v–t graph. This yields

$$0 \leq t < 10 \text{ s}; \qquad v = (2t) \text{ ft/s} \qquad a = \frac{dv}{dt} = 2 \text{ ft/s}^2$$

$$10 < t \leq 30 \text{ s}; \qquad v = 20 \text{ ft/s} \qquad a = \frac{dv}{dt} = 0$$

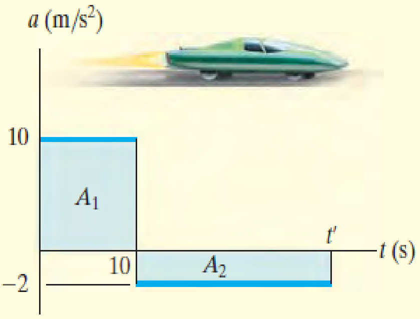
The results are plotted in Fig. 12–13*c*.



(c)

NOTE: Show that $a = 2 \text{ ft/s}^2$ when $t = 5$ s by measuring the slope of the v–t graph.

EXAMPLE 12.7



(a)

The car in Fig. 12–14*a* starts from rest and travels along a straight track such that it accelerates at 10 m/s^2 for 10 s, and then decelerates at 2 m/s^2. Draw the v–t and s–t graphs and determine the time t' needed to stop the car. How far has the car traveled?

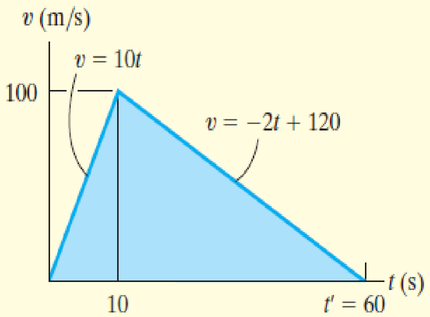
SOLUTION

v–t Graph. Since $dv = a\,dt$, the v–t graph is determined by integrating the straight-line segments of the a–t graph. Using the *initial condition* $v = 0$ when $t = 0$, we have

$$0 \le t < 10 \text{ s}; \qquad a = (10) \text{ m/s}^2; \qquad \int_0^v dv = \int_0^t 10\,dt, \qquad v = 10t$$

When $t = 10$ s, $v = 10(10) = 100$ m/s. Using this as the *initial condition* for the next time period, we have

$$10 \text{ s} < t \le t'; a = (-2) \text{ m/s}^2; \int_{100 \text{ m/s}}^{v} dv = \int_{10 \text{ s}}^{t} -2\,dt, v = (-2t + 120) \text{ m/s}$$



(b)

When $t = t'$ we require $v = 0$. This yields, Fig. 12–14*b*,

$$t' = 60 \text{ s} \qquad \textit{Ans.}$$

A more direct solution for t' is possible by realizing that the area under the a–t graph is equal to the change in the car's velocity. We require $\Delta v = 0 = A_1 + A_2$, Fig. 12–14*a*. Thus

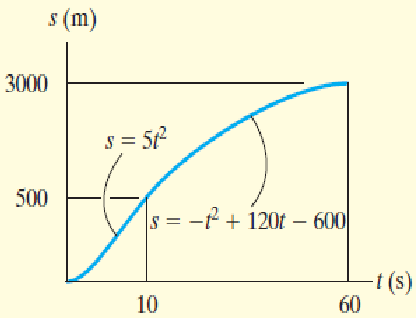
$$0 = 10 \text{ m/s}^2(10 \text{ s}) + (-2 \text{ m/s}^2)(t' - 10 \text{ s})$$

$$t' = 60 \text{ s} \qquad \textit{Ans.}$$

s–t Graph. Since $ds = v\,dt$, integrating the equations of the v–t graph yields the corresponding equations of the s–t graph. Using the *initial condition* $s = 0$ when $t = 0$, we have

$$0 \le t \le 10 \text{ s}; \qquad v = (10t) \text{ m/s}; \qquad \int_0^s ds = \int_0^t 10t\,dt, \qquad s = (5t^2) \text{ m}$$

When $t = 10$ s, $s = 5(10)^2 = 500$ m. Using this *initial condition*,



(c)

$$10 \text{ s} \le t \le 60 \text{ s}; v = (-2t + 120) \text{ m/s}; \int_{500 \text{ m}}^{s} ds = \int_{10 \text{ s}}^{t} (-2t + 120)\,dt$$

$$s - 500 = -t^2 + 120t - [-(10)^2 + 120(10)]$$

$$s = (-t^2 + 120t - 600) \text{ m}$$

When $t' = 60$ s, the position is

$$s = -(60)^2 + 120(60) - 600 = 3000 \text{ m} \qquad \textit{Ans.}$$

The s–t graph is shown in Fig. 12–14*c*.

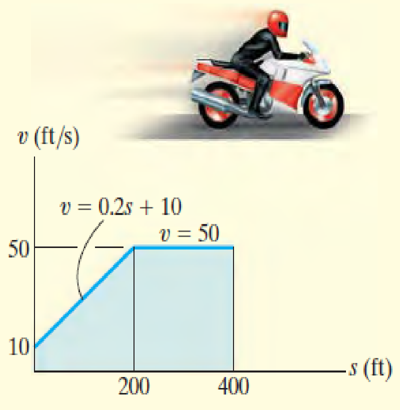
NOTE: A direct solution for s is possible when $t' = 60$ s, since the *triangular area* under the v–t graph would yield the displacement $\Delta s = s - 0$ from $t = 0$ to $t' = 60$ s. Hence,

$$\Delta s = \tfrac{1}{2}(60 \text{ s})(100 \text{ m/s}) = 3000 \text{ m} \qquad \textit{Ans.}$$

Fig. 12–14

EXAMPLE 12.8

The v–s graph describing the motion of a motorcycle is shown in Fig. 12–15a. Construct the a–s graph of the motion and determine the time needed for the motorcycle to reach the position $s = 400$ ft.



(a)

SOLUTION

a–s Graph. Since the equations for segments of the v–s graph are given, the a–s graph can be determined using $a\,ds = v\,dv$.

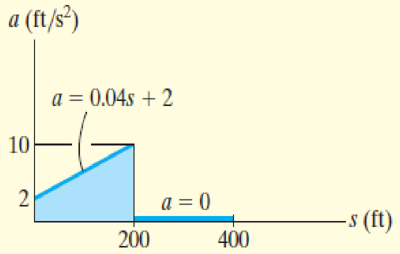
$0 \le s < 200$ ft; $\quad v = (0.2s + 10)$ ft/s

$$a = v\frac{dv}{ds} = (0.2s + 10)\frac{d}{ds}(0.2s + 10) = 0.04s + 2$$

$200 \text{ ft} < s \le 400$ ft; $\quad v = 50$ ft/s

$$a = v\frac{dv}{ds} = (50)\frac{d}{ds}(50) = 0$$

The results are plotted in Fig. 12–15b.



(b)

Fig. 12–15

Time. The time can be obtained using the v–s graph and $v = ds/dt$, because this equation relates v, s, and t. For the first segment of motion, $s = 0$ when $t = 0$, so

$0 \le s < 200$ ft; $\quad v = (0.2s + 10)$ ft/s; $\quad dt = \dfrac{ds}{v} = \dfrac{ds}{0.2s + 10}$

$$\int_0^t dt = \int_0^s \frac{ds}{0.2s + 10}$$

$$t = (5\ln(0.2s + 10) - 5\ln 10)\text{ s}$$

At $s = 200$ ft, $t = 5\ln[0.2(200) + 10] - 5\ln 10 = 8.05$ s. Therefore, using these initial conditions for the second segment of motion,

$200 \text{ ft} < s \le 400$ ft; $\quad v = 50$ ft/s; $\quad dt = \dfrac{ds}{v} = \dfrac{ds}{50}$

$$\int_{8.05\text{ s}}^t dt = \int_{200\text{ m}}^s \frac{ds}{50};$$

$$t - 8.05 = \frac{s}{50} - 4; \quad t = \left(\frac{s}{50} + 4.05\right)\text{s}$$

Therefore, at $s = 400$ ft,

$$t = \frac{400}{50} + 4.05 = 12.0\text{ s} \qquad \textit{Ans.}$$

NOTE: The graphical results can be checked in part by calculating slopes. For example, at $s = 0$, $a = v(dv/ds) = 10(50 - 10)/200 = 2\text{ m/s}^2$. Also, the results can be checked in part by inspection. The v–s graph indicates the initial increase in velocity (acceleration) followed by constant velocity ($a = 0$).