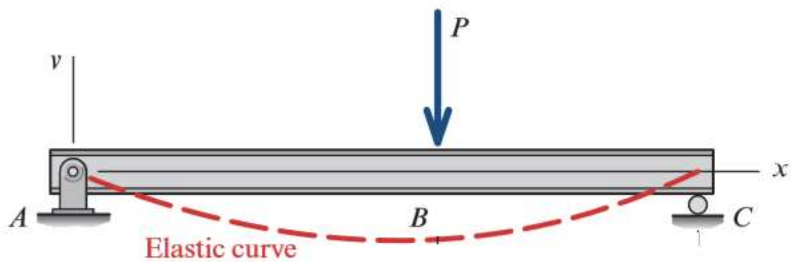
# Beam Deflection

## (Double integration Method)

Important relations between applied load and both normal and shear stresses developed in a beam were presented in previous chapters. However, a design is normally not complete until the deflection of the beam has been determined for its particular load. In building construction, excessive deflections can cause cracks in walls and ceilings. Doors and windows may not close properly.




**Steps to calculate the deflection in beams:**

1- Sketch the free-body diagram of the beam and establish the x and v coordinates.
2- Calculate the support reactions and write the moment equation as a function of the x coordinate.
3- Substitute the moment expression into the equation of the elastic curve and integrate once to obtain the slope. Integrate again to obtain the deflection in the beam.

$$\boxed{EI\,\theta = \int M dx + C_1}$$

$$\boxed{EIv = \iint M dx dx + C_1 x + C_2}$$

4- Using the boundary conditions, determine the integration constants and substitute them into the equations obtained in step 3 to obtain the slope and the deflection of the beam.

**Relationship of Derivatives:**

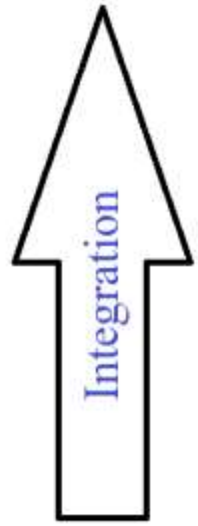

$$\text{Deflection} = v$$

$$\text{Slope} = \frac{dv}{dx} = \theta$$

$$\text{Moment } M = EI\frac{d^2v}{dx^2} \qquad (\text{for } EI \text{ constant})$$

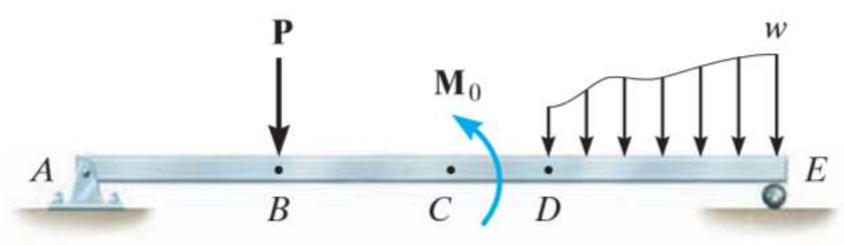
$$\text{Shear } V = \frac{dM}{dx} = EI\frac{d^3v}{dx^3} \qquad (\text{for } EI \text{ constant})$$

$$\text{Load } w = \frac{dV}{dx} = EI\frac{d^4v}{dx^4} \qquad (\text{for } EI \text{ constant})$$




**Discontinuity Functions (Macaulay Functions)**

In cases where a beam is subjected to a combination of distributed loads, concentrated loads, and moments, using the method of double integration to determine the deflections of such beams is really involving, since various segments of the beam are represented by several moment functions, and much computational efforts are required to find the constants of integration. Using the method of singularity function in such cases to determine deflections is comparatively easier and relatively quick. This method of analysis was first introduced by Macaulay in 1919, and it entails the use of one equation that contains a singularity or half-range function to describe the entire beam deflection curve. A singularity or half-range function is defined as follows:

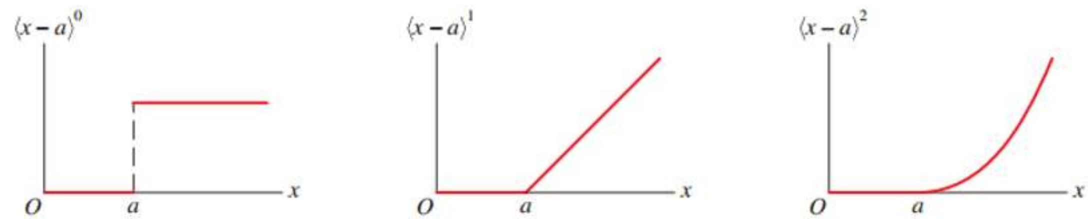


$$\langle x-a\rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases} \qquad n \geq 0$$

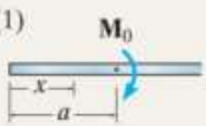
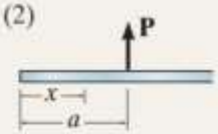
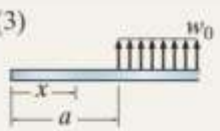
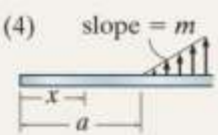
x= coordinate position of a point along the beam.

a= any location along the beam where discontinuity due to bending occurs.

n= the exponential values of the functions. for n = 0, n = 1, and n = 2 are plotted in the figure shown

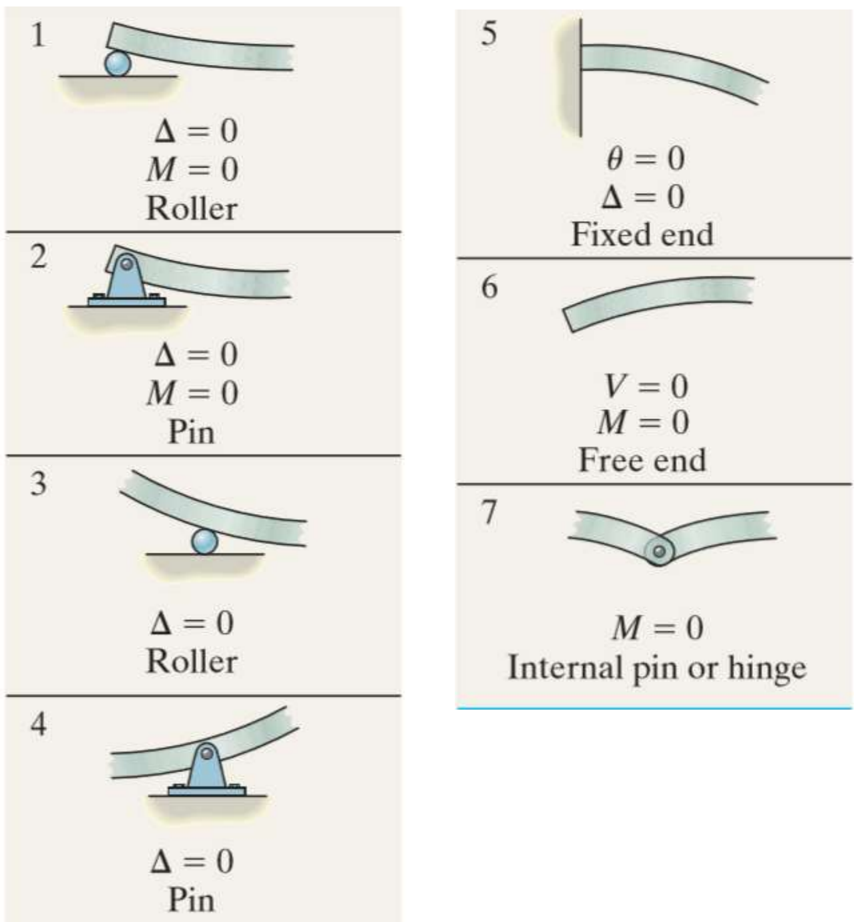



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

| Loading | Loading Function $w = w(x)$ | Shear $V = \int w(x)dx$ | Moment $M = \int V dx$ |
|---|---|---|---|
| (1)   | $w = M_0\langle x-a\rangle^{-2}$ | $V = M_0\langle x-a\rangle^{-1}$ | $M = M_0\langle x-a\rangle^0$ |
| (2)   | $w = P\langle x-a\rangle^{-1}$ | $V = P\langle x-a\rangle^0$ | $M = P\langle x-a\rangle^1$ |
| (3)   | $w = w_0\langle x-a\rangle^0$ | $V = w_0\langle x-a\rangle^1$ | $M = \frac{w_0}{2}\langle x-a\rangle^2$ |
| (4)   | $w = m\langle x-a\rangle^1$ | $V = \frac{m}{2}\langle x-a\rangle^2$ | $M = \frac{m}{6}\langle x-a\rangle^3$ |

**Boundary conditions**

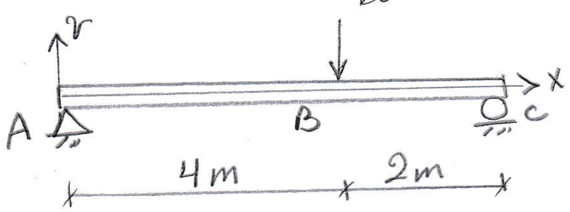
Boundary conditions are specific values of the deflection v or slope θ that are known at particular locations along the beam span. We use the boundary condition ( B.C) to determine the integration constants.



Ex 1 :-

A beam is loaded and supported as shown. Assume $EI$ is constant for the beam. Determine

1) the equation of the elastic curve in term $v$, $E$ and $I$
2) the rotation (slope) at points A, B and C
3) the deflection at point (B)
4) the max. deflection

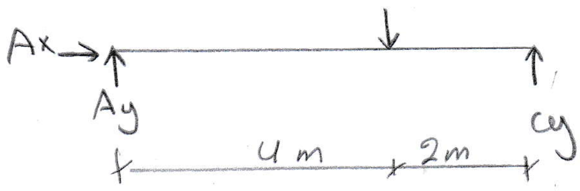



Solution :-

1) support reactions :-

$\sum F_y = 0 \; +\uparrow$

$A_y + C_y - 600 = 0$ — ①




$\sum M @ A = 0$ ↻+

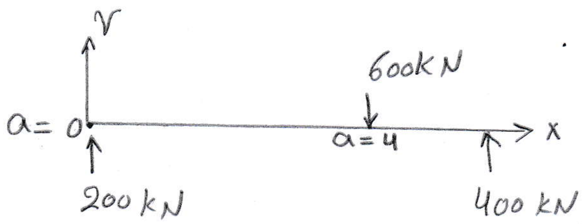
$600 * 4 - C_y * 6 = 0 \Rightarrow C_y = 400 \text{ kN} \uparrow$

$\therefore A_y + 400 - 600 = 0 \Rightarrow A_y = 200 \text{ kN} \uparrow$

$\sum F_x = 0 \; \xrightarrow{+} \quad \Rightarrow A_x = 0$

2) Moment Equation (using the table)

$$M_x = 200\langle x-0\rangle^1 - 600\langle x-4\rangle^1$$
$$= 200\langle x\rangle - 600\langle x-4\rangle$$



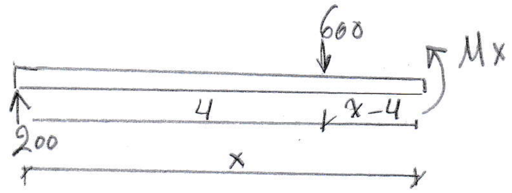

or by equilibrium

Fig(1)

$\sum M @ \text{section} = 0$ ↻+

$200\langle x\rangle - 600\langle x-4\rangle - M_x = 0$

$\therefore M_x = 200\langle x\rangle - 600\langle x-4\rangle$




Fig(2) :- F.B.D For a cut between B and C

3) Integration

$$EI\theta = \int EI \frac{d^2v}{dx} = \int M_x \, dx$$

Rotation

$$= \int 200\langle x\rangle - 600\langle x-4\rangle \, dx$$

$$= 200\frac{\langle x\rangle^2}{2} - 600\frac{\langle x-4\rangle^2}{2} + C_1 \quad — ①$$

$$EI\,v = \int EI\theta = \int 200\frac{\langle x\rangle^2}{2} - 600\frac{\langle x-4\rangle^2}{2} + C_1\,dx$$

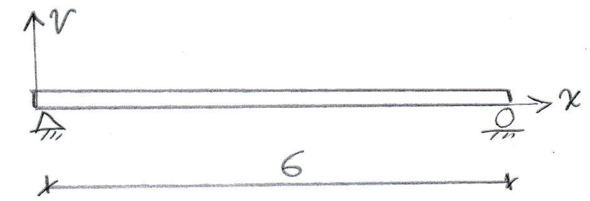
deflection

$$= 200\frac{\langle x\rangle^3}{6} - 600\frac{\langle x-4\rangle^3}{6} + C_1 x + C_2 \quad \text{---} (2)$$

4) Boundary Condition (B.C)

sub $B.C_1$ in equation (2)

$x=0$, $v=0$




at $x=0$, $v=0$, pin

$x=6$, $v=0$, roller

النتيجة داخل القوس سالب ← 0

$$EI(0) = \frac{200}{6}\langle 0\rangle - \frac{600}{6}\langle \cancel{0-4}\rangle^3 + C_1(0) + C_2$$

$$\therefore\ 0 = 0 + 100 * 0 + 0 + C_2 \Rightarrow C_2 = 0$$

$B.C_2$ at $x=6$, $v=0$ sub in equation (2)

$$EI(0) = \frac{200}{6}\langle 6\rangle^3 - \frac{600}{6}\langle 6-4\rangle^3 + C_1 * 6 + \cancel{C_2}^{\,0}$$

$$0 = 33.33 * (6)^3 - 100 * (2)^3 + 6C_1$$

$$\therefore\ C_1 = -1066.66$$

1) Equation of the elastic curve is

$$EI\,v = 33.33\langle x\rangle^3 - 100\langle x-4\rangle^3 - 1066.66x$$

2) The rotation Equation is

$$EI\theta = \frac{200}{2}\langle x\rangle^2 - \frac{600}{2}\langle x-4\rangle^2 - 1066.66$$

$$= 100\langle x\rangle^2 - 300\langle x-4\rangle^2 - 1066.66$$

– Rotation at A, sub $x=0$

تهمل

$$EI\theta_A = 100\langle 0\rangle^2 - 300\langle \cancel{0-4}\rangle^2 - 1066.66$$

$$EI\theta_A = 100 * (0) - 300 * 0 - 1066.66$$

$$EI\theta_A = -1066.66$$

$$\therefore\ \theta_A = \frac{-1066.66}{EI}$$

- Rotation at B , sub $x=4$

$$EI\,\theta_B = 100\langle 4\rangle^2 - 300\langle 4-4\rangle^2 - 1066.66$$

$$\theta_B = \frac{-533.33}{EI}$$

- Rotation at C , sub $x=6$

$$EI\,\theta_C = 100\langle 6\rangle^2 - 300\langle 6-4\rangle^2 - 1066.66$$

$$\theta_C = \frac{1333.33}{EI}$$

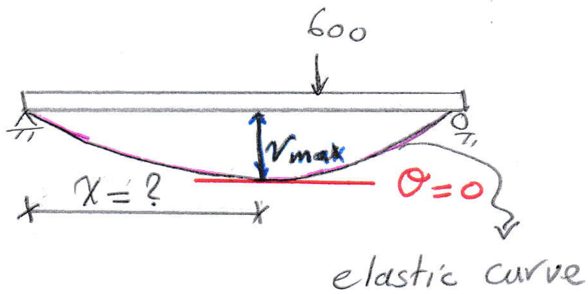
3) Deflection at B , sub $x=4$ at deflection equation

$$EI\,v = 33.33\langle x\rangle^3 - 100\langle x-4\rangle^3 - 1066.66x$$

$$= 33.33\langle 4\rangle^3 - 100\langle 4-4\rangle^3 - 1066.66 * 6$$

$$\therefore v_B = -\frac{2133.33}{EI}$$

4)




Note at $V_{max}$, $\theta = 0$

$$EI\,\theta = 100\langle x\rangle^2 - 300\langle x-4\rangle^2 - 1066.66$$

assume $V_{max}$ at $0 < x \leq 4$

$$0 = 100\langle x\rangle^2 - 300\langle \cancel{x-4}\rangle^2 - 1066.66$$

$$0 = 100x^2 - 1066.66$$

$$x^2 = \frac{1066.66}{100} \Rightarrow x = 3.26\ m \quad \because 0 < x < 4 \Rightarrow o.k$$

$\therefore$

$$EI\,V_{max} = 33.33\langle x\rangle^3 - 100\langle x-4\rangle^3 - 1066.66x$$

$$= 33.33\langle 3.26\rangle^3 - 100\langle \cancel{3.26-4}\rangle^3 - 1066.66 * 3.26$$

$$\therefore V_{max} = \frac{-2322.47}{EI}$$

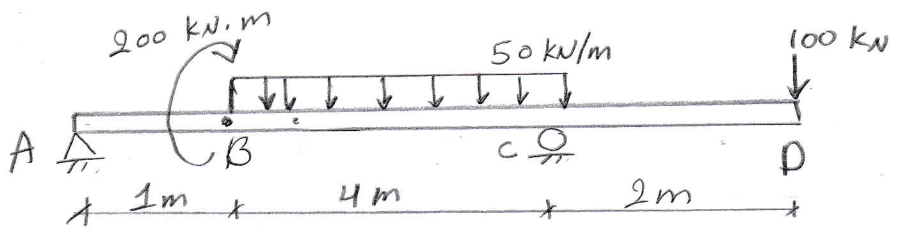
Ex (2):-

For the beam and Loading Shown in the figure Compute

1) the slope of the beam at (C)

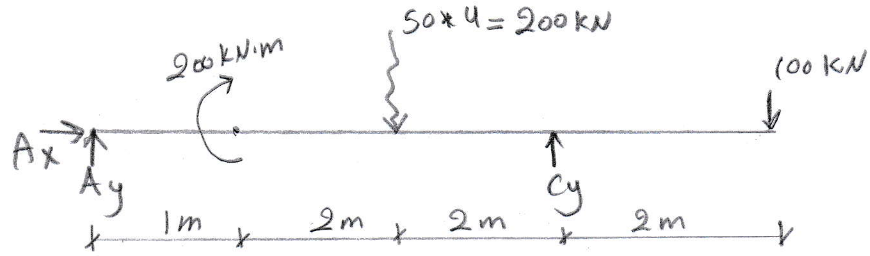
2) the deflection of the beam at (B)

Assume a Constant value of $EI = 5 * 10^{13}$ N.mm²




Solution :-

1) support reactions.




$\Sigma M @ A = 0$ +↻

$200 + 200 * 3 - Cy * 5 + 100 * 7 = 0$

$\Rightarrow Cy = 300$ kN ↑

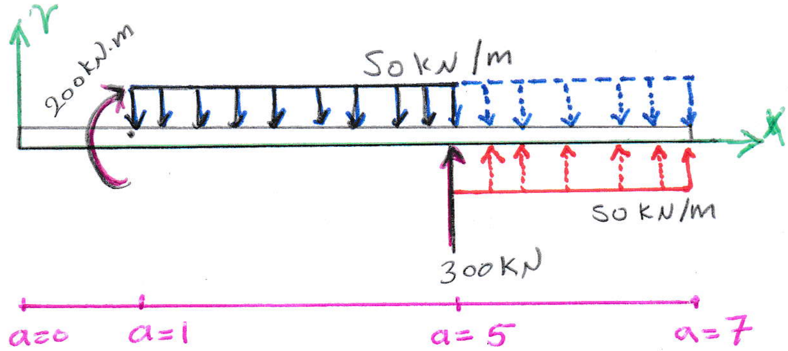
$\Sigma Fy = 0$ +↑

$Ay - 200 + 300 - 100 = 0$

$Ay = 0$

$\Sigma Fx = 0$ $\xrightarrow{+}$ $\Rightarrow Ax = 0$

2) Moment Equation :-




$$M_x = M_o \langle x-1\rangle^0 - \frac{w_o}{2}\langle x-1\rangle^2 + \frac{w_o}{2}\langle x-5\rangle^2 + P\langle x-5\rangle^1$$

$$= 200\langle x-1\rangle^0 - \frac{50}{2}\langle x-1\rangle^2 + \frac{50}{2}\langle x-5\rangle^2 + 300\langle x-5\rangle^1$$

3) Integration :-

$$EI\theta = \int EI \frac{dv}{dx} = \int M_x\, dx$$

$$= \int 200\langle x-1\rangle^0 - \frac{50}{2}\langle x-1\rangle^2 + \frac{50}{2}\langle x-5\rangle^2 + 300\langle x-5\rangle$$

$$= 200\langle x-1\rangle^1 - \frac{50}{6}\langle x-1\rangle^3 + \frac{50}{6}\langle x-5\rangle^3 + \frac{300}{2}\langle x-5\rangle^2 + C_1$$

$$EI\,v = \int EI\theta = \iint M_x\,dx$$

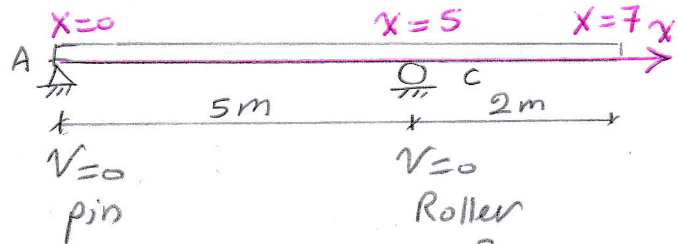
$$= \int 200\langle x-1\rangle^1 - \frac{50}{6}\langle x-1\rangle^3 + \frac{50}{6}\langle x-5\rangle^3 + \frac{300}{2}\langle x-5\rangle^2 + C_1$$

$$= \frac{200}{2}\langle x-1\rangle^2 - \frac{50}{24}\langle x-1\rangle^4 + \frac{50}{24}\langle x-5\rangle^4 + \frac{300}{6}\langle x-5\rangle^3 + C_1 x + C_2 \qquad (2)$$

4) Boundary Condition (B.C)

B.C 1, at $x=0$, $V=0$

sub in equation (2)




$$EI\,v = 100\langle x-1\rangle^2 - \frac{50}{24}\langle x-1\rangle^4 + \frac{50}{24}\langle x-5\rangle^4 + 50\langle x-5\rangle^3 + C_1 X + C_2$$

$$0 = 100\cancel{\langle 0-1\rangle^2}^{\,0} - \frac{50}{24}\cancel{\langle 0-1\rangle^4}^{\,0} + \frac{50}{24}\cancel{\langle 0-5\rangle^4}^{\,0} + 50\cancel{\langle 0-5\rangle^3}^{\,0} + C_1(0) + C_2$$

$C_2 = 0$

B.C 2, $X=5$, $V=0$

$$0 = 100\langle 5-1\rangle^2 - \frac{50}{24}\langle 5-1\rangle^4 + \frac{50}{24}\langle 5-5\rangle^4 + 50\langle 5-5\rangle^3 + C_1 * 5$$

$$= 100\langle 4\rangle^2 - \frac{50}{24}\langle 4\rangle^4 + \frac{50}{24} * 0 + 50 * 0 + 5C_1$$

$C_1 = 320$

$\therefore$ $V$ and $\theta$ equations are

$$EI\theta = 200\langle X-1\rangle - \frac{50}{6}\langle X-1\rangle^3 + \frac{50}{6}\langle X-5\rangle^3 + \frac{300}{2}\langle X-5\rangle^2 + 320$$

$$EI\,v = 100\langle X-1\rangle^2 - \frac{50}{24}\langle X-1\rangle^4 + \frac{50}{24}\langle X-5\rangle^4 + 50\langle X-5\rangle^3 + 320X$$

1) slope at C, $X=5$

$$EI\theta_c = 200\langle 5-1\rangle - \frac{50}{6}\langle 5-1\rangle^3 + \frac{50}{6}\cancel{\langle 5-5\rangle^3}^{\,0} + \frac{200}{2}\cancel{\langle 5-5\rangle^2}^{\,0} + 320$$

$$= 200\langle 4\rangle - \frac{50}{6}\langle 4\rangle^3 + 320$$

$$\theta_c = \frac{586.66}{EI}$$

$$= \frac{586.66}{50*10^{13}*10^{-6}*10^{-3}} = 0.0117\ \text{rad.}$$

$mm^2 \Rightarrow m^2$ $\quad$ $N \Rightarrow kN$

2) deflection at (B)

$$EI\,v = 100\langle x-1\rangle^2 + \frac{50}{24}\langle x-1\rangle^4 - \frac{50}{24}\langle x-5\rangle^4 + 50\langle x-5\rangle^3 + 320x$$

$$x_B = 1$$

$$EI\,v = 100\langle 1-1\rangle^2 + \frac{50}{24}\langle 1-1\rangle^4 - \frac{50}{24}\langle \cancel{1-5}\rangle^4 + 50\langle \cancel{1-5}\rangle^3 + 320*1$$

$$= 100*0 + \frac{50}{24}*0 + 320$$

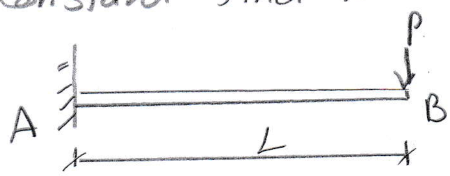
$$v = \frac{320}{EI} = \frac{320}{50*10^{13}*10^{-6}*10^{-3}} = 0.0064\ m$$

$$= 6.4\ mm$$

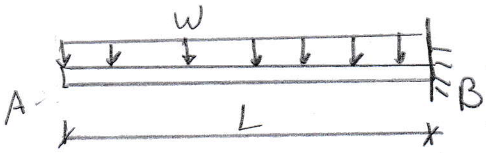
EX(3):-

A cantilever beam is loaded as shown. Assume EI is constant find the max. defletion.

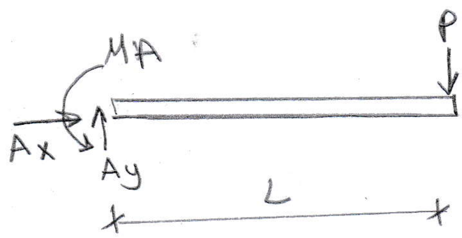
a)




b)




① Reaction




$$\sum F_x = 0 \Rightarrow A_x = 0$$

$$\sum F_y = 0 \uparrow + \Rightarrow A_y - P = 0 \Rightarrow A_y = P$$

$$\sum M@A = 0 \,+\!\!\curvearrowright \Rightarrow$$

$$P*L - M_A = 0 \Rightarrow M_A = PL$$

② Moment Equation

$$M_x = -M_A\langle x-0\rangle^0 + A_y\langle x-0\rangle^1$$

$$M_x = -PL\langle x\rangle^0 + P\langle x\rangle$$

③ Integration

$$EI\,\theta = \int M_x = \int -PL\langle x\rangle^0 + P\langle x\rangle\,dx$$

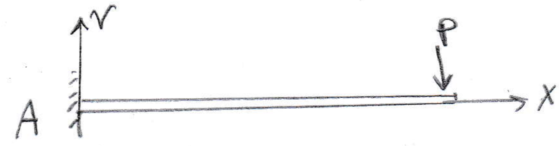
$$= -PL\langle x\rangle + P\frac{\langle x\rangle^2}{2} + C_1$$

$$EI\,v = \int EI\,\theta = \int -PL\langle x\rangle + P\frac{\langle x\rangle^2}{2} + C_1$$

$$= -\frac{PL}{2}\langle x\rangle^2 + \frac{P}{6}\langle x\rangle^3 + C_1 x + C_2$$

4) Boundary Condition B.C

at $X=0,\ \theta=0,\ V=0$




$X=0$
$\theta=0$
$V=0$ Fixed end

$$\therefore EI\theta = -PL\langle x\rangle + P\frac{\langle x\rangle^2}{2} + C_1$$

$$0 = -PL\langle 0\rangle + \frac{P}{2}\langle 0\rangle^2 + C_1$$

$$C_1 = 0$$

at $x=0,\ V=0$

$$EIv = \frac{-PL}{2}\langle x\rangle^2 + \frac{P}{6}\langle x\rangle^3 + \cancel{C_1 X}^{\,0} + C_2$$

$$0 = \frac{-PL}{2}\langle 0\rangle^2 + \frac{P}{6}\langle 0\rangle^3 + C_2 \Rightarrow C_2 = 0$$

$$\therefore EI\theta = -PL\langle x\rangle + \frac{P}{2}\langle x\rangle^2$$

$$EIv = \frac{-PL}{2}\langle x\rangle^2 + \frac{P}{6}\langle x\rangle^3$$

at $x=L$, $V_{max}$

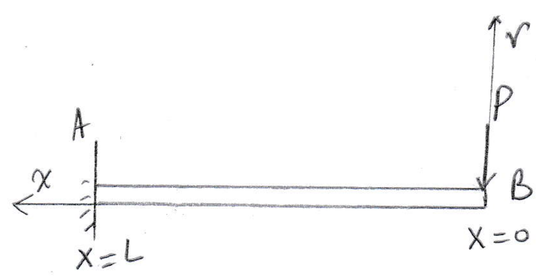
$$EI\,V_{max} = \frac{-PL}{2}\langle L\rangle^2 + \frac{P}{6}\langle L\rangle^3$$

$$= \frac{-PL^3}{2} + \frac{P}{6}L^3$$

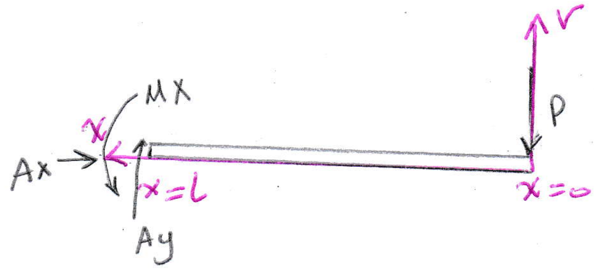
$$= \frac{-3PL^3}{6} + \frac{PL^3}{6}$$

$$EI\,V_{max} = \frac{-2PL^3}{6} = -\frac{PL^3}{3}$$

$$V_{max} = -\frac{PL^3}{3EI}$$




Or
start from the right side at point B. $\Rightarrow$ No need to fined the reaction.




2) Moment Equation

$$M_X = -P\langle X-0\rangle^1$$

$$= -P\langle X\rangle$$

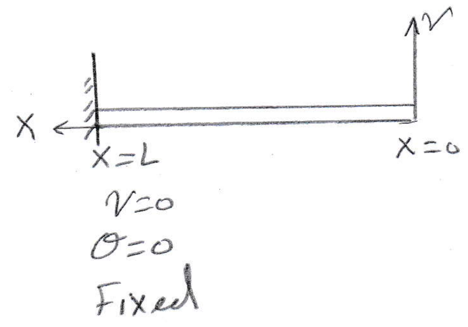
3) Integration

$$EI\theta = \int M_X = \int -P\langle x\rangle\,dx = \frac{-P}{2}\langle x\rangle^2 + C_1$$

$$EIv = \int EI\theta = \int -\frac{P}{2}\langle x\rangle^2 + C_1 = \frac{-P}{6}\langle x\rangle^3 + C_1 X + C_2$$

Boundary Condition (B.C)

at $X = L$, $V = 0$, $\theta = 0$




$$EI\theta = \frac{-P}{2} \langle X \rangle^2 + C_1$$

$$0 = \frac{-P}{2} \langle L \rangle^2 + C_1$$

$$0 = -\frac{P}{2} L^2 + C_1 \Rightarrow C_1 = \frac{PL^2}{2}$$

$$EIV = \frac{-P}{6} \langle X \rangle^3 + C_1 X + C_2$$

$$0 = \frac{-P}{6} \langle L \rangle^3 + \frac{PL^2}{2} * L + C_2$$

$$0 = \frac{-PL^3}{6} + \frac{PL^3}{2} + C_2$$

$$0 = \frac{-PL^3 + 3PL^3}{6} + C_2$$

$$0 = \frac{2PL^3}{6} + C_2 \Rightarrow C_2 = \frac{-PL^3}{3}$$

$$EIV = \frac{-P}{6} \langle x \rangle^3 + \frac{PL^2}{2} x - \frac{PL^3}{3}$$

at $X = 0$, $V = V_{max}$

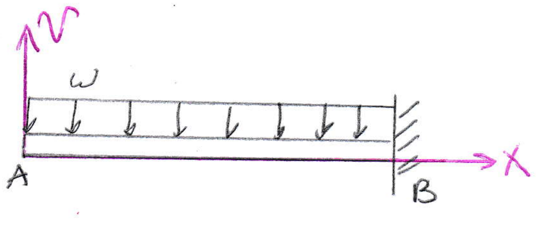
$$EI V_{max} = \frac{-P}{6} \langle 0 \rangle^3 + \frac{PL^2}{2} * 0 - \frac{PL^3}{3}$$

$$EI V_{max} = -\frac{PL^3}{3}$$

$$\therefore V_{max} = \frac{-PL^3}{3EI}$$

B)

start from the left side at A




1) Moment equation

$$M_X = -\frac{W_0}{2} \langle X - a \rangle^2$$

$$= -\frac{W_0}{2} \langle X - 0 \rangle^2$$

$$= -\frac{W}{2} \langle X \rangle^2$$

② Integration

$$EI\theta = \int M_x \, dx$$

$$= \int -\frac{w}{2} \langle x \rangle^2$$

$$= -\frac{w}{6} \langle x \rangle^3 + C_1 \quad — ①$$

$$EIv = \int EI\theta = \int -\frac{w}{6} \langle x \rangle^3 + C_1$$

$$= -\frac{w}{24} \langle x \rangle^4 + C_1 x + C_2 \quad — ②$$

③ Boundary Condition :

at $x = L$, $v = 0$, $\theta = 0$

$$EI\theta = \frac{-w}{6} \langle x \rangle^3 + C_1$$

$$0 = -\frac{w}{6} \langle L \rangle^3 + C_1$$

$$= \frac{-w}{6} L^3 + C_1 \Rightarrow C_1 = \frac{wL^3}{6}$$

at $x = l$, $v = 0$

$$EIv = \frac{-w}{24} \langle x \rangle^4 + C_1 x + C_2$$

$$0 = \frac{-w}{24} \langle L \rangle^4 + \frac{wL^3}{6} \cdot L + C_2$$

$$0 = \frac{-wL^4}{24} + \frac{wL^4}{6} + C_2$$

$$0 = \frac{-wL^4 + 4wL^4}{24} + C_2$$

$$0 = \frac{3wL^4}{24} + C_2 \Rightarrow C_2 = -\frac{3wL^4}{24}$$

$$\therefore EIv = \frac{-w}{24} \langle x \rangle^4 + \frac{wL^3}{6} x - \frac{3wL^4}{24}$$

at $x = 0$, $v = v_{max}$

$$EI v_{max} = \frac{-w}{24} \langle 0 \rangle^4 + \frac{wL^3}{6}(0) - \frac{3wL^4}{24}$$

$$v_{max} = -\frac{3wL^4}{24EI}$$

$$v_{max} = -\frac{wL^4}{8EI}$$