Homogenous:

A first order D.E in the form of

$$f(x,y)dx + g(x,y)dy = 0$$

is a homogenous type if both functions $f(x,y)$ and $g(x,y)$ are homogenous of same degree n.
That is, multiplying each verable by prameter λ, we find:

$$f(\lambda x,\lambda y) = \lambda^n f(x,y) \quad \text{and}$$

$$g(\lambda x,\lambda y) = \lambda^n g(x,y)$$

$$\frac{f(\lambda x,\lambda y)}{g(\lambda x,\lambda y)} = \frac{f(x,y)}{g(x,y)}$$

Method of solution:

1- Determine whether the equation homogenous or not.

2- use substitution $y = vx$ and

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$ in the original D.E

3- Seprate the verable x and v.

4- integrate both sides.

5- Cheng v to $\frac{y}{x}$.

Example1: Solve the D.E

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$$

solution:

Test for homogenous

$$\frac{dy}{dx} = \frac{\lambda^2 x^2 - \lambda^2 y^2}{\lambda x \lambda y} = \frac{\cancel{\lambda^2}(x^2 - y^2)}{\cancel{\lambda^2}(xy)}$$

$\therefore$ homogenous D.E

$$v = \frac{y}{x} \quad \text{or} \quad y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy} \qquad \div x^2$$

$$\frac{dy}{dx} = \frac{1 - \frac{y^2}{x^2}}{\frac{y}{x}}$$

$$v + x\frac{dv}{dx} = \frac{1-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{1-v^2-v^2}{v} = \frac{1-2v^2}{v}$$

$$-\frac{1}{4}\int \frac{-4v}{1-2v^2}\,dv = \int \frac{dx}{x}$$

$$-\frac{1}{4}\ln|1-2v^2| = \ln x + c$$

$$-\frac{1}{4}\ln\left|1-2\frac{y^2}{x^2}\right| = \ln x + c$$

Example 2: solve the D.E

$$(y^2+xy)\,dx - x^2\,dy = 0$$

solution:

Test for homogenous

$$\frac{dy}{dx} = \frac{y^2+xy}{x^2}$$

$$\frac{dy}{dx} = \frac{\lambda^2y^2+\lambda x\lambda y}{\lambda^2x^2} = \frac{\cancel{\lambda^2}(y^2+xy)}{\cancel{\lambda^2}\,x^2}$$

$\therefore$ homogenous D.E

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x}$$

$$\frac{y}{x} = v \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\cancel{v} + x\frac{dv}{dx} = v^2 + \cancel{v}$$

$$x\frac{dv}{dx} = v^2$$

$$\int \frac{dx}{x} = \int \frac{dv}{v^2}$$

$$\ln x = -\frac{1}{v} + c$$

$$\ln x = -\frac{1}{\frac{y}{x}} + c$$

$$\ln x = -\frac{x}{y} + c$$

Example 3: solve the D.E

$$2xy\,dy + (x^2 - y^2)\,dx = 0$$

Solution

Test for homogenous

$$2xy\,dy = (y^2 - x^2)\,dx$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

$$\frac{dy}{dx} = \frac{\lambda^2 y^2 - \lambda^2 x^2}{2\lambda x\lambda y} = \frac{\cancel{\lambda^2}(y^2 - x^2)}{\cancel{\lambda^2}\,2xy}$$

∴ homogeneous

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{\frac{y^2}{x^2} - 1}{2\frac{y}{x}}$$

$$v = \frac{y}{x} \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{-v^2 - 1}{2v}$$

$$\int\frac{dx}{x} = -\int\frac{-2v}{-v^2 - 1}\,dv$$

$$\ln x + c = -\ln\left|-v^2 - 1\right|$$

$$\ln x + c = -\ln\left|-\frac{y^2}{x^2} - 1\right|$$

Example 4: Solve the D.E

$$\frac{dy}{dx} = \frac{y}{x} + \cos\left(\frac{y-x}{x}\right)$$

solution:

Test for homogenous

$$\frac{dy}{dx} = \frac{\cancel{\lambda} y}{\cancel{\lambda} x} + \cos\left(\frac{\cancel{\lambda} y - \cancel{\lambda} x}{\cancel{\lambda} x}\right) \quad \therefore \text{homogenous}$$

$$\frac{dy}{dx} = \frac{y}{x} + \cos\left(\frac{y}{x} - 1\right)$$

$$v = \frac{y}{x} \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + \cos(v-1)$$

$$\int \frac{dx}{x} = \int \sec(v-1)\,dv$$

$$\int \frac{dx}{x} = \int \sec(v-1) * \frac{\sec(v-1) + \tan(v-1)}{\sec(v-1) + \tan(v-1)}\,dv$$

$$\int \frac{dx}{x} = \int \frac{\sec^2(v-1) + \sec(v-1)\tan(v-1)}{\sec(v-1) + \tan(v-1)}\,dv$$

$$\ln x + c = \ln\left|\sec(v-1) + \tan(v-1)\right|$$

$$\ln x + c = \ln\left|\sec\left(\frac{y}{x} - 1\right) + \tan\left(\frac{y}{x} - 1\right)\right|$$

Example 5: Solve the D.E

$$(2xy+3y^2)dx-(x^2+2xy)dy=0$$

solution:

Test for homogenous

$$\frac{dy}{dx}=\frac{2xy+3y^2}{x^2+2xy}=\frac{2\lambda x\lambda y+3\lambda^2y^2}{\lambda^2x^2+2\lambda x\lambda y}$$

$$\frac{dy}{dx}=\frac{\cancel{\lambda^2}(2xy+3y^2)}{\cancel{\lambda^2}(x^2+2xy)}=\frac{2xy+3y^2}{x^2+2xy}$$

$\therefore$ homogenous

$$v=\frac{y}{x} \text{ and } \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=\frac{2v+3v^2}{1+2v}$$

$$x\frac{dv}{dx}=\frac{2v+3v^2}{1+2v}-v$$

$$x\frac{dv}{dx}=\frac{2v+3v^2-v-2v^2}{1+2v}$$

$$x\frac{dv}{dx}=\frac{v+v^2}{1+2v}$$

$$\frac{dx}{x}=\frac{1+2v}{v+v^2}dv$$

$$\ln x = \ln|v + v^2| + c$$

$$\ln x = \ln\left|\frac{y}{x} + \frac{y^2}{x^2}\right| + c$$

Example 6: solve the following first order ordinary D.E:

$$(x^2 + xy)\frac{dy}{dx} = xy - y^2$$

Solution:

Test for homogenous

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2 + xy} = \frac{\lambda x \lambda y - \lambda^2 y^2}{\lambda^2 x^2 + \lambda x \lambda y}$$

$$\frac{dy}{dx} = \frac{\cancel{\lambda^2}(xy - y^2)}{\cancel{\lambda^2}(x^2 + xy)} \quad \therefore \text{ homogenous}$$

$$v = \frac{y}{x} \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{v - v^2}{1 + v}$$

$$x\frac{dv}{dx} = \frac{v - v^2}{1 + v} - v$$

$$x\frac{dv}{dx} = \frac{\cancel{v} - v^2 - \cancel{v} - v^2}{1 + v} = \frac{-2v^2}{1 + v}$$

$$\frac{1+v}{2v^2}\,dv = -\frac{dx}{x}$$

$$\frac{1+v}{v^2}\,dv = -2\frac{dv}{v}$$

$$\int\left[\frac{1}{v^2}+\frac{1}{v}\right]dv = -2\int\frac{dx}{x}$$

$$-\frac{1}{v}+\ln v = -2\ln x + c$$

$$-\frac{x}{y}+\ln\frac{y}{x} = -2\ln x + c$$

Example 7: Solve the D-E

$$-y\,dx + (x+\sqrt{xy})\,dy = 0$$

solution:

Test for homogenous

$$\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}} = \frac{\lambda y}{\lambda x+\sqrt{\lambda x\lambda y}}$$

$$\frac{dy}{dx} = \frac{\cancel{\lambda}}{\cancel{\lambda}}\,\frac{y}{x+\sqrt{xy}} \qquad \therefore \text{homogenous}$$

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{1+\sqrt{\frac{y}{x}}}$$

$v = \frac{y}{x}$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v}{1+\sqrt{v}}$$

$$x\frac{dv}{dx} = \frac{v}{1+\sqrt{v}} - v$$

$$x\frac{dv}{dx} = \frac{\cancel{v} - \cancel{v} - v^{\frac{3}{2}}}{1+\sqrt{v}}$$

$$x\frac{dv}{dx} = \frac{-v^{\frac{3}{2}}}{1+\sqrt{v}}$$

$$\frac{1+\sqrt{v}}{v^{\frac{3}{2}}}dv = -\frac{dx}{x}$$

$$\int \left[\frac{1}{v^{\frac{3}{2}}} + \frac{1}{v}\right] dv = -\int \frac{dx}{x}$$

$$-2\frac{1}{\sqrt{v}} + \ln v = -\ln x + c$$

$$-2\frac{1}{\sqrt{\frac{y}{x}}} + \ln\frac{y}{x} = \ln x + c$$

Example 8: solve the D.E

$$(x^2 - xy + y^2)dx - xy\,dy = 0$$

Solution:

Test for homogenous

$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy} = \frac{\lambda^2 x^2 - \lambda x \lambda y + \lambda^2}{\lambda x \, \lambda y}$$

$$\frac{dy}{dx} = \frac{\cancel{x^2}}{\cancel{x^2}} \quad \frac{x^2 - xy + y^2}{xy}$$ ∴ homogeneous

$$\frac{dy}{dx} = \frac{\frac{x^2}{x^2} - \frac{xy}{x^2} + \frac{y^2}{x^2}}{\frac{xy}{x^2}} = \frac{1 - \frac{y}{x} + \frac{y^2}{x^2}}{\frac{y}{x}}$$

$$v = \frac{y}{x} \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{1 - v + v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1 - v + v^2}{v} - v = \frac{1 - v + \cancel{v^2} - \cancel{v^2}}{v}$$

$$x\frac{dv}{dx} = \frac{1 - v}{v}$$

$$\frac{dx}{x} = \frac{v}{1 - v}\, dv$$

$$\frac{dx}{x} + \frac{v}{v - 1}\, dv = 0$$

$$\int \frac{dx}{x} + \int \left[1 + \frac{1}{v-1} \right] dv = 0$$

$$\ln x + v + \ln|v-1| = c$$

$$\ln x + \frac{y}{x} + \ln\left|\frac{y}{x} - 1\right| = c$$

Example 9: Solve the D.E

$$x \frac{dy}{dx} - y = \sqrt{x^2+y^2}$$

Solution:

Test for homogenous

$$x \frac{dy}{dx} = y + \sqrt{x^2+y^2}$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} = \frac{\lambda y + \sqrt{\lambda^2 x^2 + \lambda^2 y^2}}{\lambda x}$$

$$\frac{dy}{dx} = \frac{\cancel{\lambda}}{\cancel{\lambda}} \frac{y + \sqrt{x^2+y^2}}{x}$$ $\therefore$ homogenous

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}$$

$$v = \frac{y}{x} \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + \sqrt{1+v^2}$$

$$\int \frac{dx}{x} = \int \frac{1}{\sqrt{1+v^2}}\, dv$$

$$\ln x = \sinh^{-1} v + c$$

$$\ln x = \sinh^{-1}\left(\frac{y}{x}\right) + c$$

Example 10: Solve the D.E

$$\frac{dy}{dx} + \frac{3xy+y^2}{x^2+xy} = 0$$

solution:

Test for homogenous

$$\frac{dy}{dx} = \frac{-3xy-y^2}{x^2+xy} = \frac{-3\lambda x\lambda y - \lambda^2 y^2}{\lambda^2 x^2 + \lambda x \lambda y}$$

$$\frac{dy}{dx} = \frac{\cancel{\lambda^2}}{\cancel{\lambda^2}}\, \frac{-3xy-y^2}{x^2+xy}$$ ∴ homogenous

$$\frac{dy}{dx} = \frac{-3\frac{y}{x} - \frac{y^2}{x^2}}{1+\frac{y}{x}}$$

$$v = \frac{y}{x} \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{-3v - v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{-3v - v^2}{1+v} - v$$

$$x\frac{dv}{dx} = \frac{-3v - v^2 - v - v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{-4v - 2v^2}{1+v}$$

$$\int \frac{1+v}{4v+2v^2}\,dv = \int -\frac{dx}{x}$$

$$\frac{1}{4}\int \frac{4+4v}{4v+2v^2}\,dv = -\int \frac{dx}{x}$$

$$\frac{1}{4}\ln\left|4v+2v^2\right| = -\ln x + c$$

$$\frac{1}{4}\ln\left|4\frac{y}{x} + 2\frac{y^2}{x^2}\right| = -\ln x + c$$