Cumulative curves

Overview

This lecture discusses cumulative curves, also known as cumulative flow curves. It is show how traffic characteristics can be derived from these cumulative curves.

Definition

The function$N_x(t)$ is defined as the number of vehicles that have passed a point x at time t and is only used for traffic into one direction. Hence, this function only increases over time. Strictly speaking, this function is a step function increasing by one every time a vehicle passes. However, for larger time spans and higher flow rates, the function is often smoothed into a continuous differentiable function. The increase rate of this function equals the flow:

$$\frac{dn}{dt} = q \qquad (1)$$

Hence from the flow, we can construct the cumulative curve:

$$N = \int q dt \qquad (2)$$

This gives one degree of freedom, the value to start at. This can be chosen freely, or should be adapted to cumulative curves for other locations.

Vertical queuing model

A vertical queuing model is a model which assumes an unlimited inflow and an outflow which is restricted to capacity. The vehicles which cannot pass the bottleneck are stacked “vertically” and do not occupy any space. Figure 1 illustrates this principle.

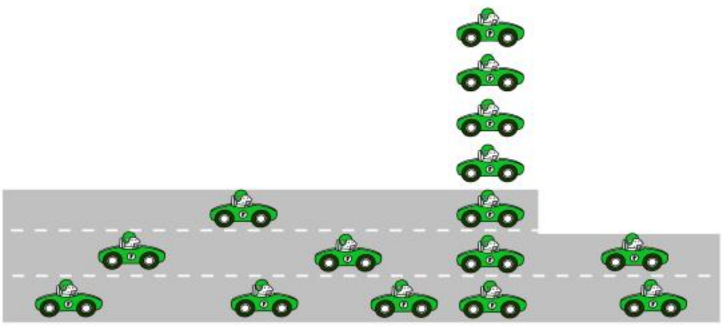


Figure 1: Illustration of a vertical queue.

Indicated here as t + 1/2, the number of vehicles in the stack is updated based on the flows q. Then, the stack provide the basis for the flows in the next time step. The stack starts at zero. Then, for each time step first the inflow to the stack is computed.

$$q_{n,t} = D \qquad 3$$

And the stack is updated accordingly, going to an intermediate state at time step t+1/2. This intermediate step is the number of vehicles in the queue if there were no outflow, so the original queue plus the inflow:

$$S_{t+1/2} = S_t + q_{in}\Delta t \qquad 4$$

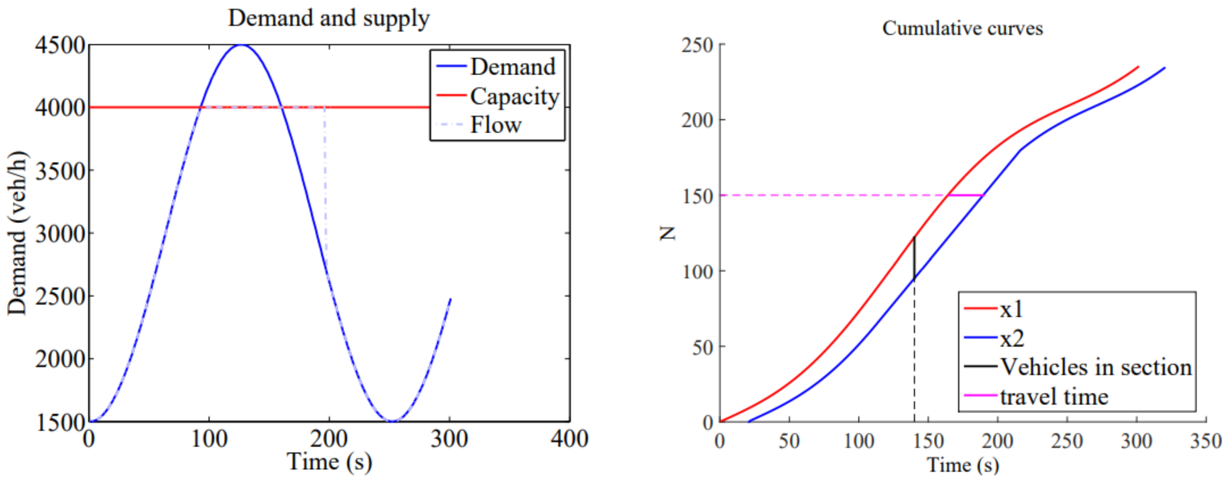
Then, the outflow out of the stack (q_{out}) is the minimum of the number of vehicles in this intermediate queue and the maximum outflow determined by the capacity C:

$$q_{out} = min\{C\Delta t, S_{i+1/2}\} \qquad 5$$

The stack after the time step is then computed as follows:

$$S_{i+1} = S_{i+1/2} - q_{out}\Delta t = S_i + (q_{ni,i} - q_{out,i})\Delta t \qquad 6$$

Let us consider a situation as depicted in Figure 1, and we are interested in the delays due to the bottleneck with a constant capacity of 4000 veh/h. The demand curve is plotted in Figure 2(a). The flows are determined using the vertical queuing model. The flows are also show in Figure 2(a). Note that the area between the flow and demand curve where the demand is higher than the flow (between approximately 90 to 160 seconds), is the same as the area between the curves where the flow is higher than the demand (between approximately 160 and 200 seconds). The reasoning is that the area represents a number of vehicles (a flow times a time). From 90 to 160 seconds the demand is higher than the flow, i.e., the inflow is higher than the outflow. The area represents the number of vehicles that cannot pass the bottleneck, and hence the number of queued vehicles. From 160 seconds, the outflow of the queue is larger than the inflow. That area represents the number of vehicles that has left the queue, and cannot be larger than the number of vehicles queued. Moreover, the flows remains at capacity until the stack is empty, so both areas must be equal.



(a) Demand and supply (b) Cumulative curves

Figure 2: Demand and cumulative curves.

Travel times, densities and delays

This section explains how travel times and delays. Delay can be computed using cumulative curves. Note that this methodology does not take spillback effects into account. If one requires this to be accounted for, please refer to shockwave theory.

Construction of cumulative curves

The cumulative curves for the above situation is shown in Figure 2(b). The curves show the flows as determined by the vertical queuing model. The inflow we hence use equation 3 and for the outflow we use 5; for both, the cumulative curves are constructed using equation 2.

Travel times, number of vehicles in the section

A black line is drawn at $t = 140s$ in Figure 2(b). The figure shows by intersection of this line with the graphs how many vehicles have passed the upstream point $x1$ and how many vehicles have passed the downstream point $x2$. Consequently, it can be determined how many vehicles are in the section between $x1$ and $x2$. This number can also be found in the graph, by taking the difference between the inflow and the outflow at that moment. This is indicated in the graph by the bold vertical black line.
Similarly, we can take a horizontal line; consider for instance the line at $N = 150$. The intersection with the inflow line shows when the 150th vehicle enters the section, and the intersection with the outflow line shows when this vehicle leaves the section. So, the horizontal distance between the two lines is the travel time of the 150th vehicle. At times where the demand is lower than the capacity, the vehicles have a free flow travel time. So without congestion, the outflow curve is the inflow curve which is translated to the right by the free flow travel time.

The vertical distance is the number of vehicles in the section (ΔN) at a moment t. In a time period dt this adds $\Delta N dt$ to the total travel time (each vehicle contributes dt). To get the total travel time, we integrate over all infinitesimal intervals dt:

$$tt = \int \Delta N dt \quad (7)$$

The horizontal distance between the two lines is the travel time for one vehicle, and vertically we find the number of vehicles. Adding up the travel times for all vehicles gives the total travel time:

$$tt = \sum_i tt_i \quad (8)$$

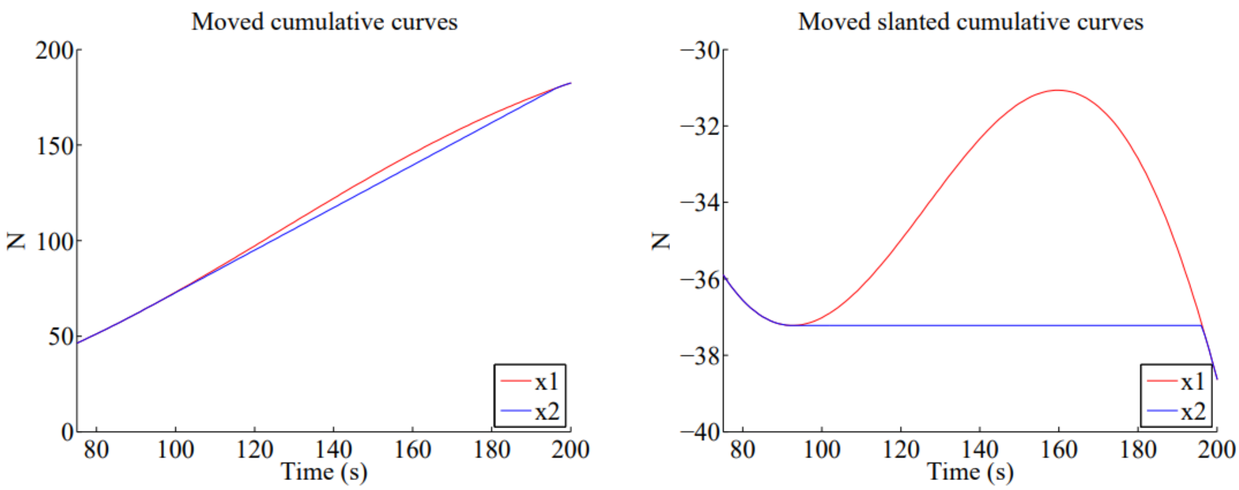
In a continuous approach, this changes into:

$$tt = \int tt_i di \quad (9)$$

Both calculation methods lead to the same interpretation: the total time spent can be determined by the area between the inflow and outflow curve.

Delays

Delays for a vehicle are the extra time it needs compared to the free flow travel time. So to calculate delay, one subtracts the free flow travel time from the actual travel time. To subtract the free flow travel time from the travel time, we can graphically move the outflow curve to the left, as is shown in Figure 3(a). For illustration purposes, the figure is zoomed at Figure 3(b). The figure shows that if the travel time equals the free flow travel time, both curves are the same, leading to 0 delay.



(a) Moved cumulative curves

(b) Slanted cumulative curves

Figure 3: Determining the delay and the flows from cumulative curves.

Similar to how the cumulative curves can be used to determine the travel time, the moved cumulative curves can be used to determine the delay. The delay for an individual vehicle can be found by the horizontal distance between the two lines. The vertical distance between the two lines can be interpreted as the number of vehicles queuing. The total delay is the area between the two lines:

$$D = \int tt_i - tt_{free\ flow}\, di \qquad 10$$

This is the area between the two lines. If we define Nqueue as the number of vehicles in the queue at moment t, we can also rewrite the total delay as:

$$D = \int N_{queue}(t)\, dt \qquad 11$$

Slanted cumulative curves

Slanted cumulative curves or oblique cumulative curves is a very powerful yet simple tool to analyse traffic streams. These are cumulative curves which are offset by a constant flow:

$$\bar{N} = \int q - q_0 dt - \int q_0 dt = \int q dt - \int q_0 dt \qquad 12$$

This means that differences with the freely chosen reference flow q_0 are amplified: in fact, only the difference with the reference flow are counted. The best choice for the reference flow q_0 is a capacity flow.
Figure 3(b) shows the slanted cumulative curves for the same situation as in Figure 3(a). The figure is offset by q_0 = 4000 veh/h. Because the demand is initially lower than the capacity, $\tilde{N}$reaches a negative value. From the moment outflow equals capacity, the slanted cumulative outflow curve is constant. Since the demand is higher than the capacity, this increases. At the moment both curves intersect again, the queue is dissolved.
The vertical distance between the two lines still shows the length of the queue, N_{queue}. That means that Equation10 still can be applied in the same way for the slanted cumulative curves, and the delay is the area between the two lines.
Slanted cumulative curves are also particularly useful to determine capacity, and to study changes of capacity, for instance the capacity drop. In that case, for one detector the slanted cumulative curves are drawn. By a change of the slope of the line a change of capacity is detected.

Exercises

Consider a road with a demand of:

$$q_{\mathrm{in}} = \begin{cases} 3600\mathrm{v/h} & \text{for } t < 1\mathrm{h} \\ 5000\mathrm{v/h} & \text{for } 1\mathrm{h} < t < 1.5\mathrm{h} \\ 2000\mathrm{v/h} & \text{for t} > 1.5\mathrm{h} \end{cases}$$

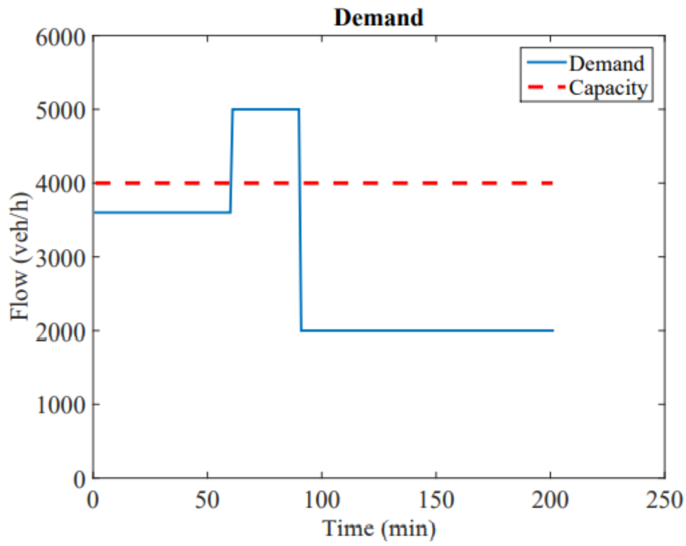


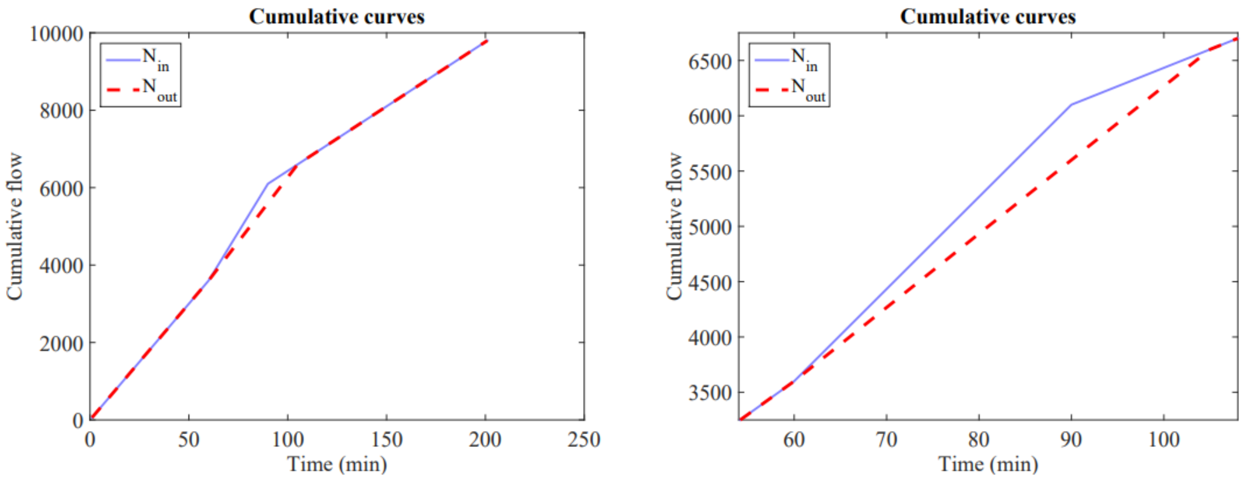
Figure 4: Demand and capacity.

The capacity of the road is 4000 veh/h. A graph of the demand and capacity is shown in Figure 4.
1. Construct the (translated=moved) cumulative curves
2. Calculate the first vehicle which encounters delay (N)
3. Calculate the time at which the delay is largest
4. Calculate the maximum number of vehicles in the queue
5. Calculate the vehicle number (N) with the largest delay
6. Calculate the delay this vehicle encounters (in h, or min)
7. Calculate the time the queue is solved
8. Calculate the last vehicle (N) which encounters delay
9. Calculate the total delay (veh-h)
10. Calculate the average delay of the vehicles which are delayed (h)

Answer:

1. For the cumulative curves, an inflow and an outflow curve needs to be constructed; both increase. For the inflow curve, the slope is equal to the demand. For the outflow

curve, the slope is restricted to the capacity. During the first hour, the demand is lower than the capacity, hence the outflow is equal to the demand. From t=1h, the inflow exceeds the capacity and the outflow will be equal to the demand. The cumulative curve hence increases with a slope equal to the capacity. As long as there remains a queue, i.e. the cumulative inflow is higher than the outflow, the outflow remains at capacity. The outflow remains hence increasing with a slope equal to the capacity until it intersects with the cumulative inflow. Then, the outflow follows the inflow: see Figure 5(a) and for a more detailed Figure 5(b).



(a) Overview of the cumulative curves (b) More detailed view of the cumulative curves

Figure 5: Cumulative curves for the example.

2. The first vehicle which encounters delay (N) Delays as soon as q>C: so after 1h at 3600 v/h = 3600 vehicles.
3. The time at which the delay is largest: A queue builds up as long as q>C, so up to 1.5 h. At that moment, the delay is largest.
4. The maximum number of vehicles in the queue: 0.5 h after the start of the queue, 0.5*5000=2500 veh entered the queue, and 0.5*4000=2000 left: so 500 vehicles are in the queue at t=0.5h.
5. The vehicle number (N) with the largest delay: N (1.5h) =3600+0.5*5000 = 6100.
6. The delay this vehicle encounters (in h, or min): It is the 2500th vehicle after t=1h. The delay is the horizontal delay between the entry and exit curve. It takes at capacity 2500/4000 = 37.5 min to serve 2500 vehicles. It entered 0.5 hours = 30 min after t=1, so the delay is 7.5 min.
7. The time the queue is solved: This is the time point that the inflow and outflow curves intersect again. 500 vehicles is the maximum queue length, and it reduces with 4000-2000=2000veh/h. So 500/2000=15 minutes after the time that q.
8. The last vehicle (N) which encounters delay. This is the vehicle number at the moment the inflow and outflow curves meet again. 15 minutes after the vehicle number with the largest delay: 6100+0.25*2000 = 6600 veh.
9. The total delay. This is the area of the triangle between inflow and outflow curve. This area is computed by 0.5 * height * base = 0.5 * 500 * (30+15)/60 = 187, 5 veh-h. Note

that here we use a generalized equation for the area of a triangle. Indeed, we transform the triangle to a triangle with a base that has the same width, and the height which is the same for all times (i.e., we skew it). The height of this triangle is 500 vehicles (the largest distance between the lines) and the width is 45 minutes.

10. The average delay of the vehicles which are delayed (h) 187,5 veh-h/ (6600-3600) veh = 0,0625 h = 3,75 min.