B- Non-homogenous 2nd order D.E:

the general form is:

$$ay'' + by' + cy = g(x)$$

the general solution is:

$$y = y_h + y_p$$

wher y_h: homogenous solution that can be determined by assuming:

$$ay'' + by' + cy = 0$$

and solving it as a homogenous 2nd order D.E.

y_p: particular solution.

There ar two methods to find the particular solution y_p:

1. Undetermined coefficent method.
2. Variation of parameters meth.

Undetermined coefficent method:

The method of undetermined coeffitcent is a method for finding a particular solution to the 2nd order non-homogenous D.E.

$$ay'' + by' + cy = g(x)$$

where $g(x)$ has a special form, involving only poloynomials, exponential, sines and cosines.

$g(x)$	$y_p(x)$
K	a_0
Ke^{rx}	a_0e^{rx} اذا كان (r) ليس من جذور y_h
Ke^{rx}	a_0xe^{rx} اذا كان (r) احد جذور y_h
Ke^{rx}	$a_0x^2e^{rx}$ اذا كان (r) جذر متكرر من جذور y_h
$K\cos(wx)$ or $K\sin(wx)$ or $K\cos(wx)+M\sin(wx)$	$a_0\cos(wx)+a_1\sin(wx)$ اذا كان (sin(wx)) ليس من جذور y_h
$K\cos(wx)$ or $K\sin(wx)$ $K\cos(wx)+M\sin(wx)$	$x[a_0\cos(wx)+a_1\sin(wx)]$ اذا كان (sin(wx)) احد جذورy_h
x^n	$a_0x^n+a(n-1)x^{(n-1)}+a(n-2)x^{(n-2)}\ldots..+a_n$ اذا كان (0) ليس من جذور y_h
x^n	$x[a_0x^n+(n-1)x^{(n-1)}+(n-2)x^{(n-2)}\ldots..+a_n]$ اذا كان (0) احد جذورy_h
x^n	$x^2[a_0x^n+(n-1)x^{(n-1)}+(n-2)x^{(n-2)}\ldots..+a_n]$ اذا كان (0) جذر متكرر من جذور y_h
x^ne^{rx}	$[a_0x^n+(n-1)x^{(n-1)}+(n-2)x^{(n-2)}\ldots..+a_n]e^{rx}$
$\cos(wx)e^{rx}$ or $\sin(wx)e^{rx}$	$[a_0\cos(wx)+a_1\sin(wx)]\,e^{rx}$
$\cos(wx)x^n$ or $\sin(wx)x^n$	$[a_0x^n+a(n-1)x^{(n-1)}+a(n-2)x^{(n-2)}\ldots..+a_n]\cos(wx)+[b_0x^n+b(n-1)x^{(n-1)}+b(n-2)x^{(n-2)}\ldots..+b_n]\sin(wx)$

Example 1: Solve the D.E

$$y'' + 2y' - 3y = 6$$ (non-homogeneous)

solution:

① Find homogeneous solution:

$$y'' + 2y' - 3y = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 2me^{mx} - 3e^{mx} = 0$$

$$(m^2 + 2m - 3)\, e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$\therefore\ m^2 + 2m - 3 = 0$$

$$(m+3)(m-1) = 0$$

either $m + 3 = 0 \longrightarrow m = -3$

or $m - 1 = 0 \longrightarrow m = 1$

$$y_h = c_1 e^{-3x} + c_2 e^{x}$$

② Find particular solution:

$y_p = a_0$ من الجدول

$$y_p = a_0$$
$$y_p' = 0$$
$$y_p'' = 0$$

نعوض بالمعادلة الاصلية

$$0 + 0 - 3a_0 = 6$$

$$\therefore a_0 = -2$$

$$y_p = -2$$

$$y = y_h + y_p$$

$$y = c_1 e^{-3x} + c_2 e^{x} - 2$$

Example 2: Find the general solution of the D.E: $y'' + 3y = e^{x}$

solution:

① Find homogeneous solution:

$$y'' + 3y = 0$$

Let $y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 3e^{mx} = 0$$

$$(m^2 + 3)\, e^{mx} = 0 \quad , e^{mx} \neq 0$$

$$m^2 + 3 = 0$$

$$m^2 = -3$$

$$m_{1,2} = \mp\sqrt{3}\, i \qquad \alpha = 0 \qquad \beta = 3$$

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

$$y_h = \cancel{e^{0}}^{\,1}(c_1 \sin\sqrt{3}x + c_2 \cos\sqrt{3}x)$$

$$y_h = c_1 \sin\sqrt{3}x + c_2 \cos\sqrt{3}x$$

② Find particular solution:

$$y_p = a_0 e^x$$

لأن e^x ليس له علاقة بجذور y_h

$$y_p' = a_0 e^x$$

$$y_p'' = a_0 e^x$$

نعوض بالمعادلة الاصلية:

$$a_0 e^x + 3a_0 e^x = e^x$$

$$a_0 + 3a_0 = 1$$

معاملات e^x

$$4a_0 = 1$$

$$a_0 = \frac{1}{4}$$

$$y_p = a_0 e^x = \frac{1}{4} e^x$$

$$y = y_h + y_p$$

$$y = c_1 \sin\sqrt{3}x + c_2 \cos\sqrt{3}x + \frac{1}{4} e^x$$

Examples: Find the general solution for the D-E using Undetermined coefficent method:

$$y'' - 6y' + 9y = 2e^{3x}$$

Solution:

① Find the homogenous solution:

$$y'' - 6y' + 9y = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} - 6m e^{mx} + 9 e^{mx} = 0$$

$$(m^2 - 6m + 9) e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0$$

$$\left.\begin{array}{ll} \text{either } m-3=0 \longrightarrow m_1 = 3 \\ \text{or } \quad m-3=0 \longrightarrow m_2 = 3 \end{array}\right\} \text{تكرر}$$

$$y_h = c_1 e^{mx} + c_2 x e^{mx}$$

$$y_h = c_1 e^{3x} + c_2 x e^{3x}$$

② Find Particular solution:

$$y_p = a_0 x^2 e^{3x}$$ [لان e^{3x} تكرر مرتين في y_h]

$$y_p' = a_0 (x^2 \cdot 3e^{3x} + e^{3x} \cdot 2x)$$

$$y_p' = a_0 (3x^2 e^{3x} + 2x e^{3x})$$

$$y_p'' = a_0 (3x^2 \cdot 3e^{3x} + e^{3x} \cdot 6x + 2x \cdot 3e^{3x} + e^{3x} \cdot 2)$$

$$y_p'' = a_0 (9x^2 e^{3x} + 12x e^{3x} + 2e^{3x})$$

نعوض y_p و y_p' و y_p'' في المعادلة الاصلية

$$9a_0x^2e^{3x} + 12a_0xe^{3x} + 2a_0e^{3x} - 6[3a_0x^2e^{3x} + 2a_0xe^{3x}] + 9a_0x^2e^{3x} = 2e^{3x}$$

$$\cancel{9a_0x^2e^{3x}} + \cancel{12a_0xe^{3x}} + 2a_0e^{3x} - \cancel{18a_0x^2e^{3x}} - \cancel{12a_0xe^{3x}} + \cancel{9a_0x^2e^{3x}} = 2e^{3x}$$

$2a_0 e^{3x} = 2e^{3x}$

$a_0 = 1$

$y_p = a_0 x^2 e^{3x} = x^2 e^{3x}$

$y = y_h + y_p$

$y = c_1 e^{3x} + c_2 x e^{3x} + x^2 e^{3x}$

Example 4: Find the general solution:

$$y'' - x^3 + 1 = 0$$

solution

$$y'' = x^3 - 1$$

① Find Homogenous solution:

$$y'' = 0$$

let $y = e^{mx}$, $y' = m e^{mx}$, $y'' = m^2 e^{mx}$

$m^2 e^{mx} = 0$, $e^{mx} \neq 0$

$$m^2 = 0$$

$$\left.\begin{matrix} m_1 = 0 \\ m_2 = 0 \end{matrix}\right] \quad \text{مكررة}$$

$$y_h = c_1 e^{mx} + c_2 x e^{mx}$$

$$y_h = c_1 e^{0x} + c_2 x e^{0x}$$

$$y_h = c_1 + c_2 x$$

③ Find particular solution:

$$y_p = [a_0 X^3 + a_1 X^2 + a_2 X + a_3] X^2$$

نضرب X^2 × لان الصفر جذر مكرر في y_h

$$y_p = a_0 X^5 + a_1 X^4 + a_2 X^3 + a_3 X^2$$

$$y_p' = 5a_0 X^4 + 4a_1 X^3 + 3a_2 X^2 + 2a_3 X$$

$$y_p'' = 20a_0 X^3 + 12a_1 X^2 + 6a_2 X + 2a_3$$

عوض بالمعادلة الاصلية

$$20a_0 X^3 + 12a_1 X^2 + 6a_2 X + 2a_3 = X^3 - 1$$

$20a_0 = 1$; معاملات X^3

$a_0 = \frac{1}{20}$

$12a_1 = 0$; معاملات X^2

$a_1 = 0$

$6a_2 = 0$; معاملات X

$a_2 = 0$

$2a_3 = -1$; الثوابت

$a_3 = -\frac{1}{2}$

$$y_p = \frac{1}{20} X^5 - \frac{1}{2} X^2$$

$$y = y_h + y_p$$

$$= c_1 + c_2 x + \frac{1}{20} x^5 - \frac{1}{2} x^2$$

Example 5: Find the general solution

of: $y'' - 4y = x e^{x} + \cos(2x)$

solution:

① Find homogenous solution:

$$y'' - 4y = 0$$

let $y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} - 4 e^{mx} = 0 \quad , \quad e^{mx} \neq 0$$

$$m^2 - 4 = 0$$

$$m = 4$$

$$m_{1,2} = \mp 2$$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_h = c_1 e^{2x} + c_2 e^{-2x}$$

② Find particular solution:

$$y_{P1} = (a_0 x + a_1) e^{x}$$

$$y_{P1}' = (a_0 x + a_1) e^{x} + a_0 e^{x}$$

$$y_{P1}'' = (a_0 x + a_1) e^{x} + 2 a_0 e^{x}$$

sub in D.E

$$(a_0 x + a_1) e^{x} + 2 a_0 e^{x} - 4 (a_0 x + a_1) e^{x} = x e^{x}$$

$$-3 a_0 x - 3 a_1 + 2 a_0 = x$$

$$-3 a_0 = 1 \qquad \text{معاملات } x:$$

$$a_0 = -\frac{1}{3}$$

$$-3 a_1 + 2 a_0 = 0 \qquad \text{الثوابت}:$$

$$a_1 = -\frac{2}{9}$$

$$y_{P1} = (a_0 x + a_1) e^{x}$$

$$y_{P1} = \left(-\frac{1}{3} x - \frac{2}{9}\right) e^{x}$$

$$y_{p2} = a_2 \cos 2x + a_3 \sin 2x$$

$$y_{p2}' = -2a_2 \sin 2x + 2a_3 \cos 2x$$

$$y_{p2}'' = -4a_2 \cos 2x - 4a_3 \sin 2x$$

sub in D.E

$$-4a_2 \cos 2x - 4a_3 \sin 2x - 4[a_2 \cos 2x + a_3 \sin 2x] = \cos 2x$$

$$-4a_2 \cos 2x - 4a_3 \sin 2x - 4a_2 \cos 2x - 4a_3 \sin 2x = \cos 2x$$

معاملات $\cos 2x$:

$$-4a_2 - 4a_2 = 1$$

$$a_2 = -\frac{1}{8}$$

معاملات $\sin 2x$:

$$-4a_3 - 4a_3 = 0$$

$$a_3 = 0$$

$y_{p2} = -\frac{1}{8}\cos 2x$

$y = y_h + y_{p1} + y_{p2}$

$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} x e^{x} - \frac{2}{9} e^{x} - \frac{1}{8}\cos 2x$

Example 6: Find the general solution for the D.E using undetermined coeffitent method: $y'' - y' = 5e^{x} - \sin(2x)$

Solution:

① Find homogenous solution:

$y'' - y' = 0$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$m^2 e^{mx} - me^{mx} = 0$

$(m^2 - m)\, e^{mx} = 0 \quad , \quad e^{mx} \neq 0$

$m^2 - m = 0$

$m(m-1) = 0$

either $m = 0$

or $m - 1 = 0 \longrightarrow m = 1$

$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y_h = c_1 e^{0} + c_2 e^{x}$

$y_h = c_1 + c_2 e^{x}$

② Find Particular solution:

$y_p = y_{p1} + y_{p2}$

$y_{p1} = a_0 x e^{x}$ نضرب بـ x لوجود e^x في احد حدود y_h

$y_{p1}' = a_0 x e^{x} + a_0 e^{x}$

$y_{p1}'' = a_0 x e^{x} + 2 a_0 e^{x}$

sub in D-E

$a_0 x e^{x} + 2a_0 e^{x} - (a_0 x e^{x} + a_0 e^{x}) = 5e^{x}$

$\cancel{a_0 x e^{x}} + 2a_0 e^{x} - \cancel{a_0 x e^{x}} - a_0 e^{x} = 5e^{x}$

$$a_0 e^x = 5 e^x$$

$$\therefore a_0 = 5$$

$$y_{p1} = a_0 x e^x = 5 x e^x$$

$$y_{p2} = a_1 \cos 2x + a_2 \sin 2x$$

$$y_{p2}' = -2a_1 \sin 2x + 2a_2 \cos 2x$$

$$y_{p2}'' = -4a_1 \cos 2x - 4a_2 \sin 2x$$

sub in D.E

$$-4a_1 \cos 2x - 4a_2 \sin 2x + 2a_1 \sin 2x - 2a_2 \cos 2x = -\sin 2x$$

معاملات $\sin 2x$:

$$-4a_2 + 2a_1 = -1 \quad (1)$$

معاملات $\cos 2x$:

$$-4a_1 - 2a_2 = 0 \quad (2)$$

نضرب المعادلة (2) بـ -2

$$2a_1 - 4a_2 = -1 \quad (1)$$

$$8a_1 + 4a_2 = 0 \quad (2)$$

بالجمع

$$10a_1 = -1$$

$$\therefore a_1 = -\frac{1}{10}$$

نعوض بالمعادلة ②

$$8\left(-\frac{1}{10}\right) + 4a_2 = 0$$

$$-\frac{1}{5} + a_2 = 0$$

$$a_2 = \frac{1}{5}$$

$$y_{p2} = a_1 \cos 2x + a_2 \sin 2x$$

$$y_{p2} = -\frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x$$

$$y = y_h + y_{p1} + y_{p2}$$

$$y = c_1 + c_2 e^{x} + 5xe^{x} - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x$$

Example 7: Solve the following D.E:

$$y'' + 9y = t^2 e^{3t} + 6$$

Solution:

① Find homogenous solution:

$$y'' + 9y = 0$$

let $y = e^{mt} \longrightarrow y' = m e^{mt} \longrightarrow y'' = m^2 e^{mt}$

$$m^2 e^{mt} + 9 e^{mt} = 0$$

$$(m^2 + 9)\, e^{mt} = 0 \quad , \; e^{mt} \neq 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$m_{1,2} = \mp 3i \qquad \alpha = 0 \qquad \beta = 3$

$$y_h = e^{\alpha t}(c_1 \sin \beta t + c_2 \cos \beta t)$$

$$y_h = e^{0}(c_1 \sin 3t + c_2 \cos 3t)$$

$$y_h = c_1 \sin 3t + c_2 \cos 3t$$

② Find particular solution:

$$y_p = y_{p1} + y_{p2}$$

$$y_{p1} = (a_0 t^2 + a_1 t + a_2)e^{3t}$$

$$y_{p1}' = (2a_0 t + a_1)e^{3t} + 3(a_0 t^2 + a_1 t + a_2)e^{3t}$$

$$y_{p1}'' = 2a_0 e^{3t} + 3(2a_0 t + a_1)e^{3t} +$$
$$3(2a_0 t + a_1)e^{3t} + 9(a_0 t^2 + a_1 t + a_2)e^{3t}$$

$$y_{p1}'' = 2a_0 e^{3t} + 6(2a_0 t + a_1)e^{3t} + 9(a_0 t^2 + a_1 + a_2)e^{3t}$$

sub in D-E

$$2a_0 e^{3t} + 6(2a_0 t + a_1)e^{3t} + 9(a_0 t^2 + a_1 t + a_2)e^{3t}$$
$$+ 9(a_0 t^2 + a_1 t + a_2)e^{3t} = t^2 e^{3t}$$

$$18a_0 t^2 e^{3t} + (12a_0 + 18a_1)t e^{3t} + (2a_0 + 6a_1 + 18a_2)e^{3t}$$
$$= t^2 e^{3t}$$

مساوي $t^2 e^{3t}$!

$$18a_0 = 1$$

$$a_0 = \frac{1}{18}$$

معامل te^{3t} :

$$12a_0 + 18a_1 = 0$$

$$12\frac{1}{18} + 18a_1 = 0$$

$$a_1 = -\frac{1}{27}$$

معامل e^{3t} :

$$2a_0 + 6a_1 + 18a_2 = 0$$

$$2\left(\frac{1}{18}\right) + 6\left(-\frac{1}{27}\right) + 18a_2 = 0$$

$$a_2 = +\frac{1}{162}$$

$$y_{p1} = (a_0t^2 + a_1t + a_2)e^{3t}$$

$$y_{p1} = \left(\frac{1}{18}t^2 - \frac{1}{27}t + \frac{1}{162}\right)e^{3t}$$

$$\left.\begin{aligned} y_{p2} &= a_3 \\ y_{p2}' &= 0 \\ y_{p2}'' &= 0 \end{aligned}\right\} \text{sub in D.E}$$

$$0 + 9a_3 = 6$$

$$a_3 = \frac{2}{3}$$

$$y_{p2} = \frac{2}{3}$$

$$y_p = y_{p1} + y_{p2}$$

$$y_p = \left(\frac{1}{18}t^2 - \frac{1}{27}t + \frac{1}{162}\right)e^{3t} + \frac{2}{3}$$

$$y = y_h + y_p$$

$$y = c_1\cos 3t + c_2\sin 3t + \left(\frac{1}{18}t^2 - \frac{1}{27}t + \frac{1}{162}\right)e^{3t} + \frac{2}{3}$$

Example 8: solve the following D.E unsig (undetermined coeffitent method):

$$3y'' + y' - 2y = 2\cos x + 3e^{-x}$$

solution:

① Find the homogeneous solution:

$$3y'' + y' - 2y = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$3m^2 e^{mx} + m e^{mx} - 2e^{mx} = 0$$

$$(3m^2 + m - 2)\, e^{mx} = 0 \quad , e^{mx} \neq 0$$

$$3m^2 + m - 2 = 0$$

$$(3m - 2)(m + 1) = 0$$

either $3m - 2 = 0 \longrightarrow m_1 = \frac{2}{3}$

or $m + 1 = 0 \longrightarrow m_2 = -1$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_h = c_1 e^{\frac{2}{3}x} + c_2 e^{-x}$$

② Find particular solution:

$$y_p = y_{p1} + y_{p2}$$

$$y_{p1} = a_0 \sin x + a_1 \cos x$$

$$y_{p1}' = a_0 \cos x - a_1 \sin x$$

$$y_{p1}'' = -a_0 \sin x - a_1 \cos x$$

sub in D-E

$$3(-a_0 \sin x - a_1 \cos x) + (a_0 \cos x - a_1 \sin x) - 2(a_0 \sin x + a_1 \cos x) = 2\cos x$$

$$-3a_0 \sin x - 3a_1 \cos x + a_0 \cos x - a_1 \sin x - 2a_0 \sin x - 2a_1 \cos x = 2\cos x$$

$a_0 - 5a_1 = 2$ ① معاملات $\cos x$:

$-5a_0 - a_1 = 0$ ② معاملات $\sin x$

$a_1 = -5a_0$ من معادلة ②

نعوض في معادلة ①

$$a_0 - 5(-5a_0) = 2$$

$$26a_0 = 2 \quad \text{or} \quad a_0 = \frac{2}{26} = \frac{1}{13}$$

$$a_1 = -5a_0 = -5\left(\frac{1}{13}\right) = -\frac{5}{13}$$

$$y_{p1} = \frac{1}{13}\sin x - \frac{5}{13}\cos x$$

$$y_{p2} = a_2 x e^{-x}$$

نضرب بـ x لوجود
e^{-x} في حدود y_h

$$y_{p2}' = a_2 x(-e^{-x}) + a_2 e^{-x}$$

$$y_{p2}' = -a_2 x e^{-x} + a_2 e^{-x}$$

$$y_{p2}'' = -a_2 x(-e^{-x}) + (-a_2 e^{-x}) - a_2 e^{-x}$$

$$y_{p2}'' = a_2 x e^{-x} - 2a_2 e^{-x}$$

sub in D.E

$$3(a_2 x e^{-x} - 2a_2 e^{-x}) + (-a_2 x e^{-x} + a_2 e^{-x})$$
$$-2a_2 x e^{-x} = 3e^{-x}$$

$$3a_2xe^{-x} - 6a_2e^{-x} - a_2xe^{-x} + a_2e^{-x}$$

$$- 2a_2xe^{-x} = 3e^{-x}$$

معاملات e^{-x} :

$$-6a_2 + a_2 = 3 \longrightarrow a_2 = -\frac{3}{5}$$

$$y_{p2} = -\frac{3}{5}xe^{-x}$$

$$y_p = y_{p1} + y_{p2}$$

$$= \frac{1}{13}\sin x - \frac{5}{13}\cos x - \frac{3}{5}xe^{-x}$$

$$y = y_h + y_p$$

$$y = c_1e^{2/3x} + c_2e^{-x} + \frac{1}{13}\sin x - \frac{5}{13}\cos x - \frac{3}{5}xe^{-x}$$

Example 9: solve: $y'' + y' = e^{x}\cos x$

solution:

① Find homogenous solution:

$$y'' + y' = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$(m^2 e^{mx} + me^{mx}) = 0$$

$$(m^2 + m)\, e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$m^2 + m = 0$$
$$m(m+1) = 0$$

either $m_1 = 0$

or $m + 1 = 0 \longrightarrow m_2 = -1$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_h = c_1 e^{0} + c_2 e^{-x}$$

$$y_h = c_1 + c_2 e^{-x}$$

② Find particular solution:

$$y_p = e^{x}(a_0 \cos x + a_1 \sin x)$$

$$y_p = a_0 e^x \cos x + a_1 e^x \sin x$$

$$y_p' = -a_0 e^x \sin x + a_0 e^x \cos x + a_1 e^x \cos x$$
$$+ a_1 e^x \sin x$$

$$y_p'' = -\cancel{a_0 e^x \cos x} - a_0 e^x \sin x$$
$$- a_0 e^x \sin x + \cancel{a_0 e^x \cos x}$$
$$- \cancel{a_1 e^x \sin x} + a_1 e^x \cos x$$
$$+ a_1 e^x \cos x + \cancel{a_1 e^x \sin x}$$

$$y_p'' = -2a_0 e^x \sin x + 2a_1 e^x \cos x$$

sub in D.E

$$-2a_0 e^x \sin x + 2a_1 e^x \cos x - a_0 e^x \sin x$$
$$+ a_0 e^x \cos x + a_1 e^x \cos x + a_1 e^x \sin x = e^x \cos x$$

$2a_1 + a_0 + a_1 = 1$; $e^x \cos x$ معاملات

$a_0 + 3a_1 = 1$ (1)

$-2a_0 - a_0 + a_1 = 0$; $e^x \sin x$ معاملات

$$-3a_0 + a_1 = 0 \quad (2)$$

نضرب معادلة ① بـ 3

$$3a_0 + 9a_1 = 3 \quad (1)$$

نجمع ① و②

$$10a_1 = 3$$

$$a_1 = \frac{3}{10}$$

نعوض بالمعادلة ②

$$a_0 = \frac{1}{10}$$

$$y_p = e^x\left(\frac{1}{10}\cos x + \frac{3}{10}\sin x\right)$$

$$y = y_h + y_p$$

$$y = c_1 + c_2e^x + \frac{1}{10}e^x\cos x + \frac{3}{10}e^x\sin x$$

Example 10: Determine the form of a particular solution of:

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}$$

solution:

① Find homogenous solution:

$$y'' - 9y' + 14y = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} - 9me^{mx} + 14e^{mx} = 0$$

$$(m^2 - 9m + 14)\, e^{mx} = 0 \quad , e^{mx} \neq 0$$

$$m^2 - 9m + 14 = 0$$

$$(m-7)(m-2) = 0$$

either $m - 7 = 0 \longrightarrow m_1 = 7$

or $m - 2 = 0 \longrightarrow m_2 = 2$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_h = c_1 e^{7x} + c_2 e^{2x}$$

(2) Find particular solution (only form)

$$y_{p1} = a_0 x^2 + a_1 x + a_2$$

$$y_{p2} = a_3 \sin 2x + a_4 \cos 2x$$

$$y_{p3} = (a_5 x + a_6) e^{6x}$$

Example11: solve the D.E:

$$y'' + 9y = 9x - \cos x$$

Solution:

① Find homogeneous solution:

$$y'' + 9y = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 9e^{mx} = 0$$

$$(m^2 + 9)e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \mp 3i \qquad \alpha = 0 \qquad \beta = 3$$

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

$$y_h = e^{0x}(c_1 \sin 3x + c_2 \cos 3x)$$

$$y_h = c_1 \sin 3x + c_2 \cos 3x$$

② Find particular solution:

$$y_P = y_{P1} + y_{P2}$$

$$y_{P1} = a_0 x + a_1$$

$$y'_{P1} = a_0$$

$$y''_{P1} = 0$$

sub in D.E

$$0 + 9(a_0 x + a_1) = 9x$$

$$9a_0 x + 9a_1 = 9x$$

$9a_0 = 9$ معاملات X :

$a_0 = 1$

$9a_1 = 0$ الثوابت :

$a_1 = 0$

$$y_{P1} = a_0 x + a_1$$

$$y_{P1} = x$$

$y_{p2} = a_2 \sin x + a_3 \cos x$

$y_{p2}' = a_2 \cos x - a_3 \sin x$

$y_{p2}'' = -a_2 \sin x - a_3 \cos x$

sub in D.E

$-a_2 \sin x - a_3 \cos x + 9a_2 \sin x$

$+ 9a_3 \cos x = -\cos x$

معاملات $\cos x$:

$-a_3 + 9a_3 = -1$

$a_3 = -\frac{1}{8}$

معاملات $\sin x$:

$-a_2 + 9a_2 = 0$

$a_2 = 0$

$y_{p2} = a_2 \sin x + a_3 \cos x$

$y_{p2} = 0 - \frac{1}{8} \cos x = -\frac{1}{8} \cos x$

$$y_p = y_{p1} + y_{p2}$$

$$y_p = x - \frac{1}{8}\cos x$$

$$y = y_h + y_p$$

$$y = c_1 \sin 3x + c_2 \cos 3x + x - \frac{1}{8}\cos x$$