2-Second order Ordinary Differential Equations:

A-Homogenous 2nd order D.E:

The general equation can be expressed in the form:

$$\left.\begin{aligned} &ay'' + by' + cy = 0 \\ \text{or}\quad &aD^2y + bDy + cy = 0 \end{aligned}\right\} \quad (1)$$

wher a, b and c are constans and

$D = \frac{d}{dx}$ the operator.

The solution:

let $y = e^{mx} \longrightarrow y' = m\,e^{mx} \longrightarrow y'' = m^2 e^{mx}$

sub in Eq. 1

$$am^2 e^{mx} + bm\,e^{mx} + c\,e^{mx} = 0$$

$$(am^2 + bm + c)\,e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$\therefore\ am^2 + bm + c = 0 \qquad (2)$$

Eq 2 is called characteristic equation that has two roots m_1 and m_2. there are three cases for the solution according to the roots of the charaterstic equation as follows:

① if $m_1 \neq m_2$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$ (real roots not equal)

② if $m_1 = m_2$

$$y_h = c_1 e^{mx} + c_2 x e^{mx}$$ (real roots equal)

③ if $m_1 = \alpha + \beta i$
$m_2 = \alpha - \beta i$

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

(complex roots)

Example 1: Solve

$$y'' + 4y' + 3y = 0$$

solution:

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 4me^{mx} + 3e^{mx} = 0$$

$$e^{mx}(m^2 + 4m + 3) = 0 \qquad e^{mx} \neq 0$$

$$m^2 + 4m + 3 = 0$$

$$(m+3)(m+1) = 0$$

either $m + 3 = 0 \longrightarrow m_1 = -3$

or $m + 1 = 0 \longrightarrow m_2 = -1$

$\therefore y_h = c_1 e^{-3x} + c_2 e^{-x}$

Example 2: solve

$$y'' - 6y' + 9y = 0$$

solution:

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} - 6me^{mx} + 9e^{mx} = 0$$

$$(m^2 - 6m + 9)\, e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0$$

either $m - 3 = 0 \longrightarrow m_1 = 3$

or $m - 3 = 0 \longrightarrow m_2 = 3$

مكررة

$$y_h = c_1 e^{mx} + c_2 x e^{mx}$$

$$y_h = c_1 e^{3x} + c_2 x e^{3x}$$

Examples: solve

$$y'' + 3y' - 10y = 0$$

solution:

Let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$m^2 e^{mx} + 3me^{mx} - 10e^{mx} = 0$

$(m^2 + 3m - 10)\, e^{mx} = 0 \quad e^{mx} \neq 0$

$m^2 + 3m - 10 = 0$

$(m+5)(m-2) = 0$

either $m + 5 = 0 \longrightarrow m_1 = -5$

or $m - 2 = 0 \longrightarrow m_2 = 2$

$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y_h = c_1 e^{-5x} + c_2 e^{2x}$

Example4: solve

$$y'' + 2y' + 5y = 0$$

solution:

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 2m e^{mx} + 5e^{mx} = 0$$

$$(m^2 + 2m + 5)\, e^{mx} = 0 \;,\; e^{mx} \neq 0$$

$$(m^2 + 2m + 5) = 0$$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-2 \mp \sqrt{4 - 20}}{2} = -1 \mp 2i$$

$\alpha = -1 \;,\; \beta = 2$ خيالي

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

$$y_h = e^{-x}(c_1 \sin 2x + c_2 \cos 2x)$$

Examples5: Solve

$$y'' + 4y' + 4y = 0$$

solution:

let $y = e^{mx} \rightarrow y' = m e^{mx} \rightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 4 m e^{mx} + 4 e^{mx} = 0$$

$$(m^2 + 4m + 4)\, e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$m^2 + 4m + 4 = 0$$

$$(m+2)(m+2) = 0$$

$$\left.\begin{array}{ll} \text{either} & m+2 = 0 \longrightarrow m_1 = -2 \\ \text{or} & m+2 = 0 \longrightarrow m_2 = -2 \end{array}\right\} \text{مكررة}$$

$$y_h = c_1 e^{mx} + c_2 x e^{mx}$$

$$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$$

Example 6: Solve

$$y'' + 9y = 0$$

solution

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 9e^{mx} = 0$$

$$(m^2 + 9)\, e^{mx} = 0 \longrightarrow e^{mx} \neq 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m_{1,2} = \mp 3i$$

$\left.\begin{matrix} \alpha = 0 \\ \beta = 3 \end{matrix}\right\}$ قالب

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

$$y_h = e^{0x}(c_1 \sin 3x + c_2 \cos 3x)$$

$$y_h = c_1 \sin 3x + c_2 \cos 3x$$

Example 7: solve

$$9y'' + 12y' + 29y = 0$$

solution:

let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$$9m^2 e^{mx} + 12m e^{mx} + 29 e^{mx} = 0$$

$$(9m^2 + 12m + 29) e^{mx} = 0 \quad , e^{mx} \neq 0$$

$$(9m^2 + 12m + 29) = 0$$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-12 \mp \sqrt{12^2 - 4 * 9 * 29}}{2 * 9}$$

$$m = \frac{-12 \mp \sqrt{144 - 1044}}{18} = \frac{-12 \mp 30i}{18}$$

$$m_1 = -\frac{2}{3} + \frac{5}{3}i \quad , m_2 = -\frac{2}{3} - \frac{5}{3}i$$

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

$$y_h = e^{-\frac{2}{3}x}\left(c_1 \sin\left(\tfrac{5}{3}x\right) + c_2 \cos\left(\tfrac{5}{3}x\right)\right)$$

Example 8: Solve

$$y'' - 5y' = 0$$

solution:

let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$$m^2 e^{mx} - 5me^{mx} = 0$$

$$(m^2 - 5m)\, e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$m^2 - 5m = 0$$

$$m(m-5) = 0$$

either $m_1 = 0$

or $m - 5 = 0 \longrightarrow m_2 = 5$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_h = c_1 e^{0} + c_2 e^{5x}$$

$$y_h = c_1 + c_2 e^{5x}$$

Example 9: Find the general solution of $4y'' + 4y' + y = 0$

solution

let $y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$4m^2 e^{mx} + 4m e^{mx} + e^{mx} = 0$$

$$(4m^2 + 4m + 1)\, e^{mx} = 0 \quad , \quad e^{mx} \neq 0$$

$$4m^2 + 4m + 1 = 0$$

$$(2m+1)^2 = 0$$

$$\left.\begin{array}{l} m_1 = -\frac{1}{2} \\ m_2 = -\frac{1}{2} \end{array}\right\}$$ مكررة

$$y_h = c_1 e^{mx} + c_2 x e^{mx}$$

$$y_h = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}$$

Example 10: solve

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$$

solution:

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2e^{mx}$

$$m^2e^{mx} + 5me^{mx} + 4e^{mx} = 0$$

$$(m^2 + 5m + 4)\, e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$m^2 + 5m + 4 = 0$$

$$(m+4)(m+1) = 0$$

either $m + 4 = 0 \longrightarrow m_1 = -4$

or $m + 1 = 0 \longrightarrow m_2 = -1$

$$y_h = c_1e^{m_1x} + c_2e^{m_2x}$$

$$y_h = c_1e^{-4x} + c_2e^{-x}$$