Linear :

First order linear O.D.E is one that can be written in the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

wher P and Q are functions of x or constants.

To solve the linear equation:

① Find the integration factor

$$I.f = e^{\int P(x)\,dx}$$

② Integrate and solve

$$y = \frac{1}{I.f}\int I.f \cdot Q(x)\,dx + \frac{c}{I.f}$$

Example 1: solve the D.E

$$x \frac{dy}{dx} = x^2 + 3y$$

solution:

$$x \frac{dy}{dx} - 3y = x^2 \qquad \div x$$

$$\frac{dy}{dx} \underbrace{- \frac{3}{x}}_{P(x)} y = \underbrace{x}_{Q(x)} \qquad \text{Linear}$$

$$\int P(x)\,dx = \int -\frac{3}{x}\,dx = -3\ln x = \ln x^{-3}$$

$$\int P(x)\,dx = \ln \frac{1}{x^3}$$

$$I.f = e^{\int P(x)dx} = e^{\ln \frac{1}{x^3}} = \frac{1}{x^3}$$

$$y = \frac{1}{I.f} \int I.f \cdot Q(x)\,dx + \frac{c}{I.f}$$

$$y = x^3 \int \frac{1}{x^3} \cdot x \cdot dx + cx^3$$

$$y = x^3 \int x^{-2}\,dx + cx^3$$

$$y = x^3 \left[-\frac{1}{x}\right] + cx^3 = -x^2 + cx^3$$

Example2: Solve the D.E

$$x\frac{dy}{dx}+(1-x)y=xe^{x}$$

Solution

$$x\frac{dy}{dx}+(1-x)y=xe^{x} \qquad \div x$$

$$\frac{dy}{dx}+\underbrace{\left(\frac{1}{x}-1\right)}_{P(x)}y=\underbrace{e^{x}}_{Q(x)} \qquad \text{Linear}$$

$$\int P(x)\,dx=\int\left(\frac{1}{x}-1\right)dx=\ln x-x$$

$$I\text{-}f=e^{\int P(x)\,dx}=e^{(\ln x-x)}=e^{\ln x}\cdot e^{-x}=\frac{x}{e^{x}}$$

$$y=\frac{1}{I\text{-}f}\int I\text{-}f\cdot Q(x)\,dx+\frac{c}{I\text{-}f}$$

$$y=\frac{e^{x}}{x}\int\frac{x}{\cancel{e^{x}}}\cdot\cancel{e^{x}}\,dx+c\,\frac{e^{x}}{x}$$

$$y=\frac{e^{x}}{x}\int x\,dx+c\,\frac{e^{x}}{x}$$

$$y=\frac{e^{x}}{\cancel{x}}\,\frac{x^{\cancel{2}}}{2}+c\,\frac{e^{x}}{x}$$

$$y=\frac{xe^{x}}{2}+c\,\frac{e^{x}}{x}$$

Example 3: Solve the D.E

$$\frac{dy}{dx} + y\cot x = \sin 2x$$

Solution:

$$\frac{dy}{dx} + \underbrace{\frac{\cos x}{\sin x}}_{p(x)} y = \underbrace{\sin 2x}_{Q(x)} \quad \text{linear}$$

$$\int p(x)\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln \sin x$$

$$I.f = e^{\int p(x)dx} = e^{\ln \sin x} = \sin x$$

$$y = \frac{1}{I.f} \cdot \int I.f \cdot Q(x)\,dx + \frac{c}{I.f}$$

$$y = \frac{1}{\sin x} \int \sin x \cdot \sin 2x\,dx + \frac{c}{\sin x}$$

$$y = \frac{1}{\sin x} \int \sin x \cdot 2\sin x\cos x\,dx + \frac{c}{\sin x}$$

$$y = \frac{2}{\sin x} \int \sin^2 x\cos x\,dx + \frac{c}{\sin x}$$

$$y = \frac{2}{3}\sin^2 x + \frac{c}{\sin x}$$

Example 4: Solve the D.E

$$\frac{dy}{dx} = \frac{3y-4x^5}{x}$$

Solution :

$$\frac{dy}{dx} \underbrace{- \frac{3}{x}}_{P(x)} y = \underbrace{-4x^4}_{Q(x)}$$ Linear

$$\int P(x)\,dx = \int -\frac{3}{x}\,dx = -3\ln x = \ln\frac{1}{x^3}$$

$$I.f = e^{\int P(x)dx} = e^{\ln\frac{1}{x^3}} = \frac{1}{x^3}$$

$$y = \frac{1}{I.f}\int I.f \cdot Q(x)\,dx + \frac{c}{I.f}$$

$$y = x^3\int \frac{1}{\cancel{x^3}}(-4x^{\cancel{4}})\,dx + cx^3$$

$$y = x^3\int(-4x)\,dx + cx^3$$

$$y = x^3\left[-\frac{4x^2}{2}\right] + cx^3$$

$$y = -2x^5 + cx^3$$

Examples: Find the general solution of the D.E

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$$

Solution:

$$\frac{dy}{dx} + \underbrace{\frac{3}{x}}_{P(x)} y = \underbrace{\frac{e^x}{x^3}}_{Q(x)} \quad \text{linear}$$

$$\int P(x)\,dx = \int \frac{3}{x}\,dx = 3\ln x = \ln x^3$$

$$I\text{-}f = e^{\int P(x)\,dx} = e^{\ln x^3} = x^3$$

$$y = \frac{1}{I\text{-}f}\int I\text{-}f \cdot Q(x)\,dx + \frac{c}{I\text{-}f}$$

$$y = \frac{1}{x^3}\int \cancel{x^3} \cdot \frac{e^x}{\cancel{x^3}}\,dx + \frac{c}{x^3}$$

$$y = \frac{1}{x^3}\int e^x\,dx + \frac{c}{x^3}$$

$$y = \frac{e^x}{x^3} + \frac{c}{x^3}$$

Example 6: solve the D.E

$$\frac{dy}{dx} - \frac{3y}{x+1} = (x+1)^4$$

solution:

$$\frac{dy}{dx} \underbrace{- \frac{3}{(x+1)}}_{P(x)} y = \underbrace{(x+1)^4}_{Q(x)}$$

$$\int P(x)\,dx = \int -\frac{3}{(x+1)}\,dx = -3\ln(x+1)$$

$$= \ln \frac{1}{(x+1)^3}$$

$$I.f = e^{\int p(x)dx} = e^{\ln \frac{1}{(x+1)^3}} = \frac{1}{(x+1)^3}$$

$$y = \frac{1}{I.f} \cdot \int I.f \cdot Q(x)\,dx + \frac{c}{I.f}$$

$$y = (x+1)^3 \int \frac{1}{(x+1)^3} \cdot (x+1)^4\,dx + c(x+1)^3$$

$$y = (x+1)^3 \int (x+1)\,dx + c(x+1)^3$$

$$y = (x+1)^3 \left[\frac{x^2}{2} + x\right] + c(x+1)^3$$

Example 7: solve the D.E

$$\bar{y} - 2y = x^2 e^{2x}$$

Solution:

$$\frac{dy}{dx} \underbrace{-2}_{P(x)} y = \underbrace{x^2 e^{2x}}_{Q(x)}$$

$$\int P(x)\,dx = \int -2\,dx = -2x$$

$$I.f = e^{\int P(x)dx} = e^{-2x}$$

$$y = \frac{1}{I.f}\int I.f \cdot Q(x)\,dx + \frac{c}{I.f}$$

$$y = \frac{1}{e^{-2x}}\int \cancel{e^{-2x}} \cdot x^2 \cancel{e^{2x}}\,dx + \frac{c}{e^{-2x}}$$

$$y = e^{2x}\int x^2\,dx + ce^{2x}$$

$$y = \frac{e^{2x}x^3}{3} + ce^{2x}$$

Example 8: solve the D.E

$$(x^2+1)\bar{y} + xy - x = 0 \qquad y(3) = 2$$

solution:

$$(x^2+1)\frac{dy}{dx} + xy = x \qquad \div (x^2+1)$$

$$\frac{dy}{dx} + \underbrace{\frac{x}{x^2+1}}_{P(x)} y = \underbrace{\frac{x}{(x^2+1)}}_{Q(x)} \quad \text{linear}$$

$$\int P(x)\,dx = \int \frac{x}{x^2+1} = \frac{1}{2}\int \frac{2x}{x^2+1}$$

$$= \frac{1}{2}\ln(x^2+1) = \ln\sqrt{x^2+1}$$

$$I\cdot f = e^{\int P(x)dx} = e^{\ln\sqrt{x^2+1}} = \sqrt{x^2+1}$$

$$y = \frac{1}{I\cdot f}\int I\cdot f\cdot Q(x)\,dx + \frac{c}{I\cdot f}$$

$$y = \frac{1}{\sqrt{x^2+1}}\int \cdot\sqrt{x^2+1}\cdot\frac{x}{(x^2+1)}\,dx + \frac{c}{\sqrt{x^2+1}}$$

$$y = \frac{1}{\sqrt{x^2+1}}\int \frac{x}{\sqrt{x^2+1}}\,dx + \frac{c}{\sqrt{x^2+1}}$$

$$y = \frac{1}{\sqrt{x^2+1}}\cdot\frac{1}{2}\int 2x(x^2+1)^{-\frac{1}{2}}\,dx + \frac{c}{\sqrt{x^2+1}}$$

$$y = \frac{1}{\sqrt{x^2+1}} \cancel{\frac{1}{2}} \frac{(x^2+1)^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + \frac{c}{\sqrt{x^2+1}}$$

$$y = \frac{1}{\cancel{\sqrt{x^2+1}}} \cdot \cancel{\sqrt{x^2+1}} + \frac{c}{\sqrt{x^2+1}}$$

$$y = 1 + \frac{c}{\sqrt{x^2+1}}$$

$$y = 2 \quad at \quad x = 3$$

$$2 = 1 + \frac{c}{\sqrt{9+1}}$$

$$c = \sqrt{10}$$

$$y = 1 + \frac{\sqrt{10}}{\sqrt{x^2+1}}$$

Example 9: solve the following first order O.D.E:

$$3x\bar{y} - y = \ln x + 1$$

solution:

$$3x\frac{dy}{dx} - y = \ln x + 1 \qquad \div 3x$$

$$\frac{dy}{dx} \underbrace{- \frac{1}{3x}}_{P(x)} y = \underbrace{\frac{\ln x + 1}{3x}}_{Q(x)} \quad \text{linear}$$

$$\int P(x)\,dx = \int -\frac{1}{3x} = -\frac{1}{3}\ln x = \ln x^{-\frac{1}{3}}$$

$$I.f = e^{\int P(x)\,dx} = e^{\ln x^{-\frac{1}{3}}} = x^{-\frac{1}{3}}$$

$$y = \frac{1}{I.f}\int I.f \cdot Q(x)\,dx + \frac{C}{I.f}$$

$$y = x^{\frac{1}{3}} \int x^{-\frac{1}{3}} \cdot \frac{\ln x + 1}{3x}\,dx + c\,x^{\frac{1}{3}}$$

$$= \frac{1}{3}x^{\frac{1}{3}} \int x^{-\frac{4}{3}} \cdot (\ln x + 1)\,dx + c\,x^{\frac{1}{3}}$$

let $u = (\ln x + 1) \longrightarrow du = \frac{1}{x}dx$

$$dv = x^{-\frac{4}{3}} \longrightarrow v = -3\,x^{-\frac{1}{3}}$$

$$y = \frac{1}{3}x^{\frac{1}{3}}\left[v \cdot u - \int v\,du\right] + c\,x^{\frac{1}{3}}$$

$$y=\frac{1}{3}x^{\frac{1}{3}}\left[-3x^{-\frac{1}{3}}(\ln x+1)+\int 3x^{-\frac{1}{3}}\frac{1}{x}\,dx\right]+c\,x^{\frac{1}{3}}$$

$$y=\frac{1}{3}x^{\frac{1}{3}}\left[-3x^{-\frac{1}{3}}(\ln x+1)-9\,x^{-\frac{1}{3}}\right]+cx^{\frac{1}{3}}$$

$$y=-\ln x-1-3+c\,x^{\frac{1}{3}}$$

$$y=-(\ln x+4)+c\,x^{\frac{1}{3}}$$

Example 10: solve the following first order O.D.E:

$$t^2\frac{dy}{dt}+3ty=t^4\ln t+1$$

solution:

$$t^2\frac{dy}{dx}+3ty=t^4\ln t+1 \qquad \div t^2$$

$$\frac{dy}{dt}+\underbrace{\frac{3}{t}}_{P(x)}y=\underbrace{\frac{t^4\ln t+1}{t^2}}_{Q(t)} \qquad \text{linear}$$

$$\int P(t)\,dt=\int \frac{3}{t}\,dt=3\ln t=\ln t^3$$

$$I\text{-}f=e^{\int P(t)dt}=e^{\ln t^3}=t^3$$

$$y=\frac{1}{I\text{-}f}\int I\text{-}f\cdot Q(t)\,dt+\frac{c}{I\text{-}f}$$

$$y=\frac{1}{t^3}\cdot\int t^3\cdot\frac{t^4\ln t+1}{t^2}\,dt+\frac{c}{t^3}$$

$$=\frac{1}{t^3}\int[t^5\ln t+t]\,dt+\frac{c}{t^3}$$

$t^5 \ln t$ by part integration

Let $u = \ln t \longrightarrow du = \frac{1}{t}\,dt$

$dv = t^5\,dt \longrightarrow v = \frac{t^6}{6}$

$v \cdot u - \int v\,du$

$$\frac{t^6}{6} \cdot \ln t - \int \frac{t^{\cancel{6}\,5}}{6} \cdot \frac{1}{\cancel{t}}\,dt$$

$$\frac{t^6}{6} \ln t - \int \frac{t^5}{6}\,dt$$

$$= \frac{t^6}{6} \ln t - \frac{t^6}{36}$$

$$y = \frac{1}{t^3}\left[\frac{t^6}{6} \ln t - \frac{t^6}{36} + \frac{t^2}{2}\right] + \frac{c}{t^3}$$

$$= \frac{t^3}{6} \ln t - \frac{t^3}{36} + \frac{1}{2t} + \frac{c}{t^3}$$