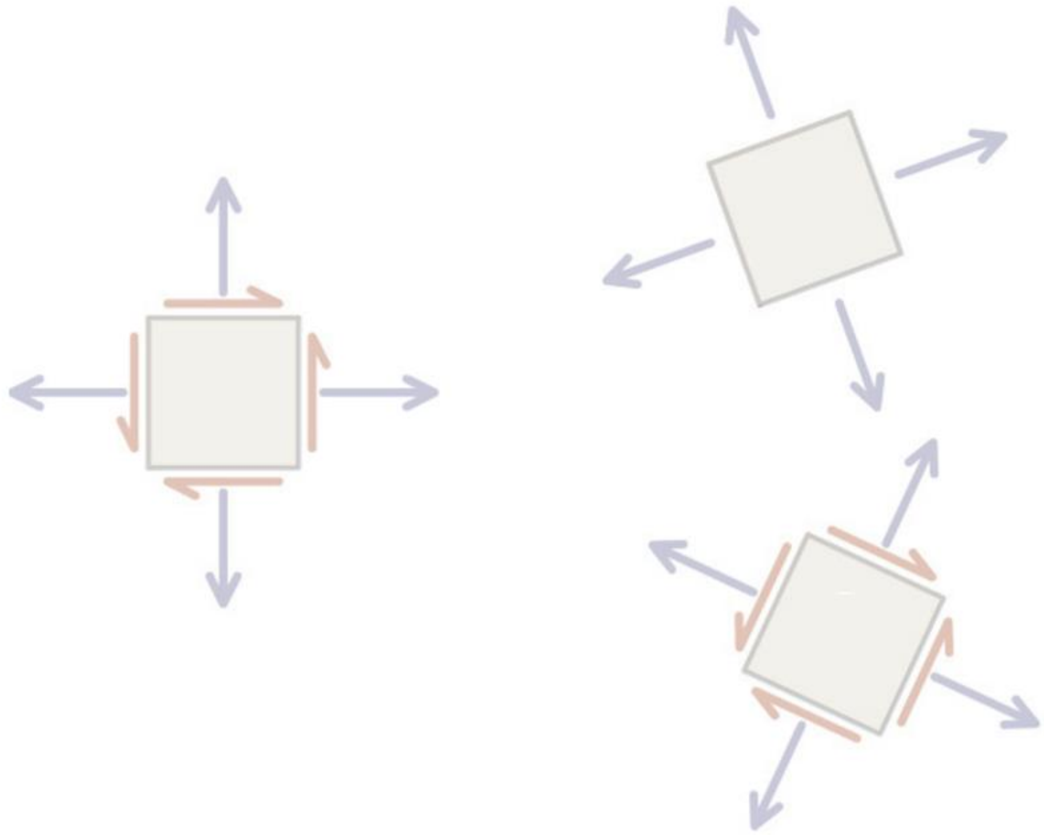
CHAPTER 12

Stress Transformations



12.7 General Equations of Plane Stress Transformation

For a successful design, an engineer must be able to determine critical stresses at any point of interest in a material object. By the mechanics of materials theory developed for axial members, torsion members, and beams, normal and shear stresses at a point in a material object can be computed with reference to a particular coordinate system, such as an x–y coordinate system. Such a coordinate system, however, has no inherent significance with regard to the material used in a structural member. Failure of the material will occur in response to the largest stresses that are developed in the object, regardless of the orientation at which those critical stresses are acting. For instance, a designer has no assurance that a horizontal bending stress computed at a point in the web of a wide-flange beam will be the largest normal stress possible at the point. To find the critical stresses at a point in a material object, methods must be developed so that stresses acting at all possible orientations can be investigated.

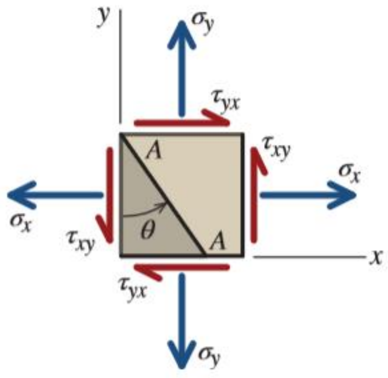


FIGURE 12.10*a*

Consider a state of stress represented by a plane stress element subjected to stresses σ_x, σ_y, and $\tau_{xy} = \tau_{yx}$, as shown in Figure 12.10*a*. *Keep in mind that the stress element is simply a convenient graphical symbol used to represent the state of stress at a specific point of interest in an object (such as a shaft or a beam).* To derive equations applicable to any orientation, we begin by defining a plane surface A–A oriented at some angle θ with respect to a reference axis x. The normal to surface A–A is termed the n axis. The axis parallel to surface A–A is termed the t axis. The z axis extends out of the plane of the stress element. Both the x–y–z and the n–t–z axes are arranged as right-hand coordinate systems. Given the σ_x, σ_y, and $\tau_{xy} = \tau_{yx}$ stresses acting on the x and y faces of the stress element, we will determine the normal and shear stress acting on surface A–A, known as the n face of the stress element. This process of changing stresses from one set of coordinate axes (i.e., x–y–z) to another set of axes (i.e., n–t–z) is termed **stress transformation**.

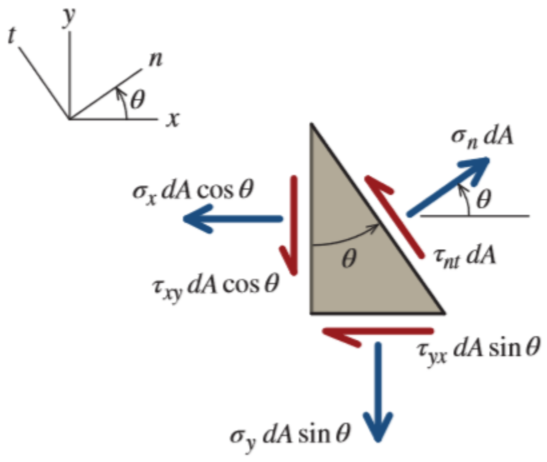


FIGURE 12.10*b*

Figure 12.10*b* is a free-body diagram of a wedge-shaped element in which the areas of the faces are dA for the inclined face (plane A–A), $dA\cos\theta$ for the vertical face (i.e., the x face), and $dA\sin\theta$ for the horizontal face (i.e., the y face). The equilibrium equation for the sum of forces in the n direction gives

$$\Sigma F_n = \sigma_n dA - \tau_{yx}(dA\ \sin\theta)\cos\theta - \tau_{xy}(dA\ \cos\theta)\sin\theta$$
$$- \sigma_x(dA\ \cos\theta)\cos\theta - \sigma_y(dA\ \sin\theta)\sin\theta = 0$$

Since $\tau_{yx} = \tau_{xy}$, this equation can be simplified to give the following expression for the normal stress acting on the n face of the wedge element:

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \tag{12.3}$$

From the free-body diagram in Figure 12.10*b*, the equilibrium equation for the sum of forces in the t direction gives

$$\Sigma F_t = \tau_{nt}\, dA - \tau_{xy}(dA\cos\theta)\cos\theta + \tau_{yx}(dA\sin\theta)\sin\theta$$
$$+ \sigma_x(dA\cos\theta)\sin\theta - \sigma_y(dA\sin\theta)\cos\theta = 0$$

Again from $\tau_{yx} = \tau_{xy}$, this equation can be simplified to give the following expression for the shear stress acting in the t direction on the n face of the wedge element:

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \tag{12.4}$$

The equations just derived for the normal stress and the shear stress can be written in an equivalent form by substituting the following double-angle identities from trigonometry:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$2\sin\theta\cos\theta = \sin 2\theta$$

Using these double-angle identities, we can write Equation (12.3) as

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{12.5}$$

and Equation (12.4) as

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{12.6}$$

Equations (12.3), (12.4), (12.5), and (12.6) are called the **plane stress transformation equations**. They provide a means for determining normal and shear stresses on any plane whose outward normal is

(a) perpendicular to the z axis (i.e., the out-of-plane axis), and
(b) oriented at an angle θ with respect to the reference x axis.

Since the transformation equations were derived solely from equilibrium considerations, they are applicable to stresses in any kind of material, whether it is linear or nonlinear, elastic or inelastic.

The normal stress acting on the n face of the stress element shown in Figure 12.11 can be determined from Equation (12.5). The normal stress acting on the t face can also be obtained from Equation (12.5) by substituting $\theta + 90°$ in place of θ, giving the following equation:

$$\sigma_t = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (12.7)$$

If the expressions for σ_n and σ_t [Equations (12.5) and (12.7)] are added, the following relationship is obtained:

$$\sigma_n + \sigma_t = \sigma_x + \sigma_y \qquad (12.8)$$

Equation (12.8) shows that the sum of the normal stresses acting on any two orthogonal faces of a plane stress element is a constant value, independent of the angle θ. This mathematical characteristic of stress is termed **stress invariance**.

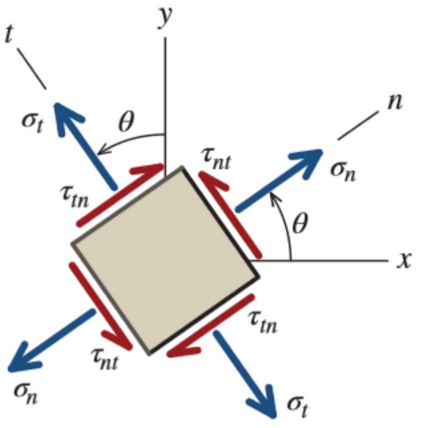


FIGURE 12.11

Sign Conventions

The sign conventions used in the development of the stress transformation equations must be rigorously followed. The sign conventions can be summarized as follows:

1. Tensile normal stresses are positive; compressive normal stresses are negative. All of the normal stresses shown in Figure 12.11 are positive.

2. A shear stress is positive if it

- acts in the positive coordinate direction on a positive face of the stress element or
- acts in the negative coordinate direction on a negative face of the stress element.

All of the shear stresses shown in Figure 12.11 are positive. Shear stresses pointing in opposite directions are negative.

An easy way to remember the shear stress sign convention is to use the directions associated with the two subscripts. The first subscript indicates the face of the stress element on which the shear stress acts. It will be either a positive face (plus) or a negative face (minus). The second subscript indicates the direction in which the stress acts, and it will be either a positive direction (plus) or a negative direction (minus).

- A positive shear stress has subscripts that are either plus–plus or minus–minus.
- A negative shear stress has subscripts that are either plus–minus or minus–plus.

3. Angles measured counterclockwise from the reference x axis are positive. Conversely, angles measured clockwise from the reference x axis are negative.

4. The n–t–z axes have the same order as the x–y–z axes. Both sets of axes form a right-hand coordinate system.

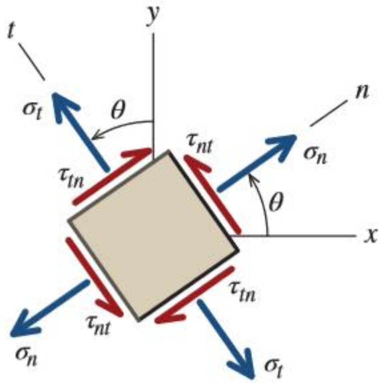
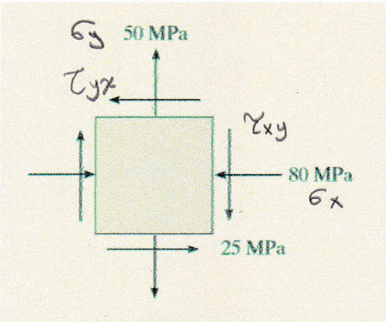


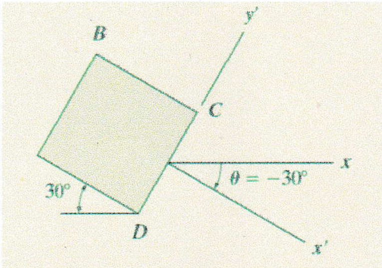
FIGURE 12.11

EXAMPLE 12.3

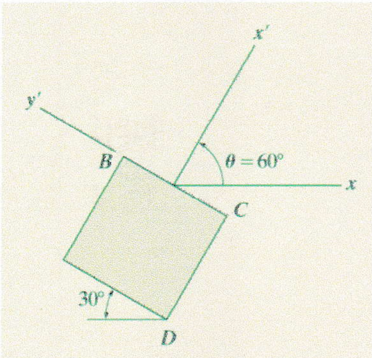
The state of plane stress at a point is represented by the element shown in Fig. 12.3*a*. Determine the state of stress at the point on another element oriented 30° clockwise from the position shown.



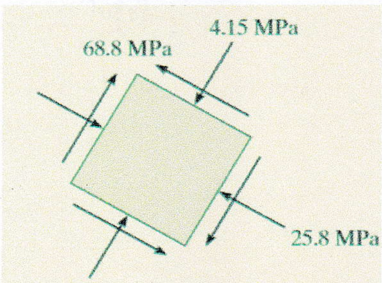
(a)



(b)



(c)



(d)

Fig. 12.3

- Shear stress (τ_{xy})
  posetive x face (+) * negative y (−)
  $\Rightarrow \tau_{xy} = -25$ Mpa , $\theta = -30°$
- $\sigma_y = +50$ Mpa , $\sigma_x = -80$ Mpa

$$\sigma_{x'} = \sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \frac{(-80)+50}{2} + \frac{(-80)-50}{2}\cos 2(-30) + (-25)\sin 2(-30)$$

$$= -25.8 \text{ Mpa.}$$

$$\sigma_{y'} = \sigma_t = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{(-80)+50}{2} - \frac{(-80)-50}{2}\cos 2(-30) - (-25)\sin 2(-30)$$

$$= -4.15 \text{ Mpa.}$$

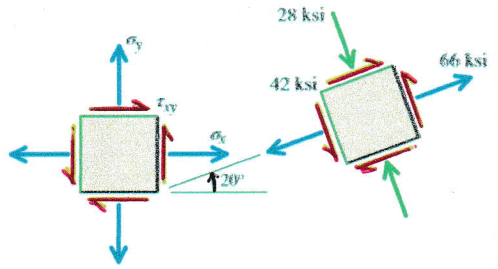
$$\tau_{x'y'} = \tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -\frac{(-80)-50}{2}\sin 2(-30) + (-25)\cos 2(-30)$$

$$= -68.8 \text{ Mpa}$$

EXAMPLE 12.4

The stresses shown act at a point on the free surface of a machine component. Determine the normal stresses σ_x and σ_y and the shear stress τ_{xy} at the point.



Plan the Solution

The stress transformation equations are written in terms of σ_x, σ_y, and τ_{xy}; however, the x and y directions do not necessarily have to be the horizontal and vertical directions, respectively. Any two orthogonal directions can be taken as x and y, as long as they define a right-hand coordinate system. To solve this problem, we will redefine the x and y axes, aligning them with the rotated element. The faces of the unrotated element will be redefined as the n and t faces.

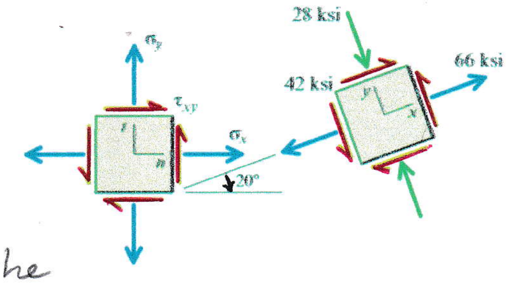
- Shear stress τ_{xy}

posetive x face (+) * posetive y (+)

$\Rightarrow \tau_{xy} = +42$ ksi

- $\sigma_y = -28$ ksi, $\sigma_x = 66$ ksi

θ from the redefined x axis to the n axis is 20° clockwise $\Rightarrow \theta = -20°$



$$\sigma_{x'} = \sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= \frac{66 + (-28)}{2} + \frac{66 - (-28)}{2} \cos 2(-20) + 42 \sin 2(-20)$$

$$= 28 \text{ ksi}$$

$$\sigma_{y'} = \sigma_t = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$= \frac{66 + (-28)}{2} - \frac{66 - (-28)}{2} \cos 2(-20) - 42 \sin 2(-20)$$

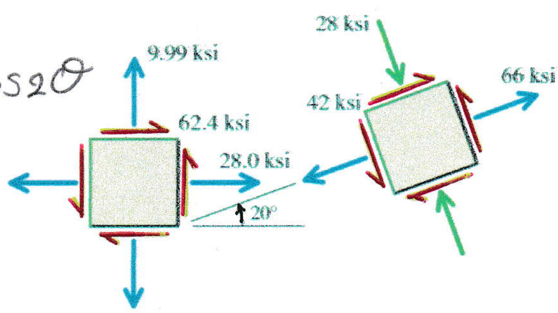
$$= 9.99 \text{ ksi}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$= -\frac{66 - (-28)}{2} \sin 2(-20)$$

$$+ 42 \cos 2(-20)$$

$$= 62.4 \text{ ksi}$$



12.8 Principal Stresses and Maximum Shear Stress

The transformation equations for plane stress [Equations (12.3), (12.4), (12.5), and (12.6)] provide a means for determining the normal stress σ_n and the shear stress τ_{nt} acting on any plane through a point in a stressed body. For design purposes, the critical stresses at a point are often the maximum and minimum normal stresses and the maximum shear stress. The stress transformation equations can be used to develop additional relationships that indicate

(a) the orientations of planes where maximum and minimum normal stresses occur,
(b) the magnitudes of maximum and minimum normal stresses,
(c) the magnitudes of maximum shear stresses, and
(d) the orientations of planes where maximum shear stresses occur.

The transformation equations for plane stress were developed in Section 12.7. Equations (12.3) and (12.4), for normal stress and shear stress are, respectively, as follows:

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

These same equations can also be expressed in terms of double-angle trigonometric functions as Equations (12.5) and (12.6):

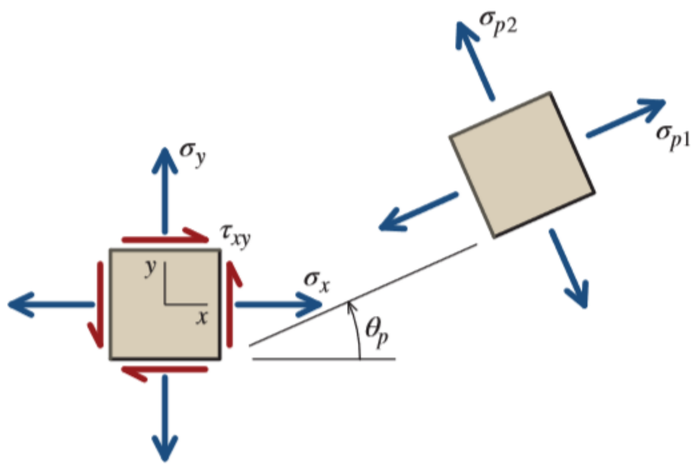
$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Principal Stresses

Planes free of shear stress are termed **principal planes**. The normal stresses acting on these planes—the maximum and minimum normal stresses—are called **principal stresses**.

$$\frac{d\sigma_n}{d\theta} = -\frac{\sigma_x - \sigma_y}{2}(2\sin 2\theta) + 2\tau_{xy}\cos 2\theta = 0 \qquad (12.10)$$



principal angles.

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (12.11)$$

principal stresses.

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (12.12)$$

Shear Stresses on Principal Planes

$$\tau_{nt} = 0$$

If a plane is a principal plane, then the shear stress acting on the plane must be zero.

The converse of this statement is also true:

If the shear stress on a plane is zero, then that plane must be a principal plane.

Maximum In-Plane Shear Stress

To determine the planes where the maximum in-plane shear stress $\tau_{\max}$ occurs, Equation (12.6) is differentiated with respect to θ and set equal to zero, yielding

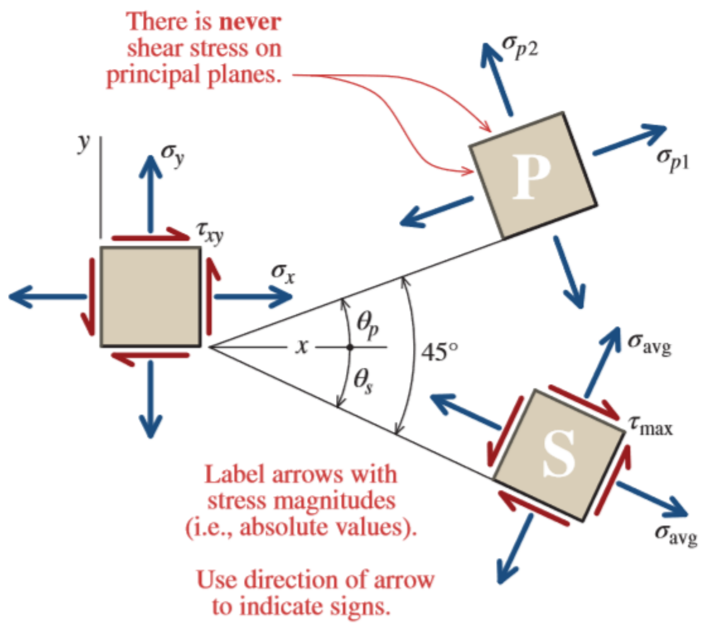
$$\frac{d\tau_{nt}}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0 \qquad (12.13)$$

The solution of this equation gives the orientation $\theta = \theta_s$ of a plane where the shear stress is either a maximum or a minimum:

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \qquad (12.14)$$

The general equation to give the magnitude of $\tau_{\max}$ is

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (12.15)$$

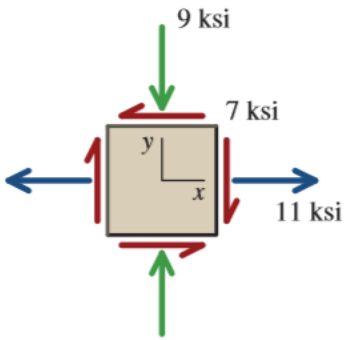


Normal Stresses on Maximum In-Plane Shear Stress Surfaces

Unlike principal planes, which are free of shear stress, planes subjected to $\tau_{\max}$ usually have normal stresses. After substituting angle functions obtained from Equation (12.14) into Equation (12.5) and simplifying, we find that the normal stress acting on a plane of maximum in-plane shear stress is

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} \qquad (12.17)$$

EXAMPLE 12.5



Consider a point in a structural member that is subjected to plane stress. Normal and shear stresses acting on horizontal and vertical planes at the point are shown.

(a) Determine the principal stresses and the maximum in-plane shear stress acting at the point.
(b) Show these stresses in an appropriate sketch.

Plan the Solution

The stress transformation equations derived in the preceding section will be used to compute the principal stresses and the maximum shear stress acting at the point.

SOLUTION

(a) From the given stresses, the values to be used in the stress transformation equations are $\sigma_x = 11$ ksi, $\sigma_y = -9$ ksi, and $\tau_{xy} = -7$ ksi. *The in-plane principal stress magnitudes* can be calculated from Equation (12.12):

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{(11\text{ ksi}) + (-9\text{ ksi})}{2} \pm \sqrt{\left(\frac{(11\text{ ksi}) - (-9\text{ ksi})}{2}\right)^2 + (-7\text{ ksi})^2}$$

$$= 13.21\text{ ksi},\quad -11.21\text{ ksi}$$

Therefore, we have the following:

$$\sigma_{p1} = 13.21\text{ ksi} = 13.21\text{ ksi (T)}$$
$$\sigma_{p2} = -11.21\text{ ksi} = 11.21\text{ ksi (C)}$$

The *maximum in-plane shear stress* can be computed from Equation (12.15):

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{\left(\frac{(11\text{ ksi}) - (-9\text{ ksi})}{2}\right)^2 + (-7\text{ ksi})^2}$$
$$= \pm 12.21\text{ ksi}$$

On the planes of maximum in-plane shear stress, the normal stress is simply the *average normal stress*, as given by Equation (12.17):

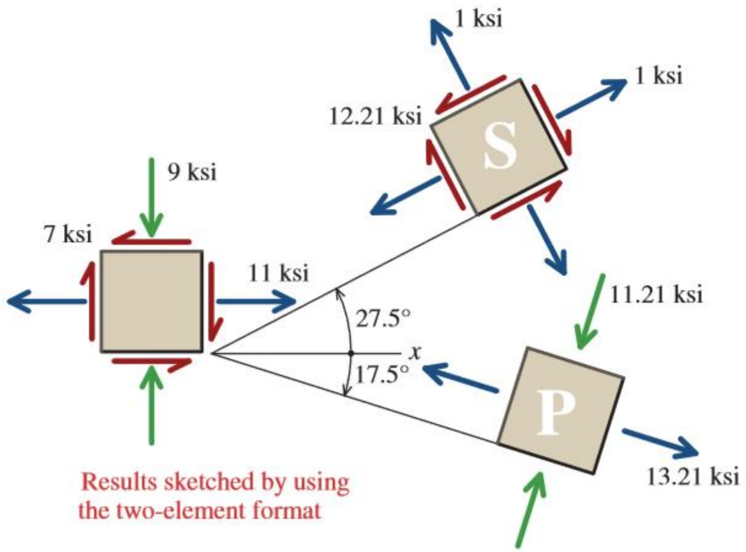
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{11\text{ ksi} + (-9\text{ ksi})}{2} = 1\text{ ksi} = 1\text{ ksi (T)}$$

(b) The principal stresses and the maximum in-plane shear stress must be shown in an appropriate sketch. The angle θ_p indicates the orientation of one principal plane relative to the reference x face. From Equation (12.11),

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-7\text{ ksi})}{11\text{ ksi} - (-9\text{ ksi})} = \frac{-14}{20}$$
$$\therefore \theta_p = -17.5^\circ$$

Since θ_p is negative, the angle is turned clockwise. In other words, the *normal* of one principal plane is rotated 17.5° below the reference x axis. One of the in-plane principal stresses—either σ_{p1} or σ_{p2}—acts on this principal plane. To determine which principal stress acts at $\theta_p = -17.5°$, use the following two-part rule:

- If the term $\sigma_x - \sigma_y$ is positive, then θ_p indicates the orientation of σ_{p1}.
- If the term $\sigma_x - \sigma_y$ is negative, then θ_p indicates the orientation of σ_{p2}.



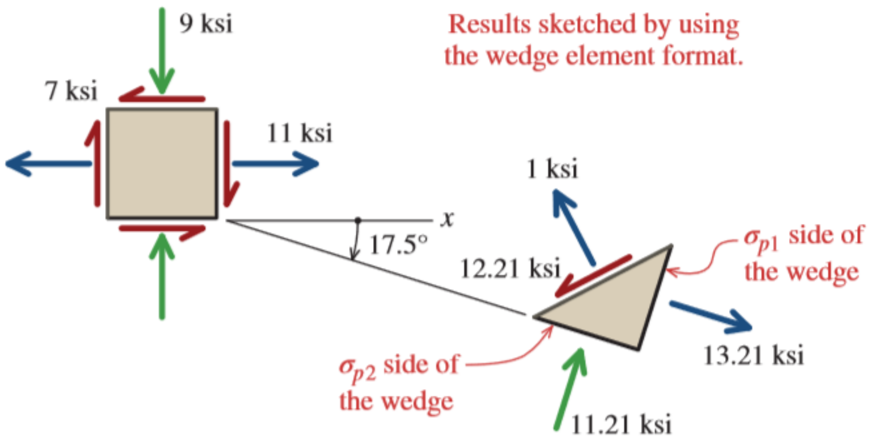
Since $\sigma_x - \sigma_y$ is positive, θ_p indicates the orientation of $\sigma_{p1} = 13.21$ ksi. The other principal stress, $\sigma_{p2} = -11.21$ ksi, acts on a perpendicular plane. The in-plane principal stresses are shown on the element labeled "P" in the figure. Note that there are never shear stresses acting on the principal planes.

The planes of maximum in-plane shear stress are always located 45° away from the principal planes; therefore, $\theta_s = 27.5°$. Although Equation (12.15) gives the magnitude of the maximum in-plane shear stress, it does not indicate the direction in which the shear stress acts on the plane defined by θ_s. To determine the direction of the shear stress, solve Equation (12.4) for τ_{nt}, using the values $\sigma_x = 11$ ksi, $\sigma_y = -9$ ksi, $\tau_{xy} = -7$ ksi, and $\theta = \theta_s = 27.5°$:

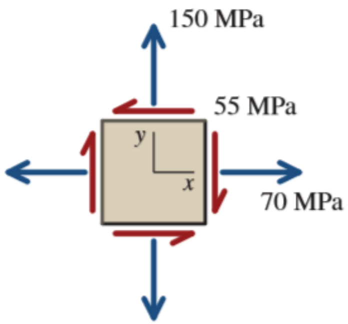
$$\begin{aligned}\tau_{nt} &= -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)\\ &= -[(11\text{ ksi}) - (-9\text{ ksi})]\sin 27.5°\cos 27.5° + (-7\text{ ksi})[\cos^2 27.5° - \sin^2 27.5°]\\ &= -12.21\text{ ksi}\end{aligned}$$

Since τ_{nt} is negative, the shear stress acts in a negative t direction on a positive n face. Once the shear stress direction has been determined for one face, the shear stress direction is known for all four faces of the stress element. The maximum in-plane shear stress and the average normal stress are shown on the stress element labeled "S." Note that, unlike the principal stress element, normal stresses will usually be acting on the planes of maximum in-plane shear stress.

The principal stresses and the maximum in-plane shear stress can also be reported on a single wedge-shaped element, as shown in the accompanying sketch. This format can be somewhat easier to use than the two-element sketch format, particularly with regard to the direction of the maximum in-plane shear stress. The maximum in-plane shear stress and the associated average normal stress are shown on the sloped face of the wedge, which is rotated 45° from the principal planes. *The shear stress arrow on this face always starts on the* σ_{p1} *side of the wedge and points toward the* σ_{p2} *side of the wedge.* Once again, there is never a shear stress on the principal planes (i.e., the σ_{p1} and σ_{p2} sides of the wedge).



EXAMPLE 12.6



Consider a point in a structural member that is subjected to plane stress. Normal and shear stresses acting on horizontal and vertical planes at the point are shown.

(a) Determine the principal stresses and the maximum in-plane shear stress acting at the point.
(b) Show these stresses in an appropriate sketch.

Plan the Solution

The stress transformation equations derived in the preceding section will be used to compute the principal stresses and the maximum shear stress acting at the point.

SOLUTION

(a) From the given stresses, the values to be used in the stress transformation equations are $\sigma_x = 70$ MPa, $\sigma_y = 150$ MPa, and $\tau_{xy} = -55$ MPa. The *in-plane principal stresses* can be calculated from Equation (12.12):

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{70\text{ MPa} + 150\text{ MPa}}{2} \pm \sqrt{\left(\frac{70\text{ MPa} - 150\text{ MPa}}{2}\right)^2 + (-55\text{ MPa})^2}$$

$$= 178.0\text{ MPa},\ 42.0\text{ MPa}$$

The *maximum in-plane shear stress* can be computed from Equation (12.15):

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{\left(\frac{70\text{ MPa} - 150\text{ MPa}}{2}\right)^2 + (-55\text{ MPa})^2}$$

$$= \pm 68.0\text{ MPa}$$

On the planes of maximum in-plane shear stress, the normal stress is simply the *average normal stress*, as given by Equation (12.17):

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{70 \text{ MPa} + 150 \text{ MPa}}{2} = 110 \text{ MPa} = 110 \text{ MPa (T)}$$

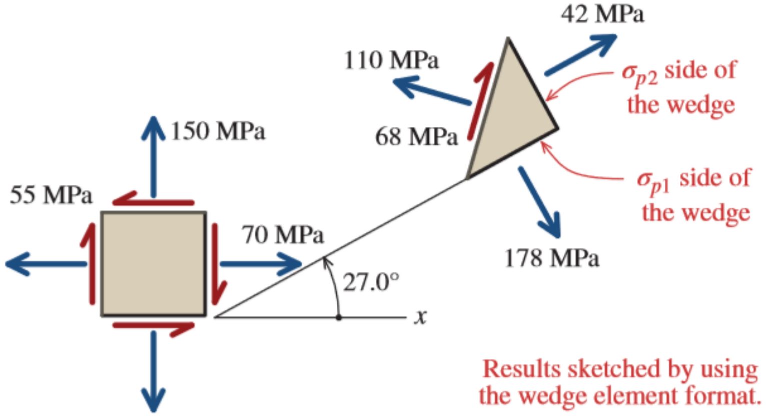
(b) The principal stresses and the maximum in-plane shear stress must be shown in an appropriate sketch. The angle θ_p indicates the orientation of one principal plane relative to the reference x face. From Equation (12.11),

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-55 \text{ MPa})}{70 \text{ MPa} - 150 \text{ MPa}} = \frac{-110}{-80}$$

$$\therefore \theta_p = 27.0°$$

The angle θ_p is positive; consequently, the angle is turned counterclockwise from the x axis. Since $\sigma_x - \sigma_y$ is negative, θ_p indicates the orientation of $\sigma_{p2} = 42.0$ MPa. The other principal stress, $\sigma_{p1} = 178.0$ MPa, acts on a perpendicular plane. The in-plane principal stresses are shown in the accompanying figure.

The maximum in-plane shear stress and the associated average normal stress are shown on the sloped face of the wedge, which is rotated 45° from the principal planes. Note that the arrow for τ_{max} starts on the σ_{p1} side of the wedge and points toward the σ_{p2} side.



Results sketched by using the wedge element format.