## Bernoulli's

It has form $\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n \qquad n \neq 1$

Division of both sides of the D.E by $y^n$ gives:

$$y^{-n}\frac{dy}{dx} + P(x)\, y^{1-n} = Q(x)$$

and reduce this D.E to be linear in $z$ we assume:

$$z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n)\, y^{-n}\frac{dy}{dx}$$

and by substitution $\frac{1}{n-1}\frac{dz}{dx} + P(x)\, z = Q(x)$

$$\frac{dz}{dx} + (1-n)P(x)\, z = (1-n)\, Q(x)$$

linear in $z$ with respect to $z$ and $\frac{dz}{dx}$

Example 1: Solve the D.E

$$y' + \frac{1}{x} y = x y^2$$ Bernoulli

solution:

$$y' + \frac{1}{x} y = x y^2 \qquad \div y^2$$

$$y^{-2} y' + \frac{1}{\cancel{x}} y^{-1} = x$$

$$z = y^{-1} \longrightarrow z = \frac{1}{y}$$

$$z' = \frac{dz}{dx} = -y^{-2} y'$$

$$y^{-2} y' + \frac{1}{x} y^{-1} = x \qquad \times -1$$

$$-y^{-2} y' - \frac{1}{x} y^{-1} = -x$$

$$z' \underbrace{\left(-\frac{1}{x}\right)}_{P(x)} z = \underbrace{\left(-x\right)}_{Q(x)}$$ linear

$$\int P(x)\,dx = \int -\frac{1}{x} = -\ln x = \ln \frac{1}{x}$$

$$I.f = e^{\int P(x)\,dx} = e^{\ln \frac{1}{x}} = \frac{1}{x}$$

$$z = \frac{1}{I.f} \int I.f \cdot Q(x)\,dx + \frac{C}{I.f}$$

$$z = x\int \frac{1}{\cancel{x}}(-\cancel{x})\,dx + cx$$

$$z = -x^2 + cx$$

$$y = \frac{1}{z}$$

$$y = \frac{1}{-x^2 + cx}$$

Example2: solve the D.E

$$3xy' + y + x^2y^4 = 0$$

solution                                        Bernoulli

$$3xy' + y = -x^2y^4 \qquad \div\, xy^4$$

$$3y^{-4}y' + \frac{1}{x}y^{-3} = -x \qquad * -1$$

$$z = y^{-3} \longrightarrow y^3 = \frac{1}{z}$$

$$z' = -3y^{-4}y'$$

$$-3y^{-4}y' - \frac{1}{x}y^{-3} = x$$

$$z' - \frac{1}{x}z = x \qquad \text{linear}$$

$P(x)$          $Q(x)$

$$\int P(x)\,dx = \int -\frac{1}{x}\,dx = -\ln x = \ln\frac{1}{x}$$

$$I\cdot f = e^{\int P(x)dx} = e^{\ln\frac{1}{x}}$$

$$z = \frac{1}{I\cdot f}\int I\cdot f\,Q(x)\,dx + \frac{c}{I\cdot f}$$

$$z = x\int \cancel{\frac{1}{x}}\,(\cancel{x})\,dx + cx$$

$$z = x^2 + cx$$

$$y = \frac{1}{\sqrt[3]{z}} = \frac{1}{\sqrt[3]{x^2+cx}}$$

Example 3 : Solve the D.E

$$\frac{dy}{dx} = \frac{2x^5y^3 - 4y}{x}$$

Solution

Bernoulli

$$\frac{dy}{dx} + \frac{4}{x}y = 2x^4y^3 \qquad \div y^3$$

$$y^{-3}\frac{dy}{dx} + \frac{4}{x}y^{-2} = 2x^4 \qquad * -2$$

$$z = y^{-2} \longrightarrow y = \frac{1}{\sqrt[2]{z}}$$

$$z' = -2y^{-3}y'$$

$$-2y^{-3}y' - \frac{8}{x}y^{-2} = -4x^4$$

$$z' - \frac{8}{x}z = -4x^4$$ linear

$P(x)$ $Q(x)$

$$\int P(x)\,dx = \int -\frac{8}{x}\,dx = -8\ln x = \ln x^{-8}$$

$$I\text{-}f = e^{\int P(x)\,dx} = e^{\ln x^{-8}} = x^{-8}$$

$$z = \frac{1}{I\text{-}f}\int I\text{-}f \cdot Q(x)\,dx + \frac{c}{I\text{-}f}$$

$$z = x^8 \int x^{\cancel{-8}\,-4}(-4\cancel{x^4})\,dx + cx^8$$

$$z = x^8 \int -4x^{-4}\,dx + cx^8$$

$$z = \frac{4}{3}x^5 + cx^8$$

$$y = \frac{1}{\sqrt[2]{z}} = \frac{1}{\sqrt[2]{\frac{4}{3}x^5 + cx^8}}$$

Example 4: Solve the D-E

$$x\frac{dy}{dx} + y = xy^3$$ Bernoulli

Solution:

$$xy^{-3}\frac{dy}{dx} + y^{-2} = x \qquad \div x$$

$$y^{-3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = 1$$

$$z = y^{-2}$$

$$\bar{z} = -2y^{-3}y'$$

$$-2y^{-3}\frac{dy}{dx} - \frac{2}{x}y^{-2} = -2$$

$$\bar{z} - \frac{2}{x}z = -2$$ linear

$P(x)$ $Q(x)$

$$\int P(x)dx = \int -\frac{2}{x}dx = -2\ln x = \ln x^{-2}$$

$$I\cdot f = e^{\int P(x)dx} = e^{\ln x^{-2}} = x^{-2}$$

$$z = \frac{1}{I\cdot f}\int I\cdot f \cdot Q(x)dx + \frac{c}{I\cdot f}$$

$$z = x^2\int x^{-2}(-2)dx + cx^2$$

$$z = 2x + cx^2 \longrightarrow y = \frac{1}{2\sqrt{2x + cx^2}}$$

Examples: Solve the D.E

$$\frac{dy}{dx} - y = x y^5$$

Bernoulli

solution

$$y^{-5} \frac{dy}{dx} - y^{-4} = x \quad * -4$$

$$z = y^{-4}$$

$$z' = -4y^{-5} y'$$

$$-4 y^{-5} y' + 4 y^{-4} = -4x$$

$$z' + \underbrace{4}_{P(x)} z = \underbrace{-4x}_{Q(x)}$$

linear

$$\int P(x)\, dx = \int 4\, dx = 4x$$

$$I \cdot f = e^{\int P(x) dx} = e^{4x}$$

$$z = \frac{1}{I \cdot f} \int I \cdot f \cdot Q(x)\, dx + \frac{c}{I \cdot f}$$

$$z = \frac{1}{e^{4x}} \int \underbrace{e^{4x}(-4x)\, dx}_{u\, dv} + \frac{c}{e^{4x}}$$

$u = x \longrightarrow du = dx$

$dv = -4e^{4x}\cancel{dx} \rightarrow v = -e^{4x}$

$\int u\,dv = v \cdot u - \int v\,du$

$\int e^{4x}(-4x)\,dx = -e^{4x} \cdot x - \int -e^{4x}\,dx$

$= -e^{4x}x + \frac{1}{4}e^{4x}$

$z = \frac{1}{\cancel{e^{4x}}}\left[\cancel{e^{4x}}\left(-x+\frac{1}{4}\right)\right] + \frac{c}{e^{4x}}$

$z = \left(-x+\frac{1}{4}\right) + ce^{-4x}$

$$y = \frac{1}{\sqrt[4]{\left(-x+\frac{1}{4}\right) + ce^{-4x}}}$$

Example 6: Solve the D.E

$$\frac{dy}{dx} + \frac{1}{3} y = e^{x} y^{4}$$ Bernoulli

solution:

$$y^{-4} \frac{dy}{dx} + \frac{1}{3} y^{-3} = e^{x}$$

$$z = y^{-3}$$

$$\bar{z} = -3 y^{-4} y'$$

$$-3 y^{-4} y' - y^{-3} = -3 e^{x}$$

$$\bar{z} - z = -3 e^{x}$$ linear

P(x) Q(x)

$$\int P(x)\, dx = \int - dx = -x$$

$$I \cdot f = e^{\int P(x) dx} = e^{-x}$$

$$I \cdot f = \frac{1}{e^{x}}$$

$$z = \frac{1}{I \cdot f} \int I \cdot f \cdot Q(x)\, dx + \frac{c}{I \cdot f}$$

$$z = e^{x} \int \frac{1}{\cancel{e^{x}}} (-3\cancel{e^{x}})\, dx + c e^{x}$$

$$z = e^{x} \int (-3)\, dx + c e^{x}$$

$$z = -3x e^{x} + c e^{x}$$

$$y = \frac{1}{\sqrt[3]{z}} = \frac{1}{\sqrt[3]{-3xe^{x} + ce^{x}}}$$

Example 7: Solve the D.E

$$\frac{dy}{dx} + \frac{y}{x} = y^{2} \qquad \text{Bernoulli}$$

solution:

$$y^{-2}\frac{dy}{dx} + \frac{1}{x} y^{-1} = 1$$

$$z = y^{-1}$$

$$\acute{z} = -y^{-2}\acute{y}$$

$$-y^{-2}\acute{y} - \frac{1}{x}\acute{y} = -1$$

$$\acute{z} \underbrace{\left(-\frac{1}{x}\right)}_{P(x)} z = \underbrace{(-1)}_{Q(x)} \qquad \text{linear}$$

$$\int P(x)\,dx = \int -\frac{1}{x}\,dx = \ln\frac{1}{x}$$

$$I.f = e^{\int P(x)dx} = e^{\ln\frac{1}{x}} = \frac{1}{x}$$

$$z = \frac{1}{I.f}\int I.f \cdot Q(x)\,dx + \frac{c}{I.f}$$

$$= x\int \frac{1}{x}(-1)\,dx + cx$$

$$z = x(-\ln x) + cx$$

$$y = \frac{1}{z} = \frac{1}{-x\ln x + c}$$

Example 8: Solve the D.E

$$\frac{dy}{dx} + \frac{2}{x}y = -x^2\cos x\; y^2$$ Bernoulli

solution

$$y^{-2}\frac{dy}{dx} + \frac{2}{x}y^{-1} = -x^2\cos x$$

$$z = y^{-1}$$

$$z' = -y^{-2}y'$$

$$-y^{-2}y' - \frac{2}{x}y^{-1} = x^2\cos x$$

$$\bar{z} - \frac{2}{x} z = x^2 \cos x \qquad \text{linear}$$

$P(x)$ $\quad Q(x)$

$$\int P(x)\,dx = \int -\frac{2}{x}\,dx = \ln x^{-2}$$

$$I.f = e^{\int P(x)\,dx} = e^{\ln x^{-2}} = x^{-2}$$

$$z = \frac{1}{I.f} \int I.f \cdot Q(x)\,dx + \frac{c}{I.f}$$

$$= x^2 \int \cancel{x^{-2}} (\cancel{x^2} \cos x)\,dx + c x^2$$

$$= x^2 \int \cos x\,dx + c x^2$$

$$z = x^2 \sin x + c x^2$$

$$y = \frac{1}{z} = \frac{1}{x^2 \sin x + c x^2}$$

Example 9: solve the D.E

$$2\frac{dy}{dx} + \tan x\, y = \frac{(4x+5)^2}{\cos x} y^3$$ Bernoulli

solution:

$$2y^{-3}\frac{dy}{dx} + \frac{\sin x}{\cos x} y^{-2} = \frac{(4x+5)^2}{\cos x}$$

$$z = y^{-2}$$

$$\bar{z} = -2y^{-3} y'$$

$$-2y^{-3}y' - \frac{\sin x}{\cos x} y^{-2} = -\frac{(4x+5)^2}{\cos x}$$

$$\bar{z} \underbrace{-\frac{\sin x}{\cos x}}_{P(x)} z = \underbrace{-\frac{(4x+5)^2}{\cos x}}_{Q(x)}$$ linear

$$\int P(x)\,dx = \int -\frac{\sin x}{\cos x}\,dx = \ln \cos x$$

$$I.f = e^{\int P(x)dx} = e^{\ln \cos x} = \cos x$$

$$z = \frac{1}{I.f}\int I.f \cdot Q(x)\,dx + \frac{c}{I.f}$$

$$z = \frac{1}{\cos x}\int \cancel{\cos x}\left(\frac{-(4x+5)^2}{\cancel{\cos x}}\right)dx + \frac{c}{\cos x}$$

$$z = \frac{1}{4}\frac{1}{\cos x}\int -4(4x+5)^2 dx + \frac{c}{\cos x}$$

$$z = -\frac{1}{12\cos x}(4x+5)^3 + \frac{c}{\cos x}$$

$$y = \frac{1}{\sqrt[2]{z}} = \frac{1}{\sqrt{-\frac{(4x+5)^3}{12\cos x} + \frac{c}{\cos x}}}$$

Example 10: Solve the D.E

$$x\frac{dy}{dx} + y = x^3 y^6$$ Bernoulli

solution

$$y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x^2$$

$$z = y^{-5}$$

$$z' = -5y^{-6}y'$$

$$-5y^{-6}y' - \frac{5}{x}y^{-5} = -5x^2$$

$$z' \underbrace{-\frac{5}{x}}_{P(x)} z = \underbrace{-5x^2}_{Q(x)}$$ linear

$$\int P(x)\,dx = \int -\frac{5}{x}\,dx = -5\ln x = \ln x^{-5}$$

$$I\text{-}f = e^{\int P(x)\,dx} = e^{\ln x^{-5}} = x^{-5}$$

$$z = \frac{1}{I\text{-}f}\int I\text{-}f \cdot Q(x)\,dx + \frac{c}{I\text{-}f}$$

$$z = x^{5}\int \cancel{x^{-5}}^{\,-3}\cdot(-5\cancel{x^{2}})\,dx + cx^{5}$$

$$z = x^{5}\int -5x^{-3}\,dx + cx^{5}$$

$$z = x^{5}\left(\frac{5}{2}x^{-2}\right) + cx^{5}$$

$$z = \frac{5}{2}x^{3} + cx^{5}$$

$$y = \frac{1}{\sqrt[5]{z}} = \frac{1}{\sqrt[5]{\frac{5}{2}x^{3} + cx^{5}}}$$