Relationships of traffic variables

Overview
Lecture 1 defined the variables and their definition. This lecture will discuss the relationship between these variables. First of all the mathematically required relationships will show, and then typical properties of traffic in equilibrium will discussed. Also discusses these relationships in the light of drivers, and expands this to non-equilibrium conditions. Finally, gives attention to the moving observer.

Fundamental relationship
In microscopic view, it is obvious that the headway (h), the spacing (s) and the speed (v) are related. The headway times the speed will give the distance covered in this time, which is the spacing. It thus suffices to know two of the three basic variables to calculate the third one.

$$s = hv \qquad (1)$$

Since headways and spacing's have macroscopic counterparts, there is a macroscopic equivalent for this relationship. After reordering, Equation 1 reads:

$$\frac{1}{h} = \frac{1}{s}v \qquad (2)$$

The macroscopic equivalent of this relationship is the average of this equation. Remembering that $q = \frac{1}{\langle h \rangle}$ and $k = \frac{1}{\langle s \rangle}$, we get:

$$\boxed{q = ku} \qquad (3)$$

This equation shows that the flow q is proportional with both the speed u and the density k. Intuitively, this makes sense because when the whole traffic stream moves twice as fast if the flow doubles. Similarly, if – at original speed – the density doubles, the flow doubles as well.

Table 1 summarizes the variables and their relationships.

Microscopic	Macroscopic
s	$k = \frac{1}{\langle s \rangle}$
h	$q = \frac{1}{\langle h \rangle}$
v	$u = \frac{1}{\langle v \rangle}$
$s = hv$	$q = ku$

Fundamental diagram

If two of the three macroscopic traffic flow variables are known, the third one can be calculated. This section will show that there is another relationship. In fact, there is an equilibrium relationship between the speed and the density. First, a qualitative understanding will be given, after that the effect will be shown for various couples of variables. Also, different shapes of the supposed relationship will be shown in next section.

Qualitative understanding of the shape

Let us, for the sake of argument, consider the relationship between density and flow. And let us furthermore start considering the most extreme cases. First, the case that there is no vehicle on the road. Since the density is 0, the flow is 0, according to Equation 3. In the other extreme case the density on the road is very high, and the speed is 0. Using again Equation 3 we find also for this case a flow of 0. In between, there are traffic states for which the traffic flow is larger than zero. Assuming a continuous relationship between the speed and the density (which is not necessarily true) there will be a curve relating the two points at flow 0. This is indicated in Figure 1(c).

This relationship is being observed in traffic. However, it is important to note that this is not a causal relationship. One might argue that due to the low speed, drivers will drive closer together. Alternatively, one might argue that due to the close spacing, drivers need to slow down.

(a) One extreme: an empty road (b) One extreme: an jammed road (c) Basic shape of the fundamental diagram

Figure 1: The extreme situations and an idea for the fundamental diagram.

Traffic state

We can define a traffic state by its density, flow and speed. Using Equation 3, we only need to specify two of the variables. Furthermore, using the fundamental diagram, one can be sufficient. It is required that the specified variable then has a unique relationship to the others. For instance, judged by Figure 1(c), specifying the density will lead to a unique flow, and a unique speed (using Equation 3, and thus a unique traffic state). However, specifying the flow (at any value between 0 and the capacity) will lead to two possible densities, two possible speeds, and hence two possible traffic states. The speed of the traffic can be derived using the Equation 3:

$$u = \frac{q}{k} \qquad 4$$

For a traffic state in the flow density plane, we can draw a line from the traffic state to the origin. The slope of this line is q/k. So the speed of the traffic can be found by the slope of a line connecting the origin to the traffic state in the flow density plane. The free flow speed can be found by the slope of the fundamental diagram at k=0, i.e. the derivative of the fundamental diagram in the origin.

Important points

The most important aspect of the fundamental diagram for practitioners is the capacity. This is the maximum flow which can be maintained for a while at a road. The same word is also used for the traffic state at which maximum flow is obtained. This point is found at the top of the fundamental diagram. Since we know that the flow can be determined from the headway, we can estimate a value for the capacity if we consider the minimum headway. For drivers on a motorway, the minimum headway is approximately 1.5 to 2 seconds, so we find a typical capacity value of 1/2 to 1/1.5 vehicles per second.
If we convert this to vehicles per hour, we find (there are 3600 seconds in an hour) 3600/2=1800 to 3600/1.5 = 2400 vehicles per hour.

The density for this point is called the critical density, and the according speed the critical speed. The capacity is found when the average headway is shortest, which is when a large part of the vehicles is in car-following mode. This happens at speeds of typically 80 km/h; this then is the critical speed. From the capacity and the critical speed, the critical density can be calculated using Equation 3. This varies from typically 20 veh/km/lane to 28 veh/km/lane.

For densities lower then the critical density, traffic is in an uncongested state; for higher densities, traffic is in a congested state. In the uncongested part, the traffic flow increases with an increase of density. In the congested branch, the traffic flow decreases with an increase of density. The part of the fundamental diagram of uncongested traffic states is called the uncongested branch of the diagram. Similarly, the congested branch gives the points for which the traffic state is congested.

The free flow speed is the speed of the vehicles at zero density. At the other end, we find the density at which the vehicles come to a complete stop, which is called the jam density. For the jam density, we can also make an estimation based on the length of the vehicles and the distance they keep at standstill. A vehicle is approximately 5 meters long, and they keep some distance even at standstill (2-3 meters), which means the jam density is 1/(5+3) to 1/(5+2) veh/m, or 1000/(5+3) =125veh/km to 1000/(5+2) =142 veh/km.

Fundamental diagram in different planes

So far, the fundamental diagram has only be presented in the flow density plane. However, since the fundamental equation (Equation 3) relates the three variables to each other, any function relating two of the three variables to each other will have the same effect. Stated otherwise, the fundamental relationship can be presented as flow-density relationship, but also as speed-density relationship or speed-flow relationship. Figure 2 shows all three representations of the fundamental diagram for a variety of functional forms.

In the speed-density plane, one can observe the high speeds for low densities, and the speed gradually decreasing with increasing density. In the speed-flow diagram, one sees two branches: the congested branch with high speeds and high flows, and also a congested branch with a low speed and lower flows.

Shapes of the fundamental diagram

There are many shapes proposed for the fundamental diagram. The data is quite scattered, so different approaches have been taken: very simple functions, functions with mathematically useful properties, or functions derived from a microscopic point of view. Even today, new shapes are proposed. In the remainder of this section, we will show some elementary shapes; the graphs are shown in Figure 2.

Greenshields

Greenshields was the first to observe traffic flows and publish on this in 1934 (Greenshields, 1934). He observed a platoon of vehicles and checked the density of the platoon and their speed. He assumed this relationship to be linear:

$$v = v_0 - ck \qquad 5$$

Note that for k = v/c the speed equals 0, hence the flow equals zero, so the jam density equals c/v_0.

Triangular

The Greenshields diagram is not completely realistic since for a range of low densities, drivers keep the same speed, possibly limited by the current speed limit. The fundamental diagram which is often used in academia is the triangular fundamental diagram, referring to the triangular shape in the flow-density plane. The equation is as follows:

$$q = \begin{cases} v_0 k & if\ k < k_c \\ q_c - \frac{k-k_c}{k_j-k_c}\, q_c & if\ k \geq k_c \end{cases} \qquad 6$$

Truncated triangular

Daganzo (1997) shows a truncated triangular fundamental diagram. That means that the flow is constant and maximized for a certain range of densities. The equation is as follows:

$$q = \begin{cases} v_0 \quad if\ k < k_1 \\ v_0 k_1 \quad if\ k_1 < k < k_c \\ q_c - \frac{k-k_c}{k_j-k_c} q_c \quad if\ k \geq k_c \end{cases} \tag{7}$$

Smulders

Smulders (1989) proposed a fundamental diagram in which the speed decreases linearly with the density for the free flow branch. In the congested branch the flow decreases linearly with density.

Drake

Drake et al. (1967) proposes a continuous fundamental diagram where the speed is an exponentially decreasing function of the density:

$$v = v_0 exp\left(-\frac{1}{2}\left(\frac{k}{k_c}\right)^2\right) \tag{8}$$

Inverse lambda

The capacity drop is not present in the fundamental diagrams presented above. Koshi et al. (1981) introduced an 'inverse lambda 'fundamental diagram. This means the traffic has a free speed up to a capacity point. The congested branch however, does not start at capacity but connects a bit lower at the free flow branch. It is assumed that traffic remains in the free flow branch and after congestion has set in, will move to the congested branch. Only after the congestion has solved, passing a density lower than the density where the congested branch connects to the free flow branch, traffic flows can grow again to higher values. The description is as follows:

$$q = \begin{cases} v_0 k \quad if\ k < k_c \\ v_0 k_1 - \frac{k-k_1}{k_i-k_1} v_0 k_1 \quad if\ k \geq k_1\ and\ traffic\ is\ congested \end{cases} \tag{9}$$

This shape of the fundamental diagram allows for two traffic states with similar densities but different flows. This can yield unrealistic solutions to the kinematic wave model.

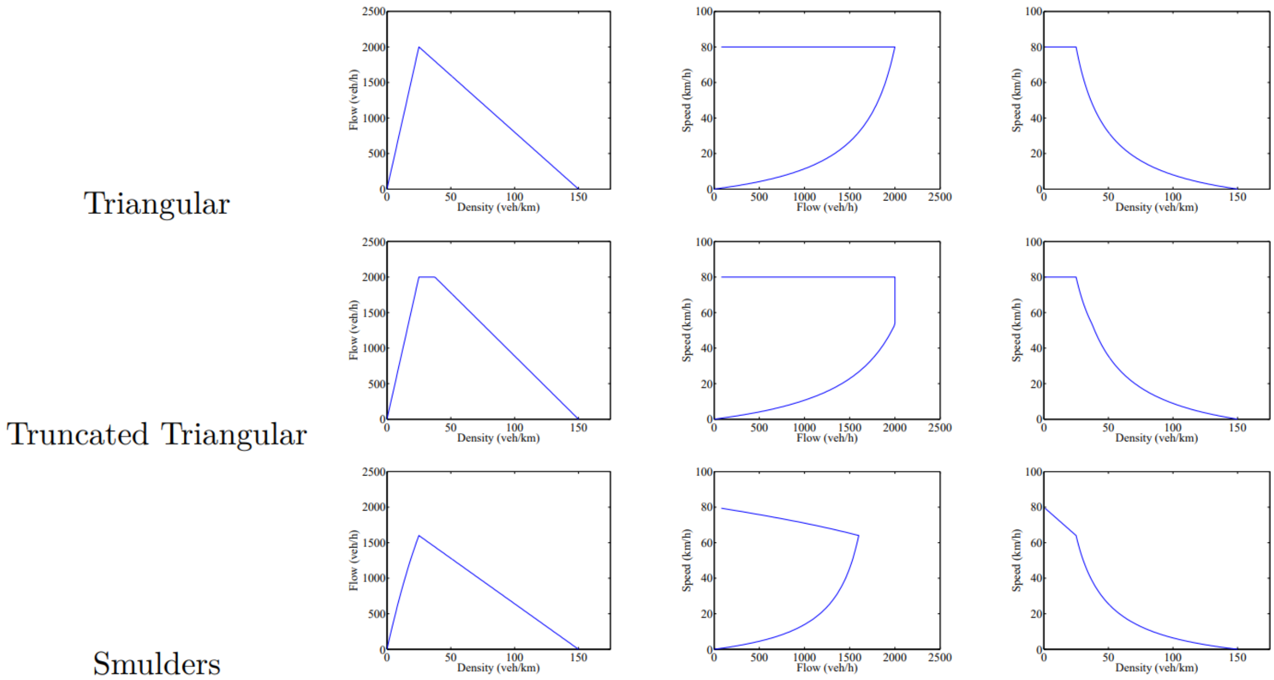
Wu

An addition to the inverse-lambda fundamental diagram is made by Wu (2002). He assumes the speed in the free flow branch to decrease with increasing density. The shape of the free flow branch is determined by the overtaking opportunities, which in turn depend on the number of lanes, l. The equation for the speed is:

$$q = \begin{cases} k\left(1-\left(\frac{k}{k_1}\right)^{l-1} \times v_0 + \left(\frac{k}{k_1}\right)^{l-1} v_p\right) & if\ v > u_p \\ v_p k_1 - \frac{k-k_1}{k_j-k_1} v_0 k_1 \qquad if\ k \geq k_1 \quad and\ traffic\ is\ congested & \end{cases} \qquad 10$$

Kerner

Kerner (2004) has proposed a different theory on traffic flow, the so-called three phase traffic flow theory. For here, is important to note that the congested branch in the three phase traffic flow theory is not a line, but an area.



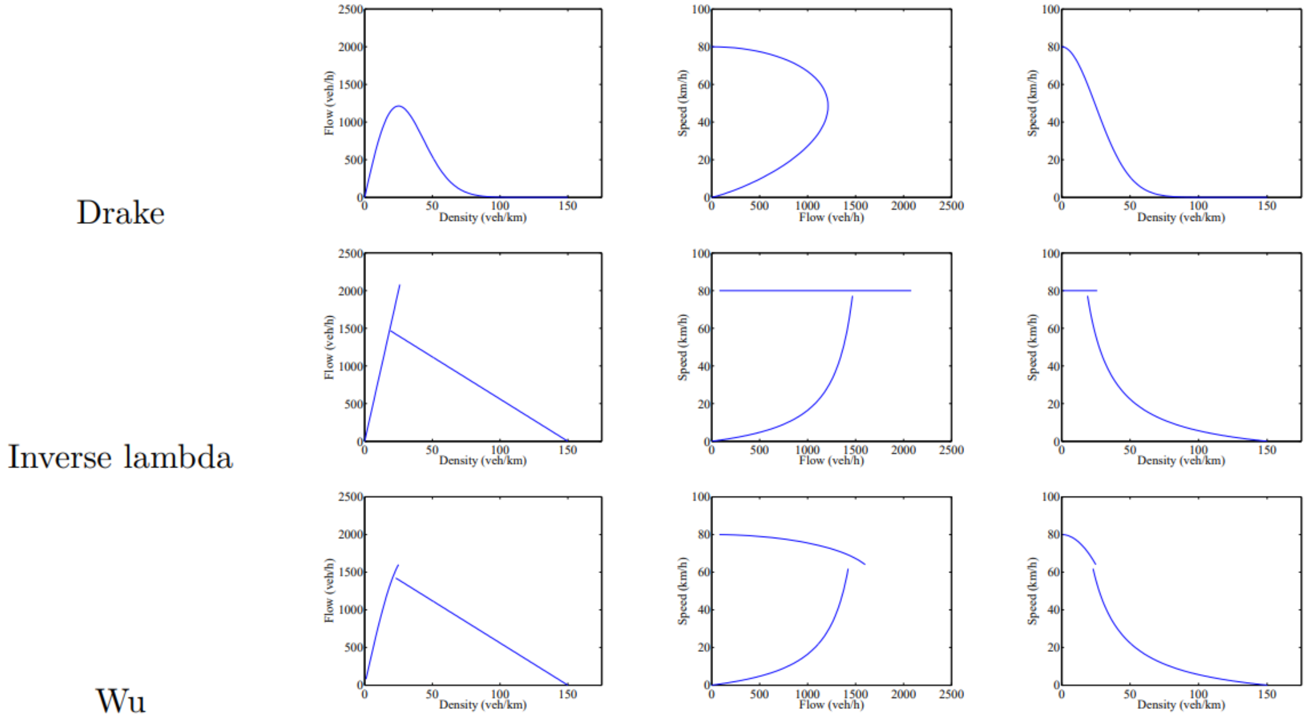


Figure 2: Different shapes of the fundamental diagram

Microscopic behaviour

Previous section showed the equilibrium relationships observed in traffic. This is a result of behavior, which can be described at the level of individual drivers as well. This section does so. First, the equilibrium behaviour will described below. And, discusses hysteresis, i.e. structural off equilibrium behaviour under certain conditions.

Equilibrium behaviour

The fundamental diagram describes traffic in equilibrium conditions. That can happen if all drivers are driving in equilibrium conditions, i.e. all drivers are driving at a headway which matches a speed. Using the relationships in Table 1 one can change a fundamental diagram on an aggregated level to a fundamental diagram on an individual level. This way, one can relate individual headways to individual speeds.

The fundamental diagram gives the average distance drivers keep. However, there is a large variation in drivers' behaviour. Some keep a larger headway, and some drivers keep a smaller headway for the same speed. These effects average out in a fundamental diagram, since the average headway for a certain speed is used. On an individual basis, there is a much larger spread in behaviour.

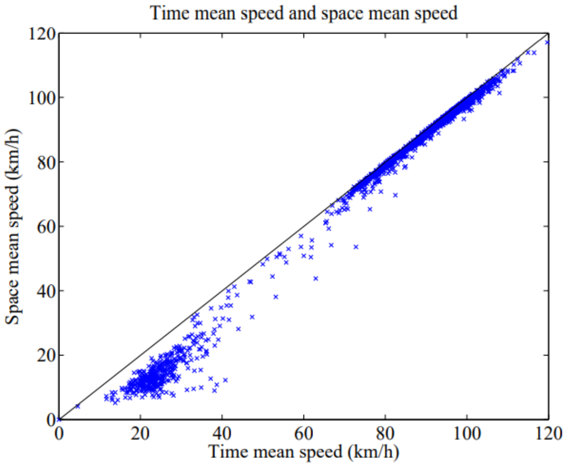
Hysteresis

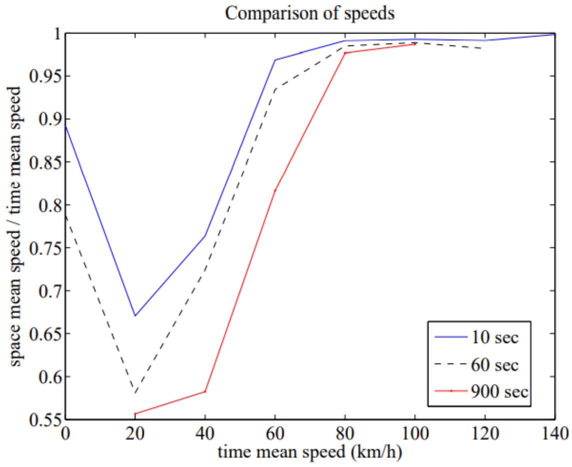
Apart from the variation between drivers, there is also a variation within a driver for a distance it keeps at a certain speed (which we assume in this section as representative of the fundamental diagram). These can be random variations, but there are also some structural variations. Usually the term Hysteresis is used to indicate that the driving behaviour (i.e. the distance) is different for drivers before they enter the congestion compared to after they come out of congestion. That is, the distance at the same speed is different in each of these conditions.
Two phenomena might play a role here:
1. Delayed reaction to a change of speed.
2. Anticipation of a change in speed.

Zhang (1999) provides an excellent introduction to hysteresis. The simplified reasoning is as follows. Let's first discuss case 1, drivers have a delayed reaction to a change of speed. That means that when driving at a speed, first the speed of the leader reduces, then the distance reduces. So during the deceleration process, the headway is shorter than the equilibrium headway. When the congestion solves, first the leader will accelerate, and the driver will react late on that. That means that the leader will shy away from the considered car, and the distance will be larger than the equilibrium distance.

In case 2, if the driver anticipates the change in speed, the exact opposite happens. Before the deceleration actually happens, a driver will already decrease speed (by definition in anticipation), leasing to a larger headway than the equilibrium headway for a certain speed. Under acceleration, the opposite happens, and a driver can already accelerate before would be suitable in case of equilibrium conditions. Hence, in the acceleration phase, the driver has a shorter headway than in equilibrium conditions for the same speed.

In traffic, we expect drivers to have a reaction time. In fact, the reaction time can be derived from the fundamental diagram, as section 8.1 will show later on. It will also show that the best way to analyze car-following behaviour is not comparing the distance-speed relationship for one pair at one moment in time, as shown in figure 4.3. It shows that the drivers have no hysteresis – they copy the movement of the leader perfectly – but still the gap changes with a constant speed. Instead, one should make the analysis of car-following behavior along the axis parallel to the wave speed. Laval (2011) provides a very good insight in the differences one can obtain using this correct technique or using the (erroneous) comparison of instantaneous headways (as in the arrows in Figure 3(b).





(a) The arithmetic mean and the harmonic mean of the speeds of the vehicles passing a cross section of a motorway (b) The arithmetic mean and the harmonic mean of the speeds of the vehicles passing a cross section of a motorway

Figure 4: The effect of inhomogeneities in speeds: the difference in arithmetic mean and harmonic mean speed.

Moving observer

An observer will only observe what is in the observation range. Many observations are taken at a point in space. This point might move with time, for instance a driver might check the number of trucks he is overtaking. In this case, the driver is moving and observing, this is called a moving observer. This section discusses the effects of the speed of the moving observer, and also discusses the effect of observed subjects passing a stationary observer with different speeds.
Basically, the movement at speed υ has no effect on density, but the relative speed of traffic changes. Therefore, applying Equation 3, the relative flow changes. Written down explicitly, one obtains a relative flow q_{rel}:

$$q_{rel} = k(v - \upsilon) \qquad 11$$

Remove so indeed if the observer moves with the speed of the traffic, the observed relative flow becomes zero. In practice if not all vehicles drive at the same speed, the flow needs to be divided into classes with the same speed and the partial flows for each of these classes should be calculated.

$$q_{rel} = \sum_{v}\left(k_{Vehicles\ at\ speed\ v}(v - \upsilon)\right) \qquad 12$$

Local measurements

Suppose there is a local detector located at location x_{detector}. Now we reconstruct which vehicles will pass in the time of one aggregation period. For this to happen, the vehicle i must be closer to the detector than the distance it travels in the aggregation time:

$$x_{detector} - x_i \leq t_{agg} v_i \qquad 13$$

In this formula, x is the position on the road and t_{agg} the aggregation time. For faster vehicles, this distance is larger. Therefore, if one takes the local (arithmetic) mean, one overestimates the influence of the faster vehicles. If the influence of the faster vehicles on speeds is overestimated, the average speed u_{t} is overestimated (compared to the space-mean speed u_{s}).

The discussion above might be conceived as academic. However, if we look at empirical data, then the differences between the time-mean speeds and space-mean speeds become apparent. Figure 4(a) shows an example where the time-mean speed and space-mean speed have been computed from motorway individual vehicle data collected on the A9 motorway near Amsterdam, The Netherlands. Figure 4(b) shows that the time mean speed can be twice as high as the space mean speed. Also note that the space-mean speeds are always lower than the time-mean speeds.

In countries where inductive loops are used to monitor traffic flow operations and arithmetic mean speeds are computed and stored, average speeds are overestimated, affecting travel time estimations. Namely, to estimate the average travel time, the average of $TT = L/v$ is needed, in which L is the length of the road stretch and v the travel speed. Since L is constant, the average travel time can be expressed as:

$$\langle tt \rangle = L \left\langle \frac{1}{v} \right\rangle \qquad 14$$

From this, it follows that the harmonically averaged speed is required for estimating the mean travel time.

Furthermore, since $q = ku$ (Equation 3) can only be used for space-mean speeds, we cannot determine the density k from the local speed and flow measurements, complicating the use of the collected data for, e.g. traffic information and traffic management purposes. As Figure 4(b) already shows, largest relative deviation is found at the lower speeds. In absolute terms, this is not too much, so one might argue this is not importing. However, a low speed means a high density or a large travel time.

For estimating the densities, this speed averaging has therefore a large effect, as can be seen in Figure 5.

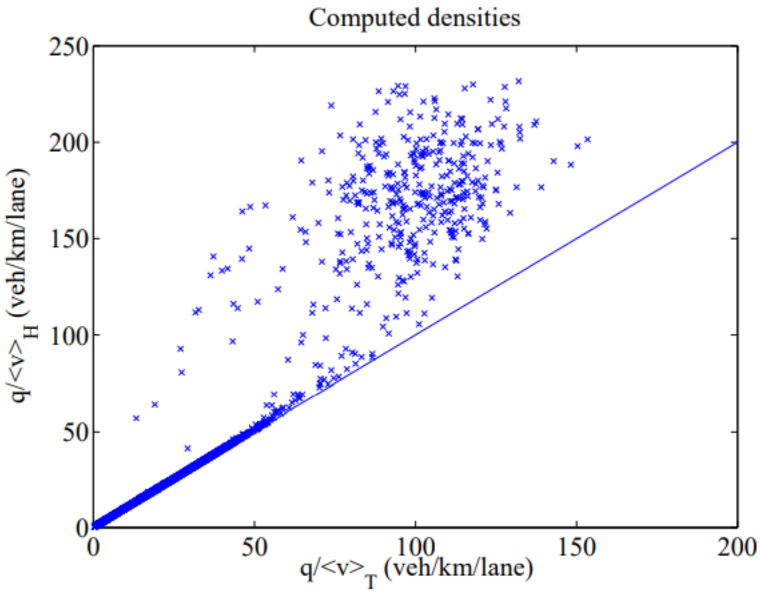


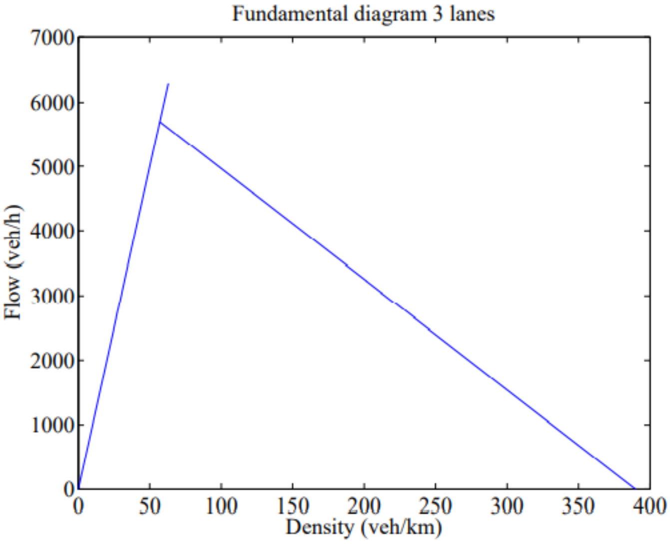
Figure 5: Densities computed using arithmic mean speeds or harmonic mean speeds.

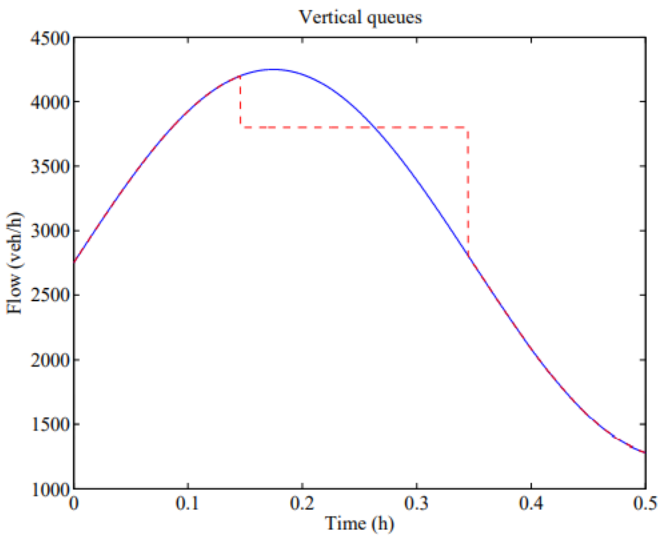
Exercises

Q1/ Draw the simplest fundamental diagram possible for the three lane section (aggregated over all lanes). Explain how you find the values for the relevant points, and give calculate them.

Answer:

The FD should have a capacity drop (see previous questions), so the simplest is to assume an inverse lambda shaped FD The free flow capacity is found in question a: 3x2100 veh/h. Also the free flow speed is known (from the speed figure): 100 km/h.
This gives the free flow branch. The density matching the queue outflow rate can be found on the free flow branch, by looking up the density for the queue outflow rate.
$Kc=qc/v_{free}$=1900/100=19 veh/km. (multiply by 3 for the 3-lane section (1)).
Furthermore, it is known that with a flow of 2 lanes x 1900 veh/h/lane (the flow through the bottleneck), the speed is 22.6 km/h (read from graph), which gives a density of 3800/22.6=168 veh/km. Now, the congested branch can be constructed.
We find the wave speed $w=\Delta q/\Delta k$= ((3*1900)-(2*1900))/ (3*19 -168) =-17 km/h.
The jam density is found by k_j= Kc-qc/w=3*(19-1900/-17.1) = 390 veh/km





Q2/ Express the relative flow in lane i f_i as function of the average headways $< h_i >$, which are given for all lanes i on a motorway.

Answer:

$$f_i = \frac{q_i}{q} = \frac{q_i}{\sum_{all\ lanes} q_i} = \frac{1/\langle h_i \rangle}{\sum_{all\ lanes} 1/\langle h_i \rangle}$$