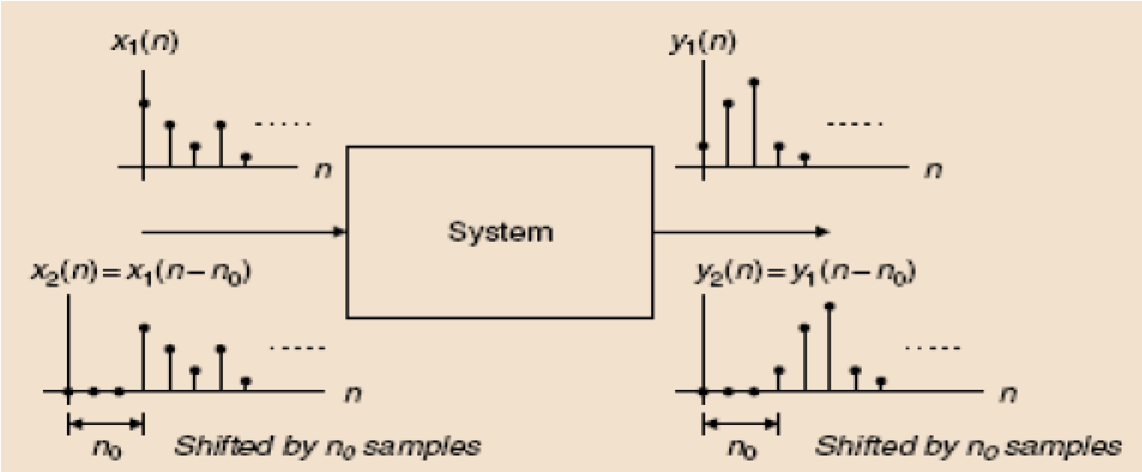


<u>Example:</u> Given the linear systems:

a. $\boldsymbol{y(n) = 2x(n-5)}$

b. $\boldsymbol{y(n) = 2x(3n)}$

c. $\boldsymbol{y(n) = n\,x(n)}$

Determine whether each of the following systems is time-invariant.

<u>sol.</u>

a. Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the system output is $y_1(n) = 2x_1(n-5)$. Again, let $x_2(n) = x_1(n-n_0)$ be the shifted input and $y_2(n)$ be the output due to the shifted input. We determine the system output using the shifted input as

$\boldsymbol{y_2(n) = 2x_2(n-5) = 2x_1(n-n_0-5)}$:

Meanwhile, shifting $y_1(n) = 2x_1(n-5)$ by n_0 samples leads to

$\boldsymbol{y_1(n-n_0) = 2x_1(n-5-n_0)}$

We can verify that $\boldsymbol{y_2(n) = y_1(n-n_0)}$. Thus the shifted input of n_0 samples causes the system output to be shifted by the same n_0 samples, thus, the system is ***time-invariant***.

b. Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the system output is $y_1(n) = 2x_1(3n)$. Again, let the input and output be $x_2(n)$ and $y_2(n)$, where $x_2(n) = x_1(n-n_0)$, a shifted version, and the corresponding output is $y_2(n)$. We get the output due to the shifted input $x_2(n) = x_1(n-n_0)$ and note that $x_2(3n) = x_1(3n-n_0)$:

$\boldsymbol{y_2(n) = 2x_2(3n) = 2x_1(3n-n_0)}$:

On the other hand, if we shift $y_1(n)$ by n_0 samples, which replaces n in

$y_1(n) = 2x_1(3n)$ by $n-n_0$, it yield

$\boldsymbol{y_1(n-n_0) = 2x_1(3(n-n_0)) = 2x_1(3n-3n_0)}$:

Clearly, we know that $\boldsymbol{y_2(n) \neq y_1(n-n_0)}$. Since the system output $y_2(n)$ using the input shifted by n_0 samples is not equal to the system output $y_1(n)$ shifted by the same n_0 samples, thus, the system is ***not time-invariant (time-varying system)***.

c. Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the output is $y_1(n) = n\mathrm{x}_1(n)$. Again, let the input and output be $x_2(n)$ and $y_2(n)$, where $x_2(n) = x_1(n - \mathrm{n}_0)$, a shifted version, and the corresponding output is $y_2(n)$. We get the output due to the shifted input $x_2(n) = x_1(n - \mathrm{n}_0)$ and note that $x_2(n) = n\,\mathrm{x}_1(n - \mathrm{n}_0)$:

$\boldsymbol{y_2(n) = n\,x_2(n) = n\,x_1(n - \mathrm{n}_0)}$:

On the other hand, if we shift $y_1(n)$ by n_0 samples, which replaces n in

$y_1(n) = n x_1(n)$ by $n - \mathrm{n}_0$, it yield

$\boldsymbol{y_1(n - \mathrm{n}_0) = (n\text{-}n_0)\,x_1(n - \mathrm{n}_0)}$:

Clearly, we know that $\boldsymbol{y_2(n) \neq y_1(n - \mathrm{n}_0)}$. Since the system output $y_2(n)$ using the input shifted by n_0 samples is not equal to the system output $y_1(n)$ shifted by the same n_0 samples, thus, the system is ***not time-invariant (time-varying system)***.

Note: Linear Time Invariant System (LTI) is the system that satisfies both the linearity and the time-invariance properties. Such systems are mathematically easy to analyze, and easy to design.

Causal and Non-Causal Systems:

A **causal** system is one in which the output $y(n)$ at time n depends only on the current input $x(n)$ at time n, its past input sample values such as $x(n - 1)$, $x(n- 2)$, . . . For example $\mathbf{y[n] = \alpha x[n] + \beta x[n\text{-}1]}$. Otherwise, if a system output depends on the future input values, such as $x(n + 1)$, $x(n + 2)$, . . . , the system is **noncausal**. For example $\boldsymbol{y[n] = \alpha x[n] + \beta x[n+1]}$. The noncausal system cannot be realized in real time.

Example: Given the linear systems:

a. $\boldsymbol{y(n) = 0.5x(n) + 2.5x(n - 2)}$**, for** $\boldsymbol{n \geq 0}$

b. $\boldsymbol{y(n) = 0.25x(n - 1) + 0.5x(n + 1) - 0.4y(n - 1)}$**, for** $\boldsymbol{n \geq 0}$**,**

c. $$\boldsymbol{y(n) = \sum_{k=-2}^{2} h(k)\,x(n-k)}$$

Determine whether each is causal.

sol.

a. Since for $n \geq 0$, the output $y(n)$ depends on the current input $x(n)$ and its past value $x(n-2)$, the system is **causal**.

b. Since for $n \geq 0$, the output $y(n)$ depends on the input's future value $x(n+1)$, the system is **noncausal**.

c. Since for $n \geq 0$, the output $y(n)$ depends on the input's future values $x(n+1)$ and $x(n+2)$, the system is **noncausal**.

Stable and Unstable Systems:

A system is said to be **bounded input-bounded output (BIBO) stable** if and only if every bounded input produces the bounded output. It means, that there exist some finite numbers say M_x and M_y, such that

$$\boxed{|x(n)| \le M_x < \infty \Rightarrow |y(n)| \le M_y < \infty}$$

For all n, If for some bounded input sequence $x(n)$, the output $y(n)$is unbounded (infinite), the system is classified as **unstable**.

Note: The system is stable, if its transfer function vanishes after a sufficiently long time. For a stable system:

$$\boxed{S = \sum_{k=-\infty}^{\infty} |\, h(k) \,| \langle \infty}$$

Where $h(k)$ = unit impulse response.

Example: Given the systems:

a. $y[n] = (x[n])^2$

b. ***Accumulator system*** $y[n] = \sum_{k=-\infty}^{n} x[k],$

c. $y[n] = e^{x[n]}$

Determine whether each is stable.

sol.

a. If $|x[n]| \le B_x < \infty$ for all n, then $|y[n]| \le B_y = B_x^2 < \infty$ for all n. Thus, the system is **stable**.

b. If $x[n] = u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \ge 0 \end{cases}$: bounded

Then $y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0 & n < 0 \\ n+1 & n \ge 0 \end{cases}$: not bounded

Thus, the accumulator system is **unstable**.

c. If $|x[n]| \le B_x < \infty$ for all n, then $|y[n]| \le B_y = e^{B_x} < \infty$ for all n. i.e., it is guaranteed that if the input is bounded by a positive number B_x, the output is bounded by a positive number e^{B_x}. Thus, the system is **stable**.

Note: $$\boxed{\sum_{k=-\infty}^{n} u[k] = \begin{cases} 0 & n < 0 \\ n+1 & n \ge 0 \end{cases}}$$

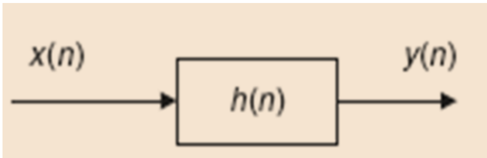
System Representation Using Its Impulse Response:

Any discrete-time can be characterized by one of the representations:

1) Difference Equation
2) Impulse Response $h(n)$
3) Transfer Function H(z)
4) Frequency Response H(W)

In this section, a Linear Time-Invariant (LTI) system will be represented by its impulse response (h(n)).

A LTI system can be completely described by its unit-impulse response, which is defined ***as the system response due to the impulse input δ(n) with zero initial conditions***, depicted in the following figure. Here $\boldsymbol{x(n) = \delta(n)}$ and $\boldsymbol{y(n) = h(n)}$.



Note: The unit step function $u[n]$ is the running sum of the unit impulse $\delta[n]$, so the **step response** $\boldsymbol{S[n]}$ of a LTI processor is the running sum of its impulse response. Therefore, if we denote the step response by $S[n]$, we have

$$\boxed{S[n] = y[n]\Big|_{x[n]=u[n]} = \sum_{m=-\infty}^{n} h[m]}$$

Alternatively, h[n] is the first order difference of $S[n]$

$$\boxed{h[n] = y[n]\Big|_{x[n]=\delta[n]} = S[n] - S[n-1]}$$

Example: For a LTI system described by the following difference equation:

$\boldsymbol{y(n) = 0.8y(n-1) + x(n)}$

a. Find and sketch the first four sample values of the impulse and step responses.
b. Determine the final value of the step response as $n \longrightarrow \infty$.

sol.

a. By setting $x(n)=\delta(n)$ in the system difference equation, then $y(n)=h(n)$ so,

$\boldsymbol{h(n)=0.8h(n-1)+\delta(n)}$

for n=0, $h(0)=(0.8)h(-1)+\delta(0)=1$

for n=1, $h(1)=(0.8)h(0)+ \delta(1)=0.8$

for n=2, $h(2)=(0.8)h(1)+\delta(2)=(0.8)^2=0.64$

for n=3, $h(3)=(0.8)h(2)+\delta(3)=(0.8)^3=0.512$