Exact :

standerd form $M(x,y)\,dx + N(x,y)\,dy = 0$

This equation is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

To solve the exact equation we first solve

$$M(x,y) = \frac{\partial f(x,y)}{\partial x} \quad , \quad N(x,y) = \frac{\partial f(x,y)}{\partial y}$$

$$df = M\,dx + N\,dy.$$

Example 1 : solve the D.E

$$x^3 - y\sin x + (\cos x + 2y)\frac{dy}{dx} = 0$$

Solution : Test for exact

$$\underbrace{(x^3 - y\sin x)}_{M}\,dx + \underbrace{(\cos x + 2y)}_{N}\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -\sin x \\ \frac{\partial N}{\partial x} &= -\sin x + 0 \end{aligned}\right\} \text{exact}$$

① $M = \frac{\partial f}{\partial x}$

$$f = \int M\,dx = \int (x^3 - y\sin x)\,dx$$

$$f = \frac{x^4}{4} + y\cos x + g(y) + c$$

② $N = \frac{\partial f}{\partial y}$

$$\cancel{\cos x} + 2y = 0 + \cancel{\cos x} + g'(y) + 0$$

$$g'(y) = 2y$$

$$g(y) = \int g'(y)\,dy = \int 2y\,dy = y^2$$

$$f = \frac{x^4}{4} + y\cos x + y^2 + c$$

Example 2: Solve the D.E

$$\underbrace{(4x^3y^3-2xy)}_{M}dx+\underbrace{(3x^4y^2-x^2)}_{N}dy=0$$

solution: Test for exact

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=12x^3y^2-2x\\ \frac{\partial N}{\partial x}&=12x^3y^2-2x\end{aligned}\right\}\therefore \text{exact}$$

① $M=\frac{\partial f}{\partial x}$

$$f=\int M\,dx=\int(4x^3y^3-2xy)\,dx$$

$$f=4y^3\frac{x^4}{4}-2y\frac{x^2}{2}+g(y)+C$$

$$f=y^3x^4-x^2y+g(y)+C$$

② $N=\frac{\partial f}{\partial y}$

$$\cancel{3x^4y^2}-\cancel{x^2}=\cancel{3x^4y^2}-\cancel{x^2}+g'(y)+0$$

$$g'(y)=0$$

$$g(y)=\int g'(y)\,dy=\int 0\,dy=C$$

$$f=y^3x^4-x^2y+C$$

Example 3: solve the D.E

$$\underbrace{(\tan x + \tan y)}_{N}\,dy + (\underbrace{y\sec^2 x + \sec x \tan x}_{M})\,dx = 0$$

solution: Test for exact

$$\frac{\partial M}{\partial y} = \sec^2 x + 0$$

$$\frac{\partial N}{\partial x} = \sec^2 x + 0$$

$\therefore$ exact

① $M = \dfrac{\partial f}{\partial x}$

$$f = \int M\,dx = \int (y\sec^2 x + \sec x \tan x)\,dx$$

$$f = y\tan x + \sec x + g(y) + c$$

② $N = \dfrac{\partial f}{\partial y}$

$$\cancel{\tan x} + \tan y = \cancel{\tan x} + 0 + g'(y) + 0$$

$$g'(y) = \tan y = \frac{\sin y}{\cos y}$$

$$g(y) = \int g'(y)\,dy = -\int \frac{-\sin y}{\cos y}\,dy$$

$$g(y) = -\ln|\cos y|$$

$$f = y\tan x + \sec x - \ln|\cos y| + C$$

Example 4: solve the D.E

$$\underbrace{2x\,e^{y}}_{M}\,dx + (\underbrace{x^2e^{y}+\cos y}_{N})\,dy = 0$$

solution: Test for exact

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2xe^{y}\\ \frac{\partial N}{\partial x} &= 2xe^{y}+0\end{aligned}\right\} \therefore \text{exact}$$

① $M = \dfrac{\partial f}{\partial x}$

$$f = \int M\,dx = \int 2xe^{y}\,dx$$

$$f = x^2e^{y} + g(y) + C$$

② $N = \dfrac{\partial f}{\partial y}$

$$\cancel{x^2e^{y}} + \cos y = \cancel{x^2e^{y}} + g'(y) + 0$$

$$g'(y) = \cos y$$

$$g(y) = \int g'(y)\,dy = \int \cos y\,dy = \sin y$$

$$f = x^2e^{y} + \sin y + C$$

Examples: Solve the D.E

$$\underbrace{2x\sin y}_{M}\,dx + \underbrace{x^2\cos y}_{N}\,dy = 0$$

solution: Test for exact

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2x\cos y\\ \frac{\partial N}{\partial x} &= 2x\cos y\end{aligned}\right]\ \therefore \text{exact}$$

(1) $M = \frac{\partial f}{\partial x}$

$$f = \int M\,dx = \int 2x\sin y\,dx$$

$$f = x^2\sin y + g(y) + C$$

(2) $N = \frac{\partial f}{\partial y}$

$$x^2\cancel{\cos y} = x^2\cancel{\cos y} + g'(y) + 0$$

$$g'(y) = 0$$

$$g(y) = \int g'(y)\,dy = \int 0\,dy = C$$

$$f = x^2\sin y + C$$

Example 6: Find the general solution to

$$\underbrace{[\sin(xy) + xy\cos(xy)]}_{M}dx + [\underbrace{x^2\cos(xy) + 2y}_{N}]dy = 0$$

solution: Test for exact

$$\frac{\partial M}{\partial y} = x\cos(xy) - x^2 y\sin(xy) + x\cos(xy)$$

$$\left.\begin{array}{l}\dfrac{\partial M}{\partial y} = 2x\cos(xy) - x^2 y\sin(xy) \\ \dfrac{\partial N}{\partial x} = -x^2 y\sin(xy) + 2x\cos(xy) + 0\end{array}\right\} \therefore \text{exact}$$

① $M = \dfrac{\partial f}{\partial x}$

$$f = \int M\,dx = \int [\sin(xy) + \underbrace{x\,y\cos(xy)}_{u \cdot dv}]\,dx$$

$$\int \sin(xy) = -\frac{1}{y}\cos(xy)$$

$$\int xy\cos(xy) = y\int x\cos(xy)$$

let $u = x \longrightarrow du = dx$

$dv = \cos(xy)\,dx \longrightarrow v = \frac{1}{y}\sin(xy)$

$$y\left[v \cdot u - \int v\,du\right] = y\left[\frac{1}{y}\sin(xy)x - \int \frac{1}{y}\sin(xy)dx\right]$$

$$= x\sin(xy) + \frac{1}{y}\cos(xy)$$

$$f = -\frac{1}{y}\cancel{\cos(xy)} + x\sin(xy) + \cancel{\frac{1}{y}\cos(xy)} + g(y) + c$$

$$f = x\sin(xy) + g(y) + c$$

(2) $N = \dfrac{\partial f}{\partial y}$

$$x^2\cancel{\cos}(xy) + 2y = x^2\cancel{\cos}(xy) + g'(y) + 0$$

$$g'(y) = 2y$$

$$g(y) = \int g'(y)\,dy = \int 2y\,dy = y^2$$

$$f = x\sin(xy) + y^2 + c$$

Example 7: Solve the D.E

$$2x + xe^{xy}\bar{y} + ye^{xy} = -2yy'$$

solution: Test for exact

$$2x + ye^{xy} + xe^{xy}y' + 2yy' = 0$$

$$2x + ye^{xy} + (xe^{xy} + 2y)\frac{dy}{dx} = 0$$

$$\underbrace{(2x + ye^{xy})}_{M}\,dx + \underbrace{(xe^{xy} + 2y)}_{N}\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 0 + yxe^{xy} + e^{xy} \\ \frac{\partial N}{\partial x} &= xye^{xy} + e^{xy} + 0 \end{aligned}\right\} \therefore \text{exact}$$

① $M = \dfrac{\partial f}{\partial x}$

$$f = \int M\,dx = \int [2x + ye^{xy}]\,dx$$

$$f = x^2 + e^{xy} + g(y) + C$$

② $N = \dfrac{\partial f}{\partial y}$

$$\cancel{x}e^{xy} + 2y = 0 + \cancel{x}e^{xy} + g'(y) + 0$$

$$g'(y) = 2y$$

$$g(y) = \int g'(y)\,dy = \int 2y\,dy = y^2$$

$$f = x^2 + e^{xy} + y^2 + c$$

Example 8: Solve the D.E

$$\left(\sqrt{y} + 2x\tan(y)\right) + \left(\frac{x}{2\sqrt{y}} + x^2\sec^2(y)\right)\frac{dy}{dx} = 0$$

solution: Test for exact

$$\underbrace{\left(\sqrt{y} + 2x\tan(y)\right)}_{M} dx + \underbrace{\left(\frac{x}{2\sqrt{y}} + x^2\sec^2(y)\right)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{2}y^{-\frac{1}{2}} + 2x\sec^2(y)$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{1}{2\sqrt{y}} + 2x\sec^2(y) \\ \frac{\partial N}{\partial x} &= \frac{1}{2\sqrt{y}} + 2x\sec^2(y) \end{aligned}\right\} \therefore \text{exact}$$

① $M = \dfrac{\partial f}{\partial x}$

$$f = \int M\,dx = \int \left(\sqrt{y} + 2x\tan(y)\right) dx$$

$f = \sqrt{y}\, x + x^2 \tan(y) + g(y) + c$

② $N = \frac{\partial f}{\partial y}$

$\cancel{\frac{x}{2\sqrt{y}}} + \cancel{x^2 \sec^2(y)} = \cancel{\frac{x}{2\sqrt{y}}} + \cancel{x^2 \sec^2(y)} + g'(y) + 0$

$g'(y) = 0$

$g(y) = \int g'(y)\,dy = \int 0\, dy = c$

$f = \sqrt{y}\, x + x^2 \tan(y) + c$

Example 9: Solve the following first order D.E:

$(1 + 2\sqrt{y}\, e^x) + \left(\frac{e^x}{\sqrt{y}} - 1\right)\frac{dy}{dx} = 0$

solution: Test for exact

$\underbrace{(1 + 2\sqrt{y}\, e^x)}_{M} dx + \underbrace{\left(\frac{e^x}{\sqrt{y}} - 1\right)}_{N} dy = 0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 0 + \frac{e^x}{\sqrt{y}} \\ \frac{\partial N}{\partial x} &= \frac{e^x}{\sqrt{y}} - 0 \end{aligned}\right] \therefore \text{exact}$$

① $M = \frac{\partial f}{\partial x}$

$$f = \int M\,dx = \int (1 + 2\sqrt{y}\,e^x)\,dx$$

$$f = x + 2\sqrt{y}\,e^x + g(y) + c$$

② $N = \frac{\partial f}{\partial y}$

$$\cancel{\frac{e^x}{\sqrt{y}}} - 1 = 0 + \cancel{\frac{e^x}{\sqrt{y}}} + g'(y) + 0$$

$$g'(y) = -1$$

$$g(y) = \int g'(y)\,dy = \int -1\,dy = -y$$

$$f = x + 2\sqrt{y}\,e^x - y + c$$

Example 10: solve the following first order O.D.E:

$$\left[e^{x}\cos(y)+(1-x)\sin(y)\right]\frac{dy}{dx}+e^{x}(1+\sin(y))+\cos(y)=0$$

Solution: Test for exact

$$\underbrace{\left[e^{x}\cos(y)+(1-x)\sin(y)\right]}_{N}dy+\underbrace{\left[e^{x}(1+\sin(y))+\cos(y)\right]}_{M}dx=0$$

$$M=e^{x}(1+\sin(y))+\cos(y)$$

$$N=e^{x}\cos(y)+(1-x)\sin(y)$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=e^{x}\cos(y)-\sin(y)\\ \frac{\partial N}{\partial x}&=e^{x}\cos(y)-\sin(y)\end{aligned}\right\}\therefore \text{exact}$$

① $M=\dfrac{\partial f}{\partial x}$

$$f=\int M\,dx=\int\left[e^{x}(1+\sin(y))+\cos(y)\right]dx$$

$$f=e^{x}(1+\sin(y))+x\cos(y)+g(y)+C$$

② $N=\dfrac{\partial f}{\partial y}$

$$\cancel{e^x \cos(y)} + \sin(y) - \cancel{x\sin(y)} = \cancel{e^x \cos(y)} - \cancel{x\sin(y)} + g'(y) + 0$$

$$g'(y) = \sin(y)$$

$$g(y) = \int g'(y)\,dy = \int \sin(y)\,dy$$

$$g(y) = -\cos(y)$$

$$f = e^x(1+\sin(y) + x\cos(y)) - \cos(y) + C$$

$$f = e^x(1+\sin y) + (x-1)\cos(y) + C$$