Variation of Parameters method:

Variation of Parameters is a method for computing a particular solution to the nonhomogenous linear second order O.D.E:

$$ay'' + by' + cy = g(x)$$

solution procedure:

① Find homogenous solution:

$$ay'' + by' + cy = 0$$

$$y_h = c_1 u_1 + c_2 u_2$$

② Use the variation parameters formula to determine the particular solution:

$$y_p = u_1 v_1 + u_2 v_2$$

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix}$$

$$v_1' = \frac{W_1}{W}$$

$$v_1 = \int v_1' \, dx$$

$$v_2' = \frac{W_2}{W}$$

$$v_2 = \int v_2' \, dx$$

تستخدم هذه الطريقة عندما لا يمكن الحل بالطريقة الاولى بسبب عدم وجود $g(x)$ في الجدول.

Example 1: Solve the D.E

$$y'' + y = \sec x$$

secX غير موجودة في الجدول لذلك نحل بطريقة
variation of parameters

solution:

① Find homogenous solution:

$$y'' + y = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + e^{mx} = 0$$

$$(m^2+1)\, e^{mx} = 0 \quad , \; e^{mx} \neq 0$$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m_{1,2} = \mp i \qquad \alpha = 0 \qquad \beta = 1$$

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

$$y_h = e^{0x}(c_1 \sin x + c_2 \cos x)$$

$$y_h = c_1 \sin x + c_2 \cos x$$

② Find Particular solution:

$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$W = -\sin^2 x - \cos^2 x = -1$$

$$W_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix}$$

$$W_1 = -\cos x * \sec x = -1$$

$$W_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} \sin x & 0 \\ \cos x & \sec x \end{vmatrix}$$

$$W_2 = \sin x * \sec x = \frac{\sin x}{\cos x}$$

$$v_1' = \frac{W_1}{W} = \frac{-1}{-1} = 1$$

$$v_1 = \int v_1' dx = \int dx = x$$

$$v_2' = \frac{W_2}{W} = \frac{-\sin x}{\cos x}$$

$$v_2 = \int v_2' dx = \int \frac{-\sin x}{\cos x} dx$$

$$v_2 = \ln \cos x$$

$y_p = u_1 v_1 + u_2 v_2$

$y_p = \sin x \cdot x + \cos x \ln \cos x$

$y = y_h + y_p = c_1 \sin x + c_2 \cos x + \sin x\, x + \cos x \ln \cos x$

Example 2: Solve the D.E

$$y'' - 2y' + y = \frac{e^x}{x^2+1}$$

solution:

(1) Find homogenous solution:

$y'' - 2y' + y = 0$

let $y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 e^{mx}$

$m^2 e^{mx} - 2m e^{mx} + e^{mx} = 0$

$(m^2 - 2m + 1) e^{mx} = 0 \quad , \quad e^{mx} \neq 0$

$m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0 \longrightarrow m_1 = m_2 = 1$

$y_h = c_1 e^{mx} + c_2 x e^{mx}$

$y_h = c_1 \underbrace{e^x}_{u_1} + c_2 \underbrace{x e^x}_{u_2}$

② Find the particular solution:

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix}$$

$$w = e^x(x e^x + e^x) - x e^x \cdot e^x$$

$$w = \cancel{x e^{2x}} + e^{2x} - \cancel{x e^{2x}} = e^{2x}$$

$$w_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & x e^x \\ \frac{e^x}{x^2+1} & x e^x + e^x \end{vmatrix}$$

$$w_1 = -x e^x \left(\frac{e^x}{x^2+1} \right) = \frac{-x e^{2x}}{x^2+1}$$

$$w_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x^2+1} \end{vmatrix}$$

$$w_2 = e^x \left(\frac{e^x}{x^2+1} \right) = \frac{e^{2x}}{x^2+1}$$

$$v_1' = \frac{w_1}{w} = \frac{-\frac{x \cancel{e^{2x}}}{x^2+1}}{\cancel{e^{2x}}} = -\frac{x}{x^2+1}$$

$$v_1 = \int v_1' \, dx = -\frac{1}{2} \int \frac{2x}{x^2+1} \, dx$$

$$v_1 = -\frac{1}{2} \ln |x^2+1|$$

$$v_2' = \frac{W_2}{W} = \frac{\frac{\cancel{e^{2x}}}{x^2+1}}{\cancel{e^{2x}}} = \frac{1}{x^2+1}$$

$$v_2 = \int v_2' \, dx = \int \frac{1}{x^2+1} \, dx = \tan^{-1} x$$

$$y_p = u_1 v_1 + u_2 v_2$$

$$y_p = e^x \left(-\frac{1}{2} \ln |x^2+1|\right) + x e^x \tan^{-1} x$$

$$y_p = -\frac{1}{2} e^x \ln |x^2+1| + x e^x \tan^{-1} x$$

$$y = y_h + y_p$$

$$y = c_1 e^x + c_2 x e^x - \frac{1}{2} e^x \ln |x^2+1| + x e^x \tan^{-1} x$$

Example 3: Solve the D.E using variation of parameters method.

$$y'' + \pi^2 y = \sin(\pi x)$$

solution:

① Find homogenous solution:

$$y'' + \pi^2 y = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + \pi^2 e^{mx} = 0$$

$$(m^2 + \pi^2)\, e^{mx} = 0 \;,\; e^{mx} \neq 0$$

$$m^2 + \pi^2 = 0$$

$$m^2 = -\pi^2$$

$m_{1,2} = \mp \pi i$ $\quad \alpha = 0 \quad \beta = \pi$

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \pi x)$$

$$y_h = e^{0x}(c_1 \sin(\pi x) + c_2 \cos(\pi x))$$

$$y_h = c_1 \sin \pi x + c_2 \cos \pi$$

② Find particular solution:

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} \sin(\pi x) & \cos(\pi x) \\ \pi\cos(\pi x) & -\pi\sin(\pi x) \end{vmatrix}$$

$$w = -\pi\sin^2 x - \pi\cos^2 \pi x = -\pi$$

$$w_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & \cos(\pi x) \\ \sin(\pi x) & -\pi\sin(\pi x) \end{vmatrix}$$

$$w_1 = -\sin(\pi x)\cos(\pi x)$$

$$w_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} \sin(\pi x) & 0 \\ \pi\cos(\pi x) & \sin(\pi x) \end{vmatrix}$$

$$w_2 = \sin^2 \pi x$$

$$v_1' = \frac{w_1}{w} = \frac{-\sin(\pi x)\cos(\pi x)}{-\pi} = \frac{\sin(\pi x)\cos(\pi x)}{\pi}$$

$$v_1 = \int v_1'\,dx = \frac{1}{\pi}\int \sin(\pi x)\cos(\pi x)\,dx$$

$$v_1 = \frac{1}{2\pi^2}\sin^2(\pi x)$$

$$v_2 = \frac{w_2}{w} = \frac{\sin^2(\pi x)}{-\pi}$$

$$\sin^2 x = \frac{1}{2}(1-\cos(2x))$$

$$v_2' = \frac{-1}{2\pi}(1-\cos(2\pi x))$$

$$v_2 = \int v_2'\,dx = -\frac{1}{2\pi}\int (1-\cos(2\pi x))\,dx$$

$$v_2 = -\frac{1}{2\pi}\left(x - \frac{1}{2\pi}\sin(2\pi x)\right)$$

$$y_p = u_1 v_1 + u_2 v_2$$

$$y_p = \sin\pi x \cdot \frac{1}{2\pi^2}\sin^2(\pi x) + \cos\pi x \cdot \left(\frac{-1}{2\pi}\right)\left[x - \frac{1}{2\pi}\sin(2\pi x)\right]$$

$$y = y_h + y_p$$

$$= c_1\sin(\pi x) + c_2\cos(\pi x) + \frac{\sin^3(\pi x)}{2\pi^2} - \frac{\cos(\pi x)}{2\pi}\left[x - \frac{1}{2\pi}\sin(2\pi x)\right]$$

Example 4: solve the D.E

$$y'' + 9y = \csc(3x)$$

solution:

① Find homogenous solution:

$$y'' + 9y = 0$$

let $y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 9 e^{mx} = 0$$

$$(m^2 + 9) e^{mx} = 0 , \quad e^{mx} \neq 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$m_{1,2} = \mp 3i \qquad \alpha = 0 \qquad \beta = 3$

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

$$y_h = e^{0x}(c_1 \sin(3x) + c_2 \cos(3x))$$

$$y_h = c_1 \underbrace{\sin(3x)}_{u_1} + c_2 \underbrace{\cos(3x)}_{u_2}$$

||

② Find particular solution:

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} \sin(3x) & \cos(3x) \\ 3\cos(3x) & -3\sin(3x) \end{vmatrix}$$

$$w = -3\cos^2(3x) - 3\sin^2(3x) = -3$$

$$w_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & \cos(3x) \\ \csc(3x) & -3\sin(3x) \end{vmatrix}$$

$$w_1 = -\cos(3x) \cdot \csc(3x) = \frac{-\cos(3x)}{\sin(3x)}$$

$$w_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} \sin(3x) & 0 \\ 3\cos(3x) & \csc(3x) \end{vmatrix}$$

$$w_2 = \sin(3x) \cdot \csc(3x) = \frac{\sin(3x)}{\sin(3x)} = 1$$

$$v_1' = \frac{w_1}{w} = \frac{-\frac{\cos(3x)}{\sin(3x)}}{-3} = \frac{1}{3} \frac{\cos(3x)}{\sin(3x)}$$

$$v_1 = \int v_1' \, dx = \frac{1}{3} \int \frac{\cos(3x)}{\sin(3x)} = \frac{1}{9} \ln \sin(3x)$$

$$v_2' = \frac{w_2}{w} = \frac{1}{-3}$$

$$v_2 = \int v_2' \, dx = \int -\frac{1}{3} \, dx = -\frac{1}{3} x$$

$$y_p = u_1 U_1 + u_2 U_2$$

$$= \sin(3x) \cdot \frac{1}{9} \ln \sin(3x) + \cos(3x)\left(-\frac{1}{3}x\right)$$

$$y_p = \frac{1}{9}\sin(3x) \ln \sin(3x) - \frac{x\cos(3x)}{3}$$

$$y = y_h + y_p$$

$$y = c_1 \sin(3x) + c_2 \cos(3x) + \frac{1}{9}\sin(3x) \ln \sin(3x) - \frac{x\cos(3x)}{3}$$

Example 5: Solve the D.E

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$$

solution:

① Find homogeneous solution:

$$y'' + 4y' + 4y = 0$$

let $y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 4m e^{mx} + 4 e^{mx} = 0$$

$$(m^2 + 4m + 4) e^{mx} = 0 \,,\; e^{mx} \neq 0$$

$$m^2 + 4m + 4 = 0$$

$$(m+2)(m+2) = 0$$

$$\left.\begin{array}{ll} \text{either} & m + 2 = 0 \longrightarrow m_1 = -2 \\ \text{or} & m + 2 = 0 \longrightarrow m_2 = -2 \end{array}\right\} \text{مكرر}$$

$$y_h = c_1 e^{mx} + c_2 x\, e^{mx}$$

$$y_h = c_1 \underbrace{e^{-2x}}_{u_1} + c_2 \underbrace{x e^{-2x}}_{u_2}$$

② Find particular solution:

$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & -2xe^{-2x} + e^{-2x} \end{vmatrix}$$

$$W = \cancel{-2xe^{-4x}} + e^{-4x} \cancel{+ 2xe^{-4x}} = e^{-4x}$$

$$W_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{-2x} \\ \frac{e^{-2x}}{x^2} & -2xe^{-2x} + e^{-2x} \end{vmatrix}$$

$$W_1 = -\frac{xe^{-4x}}{x^2} = -\frac{e^{-4x}}{x}$$

$$W_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{e^{-2x}}{x^2} \end{vmatrix}$$

$$W_2 = \frac{e^{-4x}}{x^2}$$

$$v_1' = \frac{W_1}{W} = \frac{-\frac{e^{-4x}}{x}}{e^{-4x}} = -\frac{1}{x}$$

$$v_1 = \int v_1'\,dx = \int -\frac{1}{x}\,dx = -\ln x$$

$$v_2' = \frac{W_2}{W} = \frac{e^{-4x}/x^2}{e^{-4x}} = \frac{1}{x^2}$$

$$v_2 = \int v_2'\,dx = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$y_p = u_1 v_1 + u_2 v_2$$

$$= e^{-2x} \ln x + x e^{-2x}\left(-\frac{1}{x}\right)$$

$$y_p = e^{-2x} \ln x - e^{-2x}$$

$$y = y_h + y_p$$

$$= c_1 e^{-2x} + c_2 x e^{-2x} + e^{-2x} \ln x - e^{-2x}$$

Example 6: Find the general solution of the second order D-E by using (variation of Parameters method):

$$y'' + 2y' + y = e^{-x} \ln x$$

Solution:

① Find homogenous solution:

$$y'' + 2y' + y = 0$$

let $y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$(m^2 + 2m + 1) e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$m^2 + 2m + 1 = 0$$

$$(m+1)(m+1) = 0$$

$$\left.\begin{array}{lll} \text{either} & m+1=0 \longrightarrow & m_1 = -1 \\ \text{or} & m+1=0 \longrightarrow & m_2 = -1 \end{array}\right\} \text{مكرر}$$

$$y_h = c_1 e^{mx} + c_2 x e^{mx}$$

$$y_h = c_1 \underbrace{e^{-x}}_{u_1} + c_2 \underbrace{x e^{-x}}_{u_2}$$

② Find particular solution:

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x}+e^{-x} \end{vmatrix}$$

$$w = -\cancel{x e^{-2x}} + e^{-2x} + \cancel{x e^{-2x}} = e^{-2x}$$

$$w1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\ln x & -xe^{-x}+e^{-x} \end{vmatrix}$$

$$w_1 = -xe^{-2x}\ln x$$

$$w2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x}\ln x \end{vmatrix}$$

$$w2 = e^{-2x}\ln x$$

$$v_1' = \frac{w1}{w} = \frac{-xe^{-2x}\ln x}{e^{-2x}} = -x\ln x$$

$$v_1 = \int v_1'\,dx = \int -x\ln x\,dx$$

using integration by parats

$u = \ln x \longrightarrow du = \frac{1}{x} dx$

$dv = -x dx \longrightarrow v = \frac{-x^2}{2}$

$\int u\, dv = v \cdot u - \int u\, dv$

$v_1 = \int -x \ln x\, dx = -\frac{x^2}{2} \ln x - \int \left(-\frac{x^2}{2}\right)\left(\frac{1}{x}\right) dx$

$v_1 = \frac{-x^2}{2} \ln x + \frac{x^2}{4}$

$v_2' = \frac{w_2}{w} = \frac{e^{-2x} \ln x}{e^{-2x}} = \ln x$

$v_2 = \int v_2' dx = \int \ln x\, dx$

using integration by parts

$u = \ln x \longrightarrow du = \frac{1}{x} dx$

$dv = dx \longrightarrow v = x$

$v_2 = \int \ln x\, dx = x \ln x - \int x \frac{1}{x} dx$

$v_2 = x \ln x - x$

$y_p = u_1 v_1 + u_2 v_2$

$$y_p = e^{-x}\left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right) + x e^{-x}(x\ln x - x)$$

$$y_p = \left(\pm\frac{x^2}{2}\ln x - \frac{3}{4}x^2\right)e^{-x}$$

$$y = y_h + y_p$$

$$y = c_1 e^{-x} + c_2 x e^{-x} + \left(\frac{x^2}{2}\ln x - \frac{3}{4}x^2\right)e^{-x}$$

Example 7: Find the general solution of the second order D.E by using (variation of parameters method):

$$y'' - 6y' + 9y = x^{-3} e^{3x}$$

solution:

① Find homogenous solution:

$$y'' - 6y' + 9y = 0$$

let $y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} - 6m e^{mx} + 9 e^{mx} = 0$$

$$(m^2 - 6m + 9)\, e^{mx} = 0 \quad , e^{mx} \neq 0$$

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0$$

$$\left.\begin{array}{lll} \text{either} & m-3=0 \longrightarrow m_1 = 3 \\ \text{or} & m-3=0 \longrightarrow m_2 = 3 \end{array}\right\} \text{مكرر}$$

$$y_h = c_1 e^{mx} + c_2\, x\, e^{mx}$$

$$y_h = c_1 \underbrace{e^{3x}}_{u_1} + c_2 \underbrace{x e^{3x}}_{u_2}$$

② Find particular solution:

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & 3x e^{3x} + e^{3x} \end{vmatrix}$$

$$w = \cancel{3x} e^{6x} + e^{6x} - \cancel{3x} e^{6x} = e^{6x}$$

$$w_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & x e^{3x} \\ x^{-3} e^{3x} & 3x e^{3x} + e^{3x} \end{vmatrix}$$

$$w_1 = -x^{-2} e^{6x}$$

$$w_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & x^{-3} e^{3x} \end{vmatrix}$$

$$w_2 = x^{-3} e^{6x}$$

$$v_1' = \frac{w_1}{w} = \frac{-x^{-2} e^{6x}}{e^{6x}} = -x^{-2}$$

$$v_1 = \int v_1' \, dx = \int -x^{-2} \, dx = \frac{1}{x}$$

$$v_2' = \frac{w_2}{w} = \frac{x^{-3} e^{6x}}{e^{6x}} = x^{-3}$$

$$v_2 = \int v_2' \, dx = \int x^{-3} dx = -\frac{x^{-2}}{2}$$

$$y_p = u_1 v_1 + u_2 v_2$$

$$y_p = e^{3x} \cdot \frac{1}{x} + (x e^{3x}) \left(-\frac{x^{-2}}{2}\right)$$

$$y_p = \frac{e^{3x}}{x} - \frac{e^{3x}}{2x} = \frac{e^{3x}}{2x}$$

$$y = y_h + y_p$$

$$y = c_1 e^{3x} + c_2 x e^{3x} + \frac{e^{3x}}{2x}$$

example 8: Find the general solution for the D.E using (variation of parameters method):

$$y'' + 12y' + 32y = \sin(e^{4x})$$

solution:

① Find homogenous solution:

$$y'' + 12y' + 32y = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 12me^{mx} + 32e^{mx} = 0$$

$$(m^2 + 12m + 32)\, e^{mx} = 0 \,,\; e^{mx} \neq 0$$

$$m^2 + 12m + 32 = 0$$

$$(m+4)(m+8) = 0$$

either $m + 4 = 0 \longrightarrow m_1 = -4$

or $m + 8 = 0 \longrightarrow m_2 = -8$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_h = c_1 \underbrace{e^{-4x}}_{u_1} + c_2 \underbrace{e^{-8x}}_{u_2}$$

② Find particular solution:

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} e^{-4x} & e^{-8x} \\ -4e^{-4x} & -8e^{-8x} \end{vmatrix}$$

$$w = -8e^{-12x} + 4e^{-12x} = -4e^{-12x}$$

$$w_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{-8x} \\ \sin(e^{4x}) & -8e^{-8x} \end{vmatrix}$$

$$w_1 = -e^{-8x} \sin(e^{4x})$$

$$w_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} e^{-4x} & 0 \\ -4e^{-4x} & \sin(e^{4x}) \end{vmatrix}$$

$$w_2 = e^{-4x} \sin(e^{4x})$$

$$u_1' = \frac{w_1}{w} = \frac{-e^{-8x} \sin(e^{4x})}{-4e^{-12x}}$$

$$u_1' = \frac{1}{4} e^{4x} \sin(e^{4x})$$

$$v_1 = \int v_1' \, dx = \int \frac{1}{4} e^{4x} \sin(e^{4x}) \, dx$$

$$v_1 = \frac{-1}{16} \cos(e^{4x})$$

$$v_2' = \frac{w_2}{w} = \frac{e^{-4x} \sin(e^{4x})}{-4 e^{-12x}}$$

$$v_2' = -e^{8x} \sin(e^{4x}) / 4$$

$$v_2 = \int v_2' \, dx$$

$$v_2 = -\frac{1}{4} \int e^{8x} \sin(e^{4x}) \, dx$$

$$= -\frac{1}{4} \int e^{4x} \cdot e^{4x} \sin(e^{4x}) \, dx$$

using integration by parts

$$u = e^{4x} \longrightarrow du = 4 e^{4x} \, dx$$

$$dv = e^{4x} \sin(e^{4x}) dx \Rightarrow v = -\frac{1}{4} \cos(e^{4x})$$

$$v \cdot u - \int v \, du$$

$$v_2 = -\frac{1}{4} \left[-\frac{1}{4} \cos(e^{4x}) e^{4x} - \int (-\frac{1}{4} \cos(e^{4x}) 4 e^{4x} dx \right]$$

$$v_2 = -\frac{1}{4}\left[-\frac{1}{4}\cos(e^{4x})e^{4x} + \frac{1}{4}\int 4e^{4x}\cos(e^{4x})\,dx\right]$$

$$= \frac{1}{16}\cos(e^{4x})e^{4x} - \frac{1}{16}\sin(e^{4x})$$

$$y_p = u_1 v_1 + u_2 v_2$$

$$y_p = e^{-4x}\left(-\frac{1}{16}\cos(e^{4x})\right) + e^{-8x}\left[\frac{1}{16}e^{4x}\cos(e^{4x}) - \frac{1}{16}\sin(e^{4x})\right]$$

$$y = y_h + y_p$$

$$= c_1 e^{-4x} + c_2 e^{-8x} - \frac{1}{16}e^{-4x}\cos(e^{4x}) + \frac{1}{16}e^{-4x}\cos(e^{4x}) - \frac{1}{16}e^{-8x}\sin(e^{4x})$$

Example 9: solve the D.E

$$y'' + y = \tan x + 5e^{3x}$$

Solution:

① Find homogenous solution:

$$y'' + y = 0$$

let $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$$m^2 e^{mx} + e^{mx} = 0$$

$$(m^2 + 1)e^{mx} = 0 \quad , \quad e^{mx} \neq 0$$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m_{1,2} = \mp i \qquad \alpha = 0 \quad \beta = 1$$

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

$$y_h = e^{0x}(c_1 \sin x + c_2 \cos x)$$

$$y_h = c_1 \underbrace{\sin x}_{u_1} + c_2 \underbrace{\cos x}_{u_2}$$

② Find particular solution:

$$y_P = \underbrace{y_{P1}}_{\text{variation of parameters method}} + \underbrace{y_{P2}}_{\text{undetermined coefficient method}}$$

To find y_{P1}

$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$W = -\sin^2 x - \cos^2 x = -1$$

$$W_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & \cos x \\ \frac{\sin x}{\cos x} & -\sin x \end{vmatrix}$$

$$W_1 = -\frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos} x$$

$$W_1 = -\sin x$$

$$W_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} \sin x & 0 \\ \cos x & \frac{\sin x}{\cos x} \end{vmatrix}$$

$$W_2 = \frac{\sin^2 x}{\cos x}$$

$$v_1' = \frac{w_1}{w} = \frac{-\sin x}{-1} = \sin x$$

$$v_1 = \int v_1' \, dx = \int \sin x \, dx = -\cos x$$

$$v_2' = \frac{w_2}{w} = \frac{\sin^2 x / \cos x}{-1}$$

$$v_2' = -\frac{\sin^2 x}{\cos x} = \frac{-(1-\cos^2 x)}{\cos x}$$

$$v_2' = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

$$v_2 = \int v_2' \, dx = \int [\cos x - \sec x] \, dx$$

$$= \int \cos x \, dx - \int \sec x \, dx$$

$$= \sin x - \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \sin x - \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$v_2 = \sin x - \ln|\sec x + \tan x|$$

$$y_{p1} = u_1 v_1 + u_2 v_2$$

$$= \sin x(-\cos x) + \cos x\left[\sin x - \ln|\sec x + \tan x|\right]$$

$$y_{p1} = -\cancel{\sin x \cos x} + \cancel{\sin x \cos x} - \cos x \ln|\sec x + \tan x|$$

$$y_{p1} = -\cos x \ln|\sec x + \tan x|$$

To find y_{p2}

$$y_{p2} = a_0 e^{3x}$$

$$y_{p2}' = 3a_0 e^{3x}$$

$$y_{p2}'' = 9a_0 e^{3x}$$

sub in D.E

$$9a_0 e^{3x} + a_0 e^{3x} = 5 e^{3x}$$

$$10 a_0 = 5$$

$$a_0 = \frac{1}{2}$$

$$y_{p2} = \frac{1}{2} e^{3x}$$

$$y_p = y_{p1} + y_{p2} = -\cos x \ln|\sec x + \tan x| + \frac{1}{2} e^{3x}$$

$$y = y_h + y_p$$

$$y = c_1 \sin x + c_2 \cos x - \cos x \ln|\sec x + \tan x| + \frac{1}{2} e^{3x}$$

Example 10: Solve the D.E

$$y'' - 2y' + 2y = \frac{e^x}{\sin x} + 3\cos x$$

Solution:

① Find homogenous solution

$$y'' - 2y' + 2y = 0$$

let $y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$(m^2 - 2m + 2)e^{mx} = 0 \quad , \quad e^{mx} \neq 0$$

$$m^2 - 2m + 2 = 0$$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{2 \mp \sqrt{4 - 4 \times 2}}{2}$$

$$m = 1 \mp \frac{\sqrt{-4}}{2} = 1 \mp \frac{2i}{2} = 1 \mp 1i$$

$$\alpha = 1$$
$$\beta = 1$$

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

$$y_h = e^{x}(c_1 \sin x + c_2 \cos x)$$

$$y_h = c_1 \underbrace{e^{x}\sin x}_{u_1} + c_2 \underbrace{e^{x}\cos x}_{u_2}$$

② Fin particular solution

$$y_p = y_{p1} + y_{p2}$$

y_{p1}: varation of prameters method

y_{p2}: undetermined coefficent method

To find y_{p1}

$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} e^{x}\sin x & e^{x}\cos x \\ e^{x}\cos x + e^{x}\sin x & -e^{x}\sin x + e^{x}\cos x \end{vmatrix}$$

$$W = -e^{2x}\sin^2 x + \cancel{e^{2x}\sin x\cos x} - e^{2x}\cos^2 x - \cancel{e^{2x}\sin x\cos x}$$

$$W = -e^{2x}\underbrace{(\sin^2 x + \cos^2 x)}_{1} = -e^{2x}$$

$$w_1 = \begin{vmatrix} 0 & u_2 \\ g(x) & u_2' \end{vmatrix} = \begin{vmatrix} 0 & e^x \cos x \\ \frac{e^x}{\sin x} & -e^x \sin x + e^x \cos x \end{vmatrix}$$

$$w_1 = \frac{-e^{2x} \cos x}{\sin x}$$

$$w_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & g(x) \end{vmatrix} = \begin{vmatrix} e^x \sin x & 0 \\ e^x \cos x + e^x \sin x & \frac{e^x}{\sin x} \end{vmatrix}$$

$$w_2 = e^x \sin x \cdot \frac{e^x}{\sin x} = e^{2x}$$

$$v_1' = \frac{w_1}{w} = \frac{\frac{-e^{2x}\cos x}{\sin x}}{-e^{2x}} = \frac{\cos x}{\sin x}$$

$$v_1 = \int v_1' \, dx = \int \frac{\cos x}{\sin x} = \ln \sin x$$

$$v_2' = \frac{w_2}{w} = \frac{e^{2x}}{-e^{2x}} = -1$$

$$v_2 = \int v_2' \, dx = \int -dx = -x$$

$$y_{p_1} = u_1 v_1 + u_2 v_2$$

$$y_{p_1} = e^x \sin x \cdot \ln \sin x - e^x x \cos x$$

To find y_{p2}

$$y_{p_2} = a_0 \sin x + a_1 \cos x$$

$$y_{p_2}' = a_0 \cos x - a_1 \sin x$$

$$y_{p_2}'' = -a_0 \sin x - a_1 \cos x$$

sub in D.E

$$-a_0 \sin x - a_1 \cos x - 2\left[a_0 \cos x - a_1 \sin x\right]$$
$$+2\left[a_0 \sin x + a_1 \cos x\right] = 3\cos x$$

$$-a_0 \sin x - a_1 \cos x - 2a_0 \cos x + 2a_1 \sin x$$
$$+2a_0 \sin x + 2a_1 \cos x = 3\cos x$$

معاملات $\cos x$:

$$-a_1 - 2a_0 + 2a_1 = 3$$

$$-2a_0 + a_1 = 3 \quad (1)$$

معاملات $\sin x$:

$$-a_0 + 2a_1 + 2a_0 = 0$$

$$a_0 + 2a_1 = 0 \quad (2)$$

نضرب معادلة 2 بـ 2

$$2a_0 + 4a_1 = 0 \quad (2)$$

$$-2a_0 + a_1 = 3 \quad (1)$$

بالجمع

$$5a_1 = 3$$

$$a_1 = \frac{3}{5}$$

نعوض بالمعادلة (2) لايجاد a_0

$$a_0 = -\frac{6}{5}$$

$$y_{p2} = -\frac{6}{5}\sin x + \frac{3}{5}\cos x$$

$$y_p = y_{p1} + y_{p2}$$

$$= e^x \sin x \ln \sin x - e^x x \cos x$$

$$-\frac{6}{5}\sin x + \frac{3}{5}\cos x$$

$$y = y_h + y_p$$

$$y = e^x(c_1 \sin x + c_2 \cos x) + e^x \sin x \ln \sin x$$

$$- e^x x \cos x - \frac{6}{5}\sin x + \frac{3}{5}\cos x$$