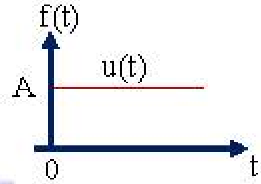
13.1 The Laplace Transform in Electric Circuits Transient:

Laplace Transform has two characteristics:

1. It transforms a set of linear constant-coefficient differential equations into a set of linear polynomial equations.
2. It automatically introduces into the polynomial equations the initial values of the current and voltage variables.

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$

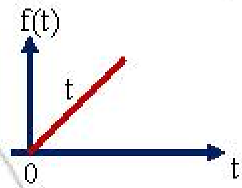
a. Step Function:



$f(t) = u(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$ Apply L.T method:

$\mathcal{L}\{u(t)\} = \int_0^\infty Ae^{-st}dt = \frac{A}{s} = \frac{1}{s} = F(s)$

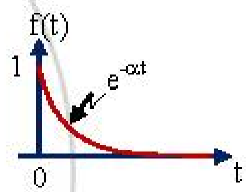
b. Ramp Function:



$f(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$ Apply L.T method:

$\mathcal{L}\{u(t)\} = \int_0^\infty te^{-st}dt = \frac{1}{s^2} = F(s)$

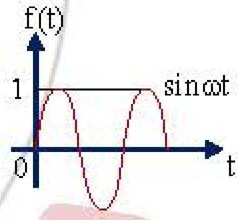
c. Exponential Function:



$f(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ Apply L.T method:

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\alpha t}e^{-st}dt = \left[\frac{e^{-(\alpha+s)t}}{-(\alpha+s)}\right]_0^\infty = \frac{1}{s+\alpha} = F(s)$

d. Sine and Cosine Wave Functions:



$f(t) = \begin{cases} sin\omega t, & t \geq 0 \\ 0, & t < 0 \end{cases}$ Apply L.T method:

$\mathcal{L}\{f(t)\} = \int_0^\infty sin\omega te^{-st}dt$ where $e^{j\theta} = cos\theta + jsin\theta$

$\therefore \mathcal{L}\{f(t)\} = \int_0^\infty Im e^{j\omega t}e^{-st}dt = Im\left[\frac{e^{-(s-j\omega)t}}{-(s-j\omega)}\right]_0^\infty$

$= Im\left[\frac{1}{s-j\omega}\right] = Im\left[\frac{s}{s^2+\omega^2} + j\frac{\omega}{s^2+\omega^2}\right] = \frac{\omega}{s^2+\omega^2} = F(s)$

in the same way: $\mathcal{L}\{cos\omega t\} = \frac{s}{s^2+\omega^2} = F(s)$

e. Special Functions:

$\mathcal{L}\{te^{-\alpha t}\} = \frac{1}{(s+\alpha)^2}$

$\mathcal{L}\{e^{-\alpha t}sin\omega t\} = \frac{\omega}{(s+\alpha)^2+\omega^2}$

$\mathcal{L}\{e^{-\alpha t}cos\omega t\} = \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

f. Differential Function:

$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$

g. Integration Function:

$\mathcal{L}\left\{\int_0^t f(x)dx\right\} = \frac{F(s)}{s} + \frac{1}{s}\int f(t)|_{t=0}dt$

<u>Example:</u> Find the Laplace Transform for 100V DC supply.

Solution:

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty 100e^{-st}dt = 100\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{100}{s}$

13.2 Inverse Laplace Transform:

a. Using Properties:

$\mathcal{L}^{-1}\left[\frac{100}{s}\right] = 100 \quad \mathcal{L}^{-1}\left[\frac{1}{s+\alpha}\right] = e^{-\alpha t} \quad \mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = sin\omega t$

b. Using Partial-Fraction Expansion:

$$F(s) = \frac{s+6}{s(s+3)(s+1)^2} = \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s+6}{s(s+3)(s+1)^2}\right\} = (K_1 + K_2e^{-3t} + K_3te^{-t} + K_4e^{-t})u(t)$$

$$K_1 = \left.\frac{s+6}{(s+3)(s+1)^2}\right|_{s=0} = 2$$

$$K_2 = \left.\frac{s+6}{s(s+1)^2}\right|_{s=-3} = \frac{-1}{4}$$

$$K_3 = \left.\frac{s+6}{s(s+3)}\right|_{s=-1} = \frac{-5}{2}$$

$$K_4 = \frac{d}{ds}\left[\frac{s+6}{s(s+3)}\right]_{s=-1} = \frac{-17}{4}$$

c. Using Long Division:

$$F(s) = \frac{s^4+13s^3+66s^2+200s+300}{s^2+9s+20} = s^2 + 4s + 10 + \frac{30s+100}{s^2+9s+20}$$

$$= s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5}$$

$$\therefore f(t) = \frac{d^2\delta}{dt^2} + 4\frac{d\delta}{dt} + 10\delta - (20e^{-4t} - 50e^{-5t})u(t)$$

13.3 Initial and Final Value Theorems:

a- Initial Value Theorem:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt = sF(s) - f(0)$$

$$\lim_{s\to\infty}\int_0^\infty \frac{df(t)}{dt}e^{-st}dt = \lim_{s\to\infty}[sF(s) - f(0)]$$

as $s \to \infty$ then $\frac{df(t)}{dt}e^{-st} \to 0$

$$\therefore \lim_{s\to\infty}[sF(s) - f(0)] = 0 \quad \Rightarrow \quad \lim_{s\to\infty}[sF(s)] = f(0)$$

$$\boxed{\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[sF(s)]}$$

b- Final Value Theorem:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt = sF(s) - f(0)$$

$$\lim_{s\to 0}\int_0^\infty \frac{df(t)}{dt}e^{-st}dt = \lim_{s\to 0}[sF(s) - f(0)]$$

as $s \to 0$ then $e^{-st} \to 1$

$$\lim_{s\to 0}\int_0^\infty \frac{df(t)}{dt}dt = \lim_{s\to 0}\int_0^\infty df(t) = f(\infty) - f(0)$$

$$\therefore f(\infty) - f(0) = \lim_{s\to 0}[sF(s)] - f(0) \quad \Rightarrow \quad \lim_{s\to 0}[sF(s)] = f(\infty)$$

$$\boxed{\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]}$$

Example: Find the initial and final values of $I_1(s) = \frac{6.67(s+250)}{s(s+166.7)}$, $I_2(s) = \frac{5}{s+2000}$.

Solution:

$i_1(0) = \lim_{s\to\infty}\frac{6.67s(s+250)}{s(s+166.7)} = 6.67\,Amp. \qquad i_1(\infty) = \lim_{s\to 0}\frac{6.67s(s+250)}{s(s+166.7)} = 10\,Amp.$

$i_2(0) = \lim_{s\to\infty}\frac{5s}{s+2000} = 5\,Amp. \qquad i_2(\infty) = \lim_{s\to 0}\frac{5s}{s+2000} = 0\,Amp.$

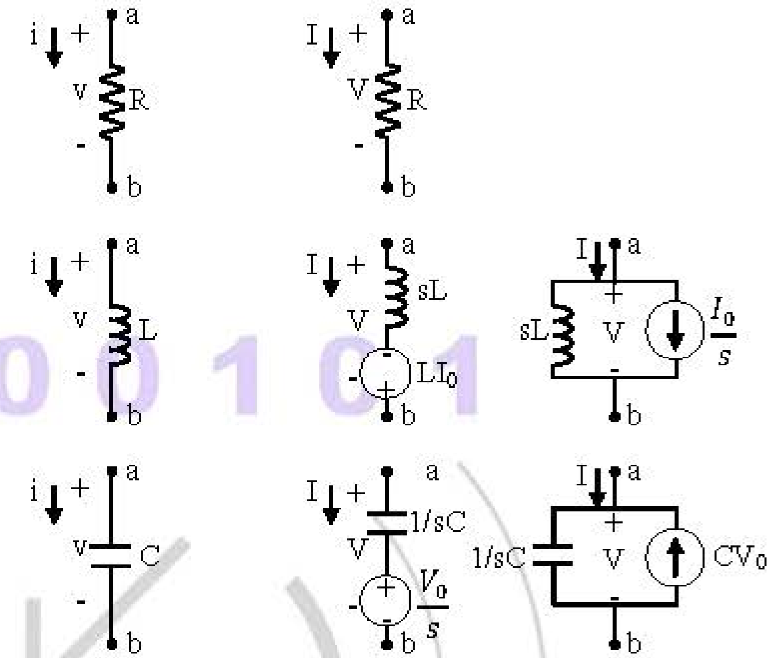
13.4 The Laplace Transform in Circuit Analysis:

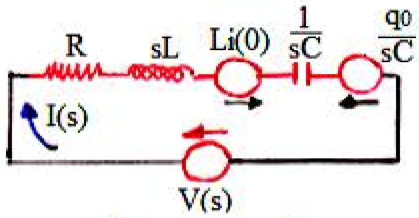
Time Domain | Frequency Domain

$v = Ri$
$V = RI$

$v = L\frac{di}{dt}$
$V = sLI - LI_0$
$I = \frac{V}{sL} + \frac{I_0}{s}$

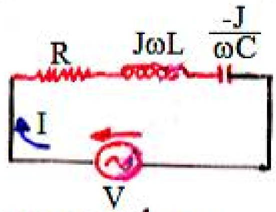
$i = C\frac{dv}{dt}$
$I = sCV - CV_0$
$V = \frac{I}{sC} + \frac{V_0}{s}$





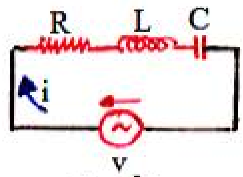
$(R+sL+\frac{1}{sC})I(s)=V(s)-\frac{q_0}{sC}+Li(0)$

Laplace Domain



$RI+J\omega L-J\frac{1}{\omega C}I=V$

Frequency Domain



$Ri+L\frac{di}{dt}+\int\frac{1}{C}idt=v$

Time Domain

a. RLC Series Circuit:

$v = v_R + v_L + v_C$

$v = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt$ Take L.T. for both sides,

$V(s) = RI(s) + L\big(sI(s) - i(0)\big) + \frac{1}{C}\left(\frac{I(s)}{s} + \frac{\int idt|_{t=0}}{s}\right)$

$\left(R + Ls + \frac{1}{Cs}\right)I(s) = V(s) + Li(0) - \frac{q_0}{Cs}$

$\therefore I(s) = \frac{V(s)+Li(0)-\frac{q_0}{Cs}}{R+Ls+\frac{1}{Cs}}$

b. RLC Parallel Circuit:

$i = i_R + i_L + i_C$

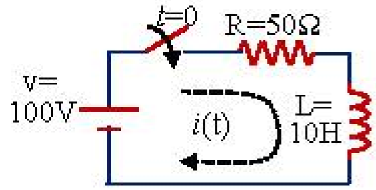
$i = \frac{v}{R} + \frac{1}{L}\int vdt + C\frac{dv}{dt}$ Take L.T. for both sides,

$I(s) = \frac{V(s)}{R} + \frac{1}{L}\left(\frac{V(s)}{s} + \frac{\int vdt|_{t=0}}{s}\right) + C\big(sV(s) - v(0)\big)$

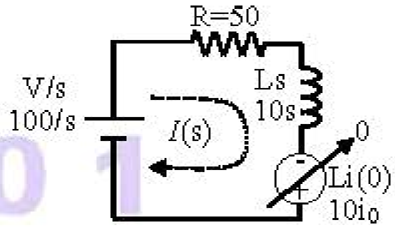
$\left(\frac{1}{R} + \frac{1}{Ls} + Cs\right)V(s) = I(s) + Cv(0) - \frac{i_0}{Ls}$

$\therefore V(s) = \frac{I(s)+Cv(0)-\frac{i_0}{Ls}}{\frac{1}{R}+\frac{1}{Ls}+Cs}$

Example: For the circuit shown, Assuming zero initial condition when the switch closed at t=0. Find:
1. The transient current equation and the voltage across each element.
2. The value of resistance to drop the inductance voltage to 25V at t=25msec.



Solution:
Apply L.T. to the circuit:



1. $i_0 = 0 = i(0)$ no initial energy stored in the coil

$$I(s) = \frac{V/s}{R+Ls} = \frac{100/s}{50+10s} = \frac{10}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

$$A = \left.\frac{V/L}{s+R/L}\right|_{s=0} = \left.\frac{10}{s+5}\right|_{s=0} = 2 = \frac{V}{R}$$

$$B = \left.\frac{V/L}{s}\right|_{s=-R/L} = \left.\frac{10}{s}\right|_{s=-5} = -2 = -\frac{V}{R}$$

$$\therefore I(s) = \frac{V}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right) = 2\left(\frac{1}{s} - \frac{1}{s+5}\right)$$

$$\therefore i(t) = \mathcal{L}^{-1}\{I(s)\} = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right) = 2(1 - e^{-5t}) \quad Amp. \quad t \geq 0$$

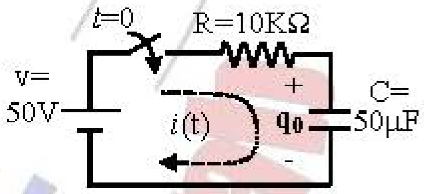
$$V_R = 50 * I(s) = \frac{500}{s(s+5)} = \frac{100}{s} - \frac{100}{s+5}$$

$$\therefore V_R(t) = \mathcal{L}^{-1}\{V_R(s)\} = V\left(1 - e^{-\frac{R}{L}t}\right) = 100(1 - e^{-5t})\ Volt \quad t \geq 0$$

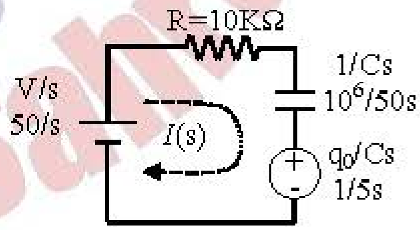
$$V_L = 10s * I(s) = \frac{100}{s+5} \quad \Rightarrow \quad V_L(t) = \mathcal{L}^{-1}\{V_L(s)\} = Ve^{-\frac{R}{L}t} = 100e^{-5t}\ Volt \quad t \geq 0$$

2. $V_L(25msec) = 25 = 100e^{-\frac{R}{10}*25*10^{-3}} \quad \Rightarrow \quad R = 554.52\Omega$

Example: For the circuit shown, find the transient current equation and the voltage across each element when the switch closed at t=0 with q_0=10μCoulomb. Also find the value of resistance to drop the capacitance voltage to 25V at t=25msec.



Solution:
Apply L.T. to the circuit:



$$I(s) = \frac{\frac{V}{s} - \frac{q_0}{Cs}}{R + \frac{1}{Cs}} = \frac{\frac{50}{s} - \frac{1}{5s}}{10*10^3 + \frac{10^6}{50s}} = \frac{0.00498}{s+2} = \frac{v - \frac{q_0}{C}}{R} \times \frac{1}{s + \frac{1}{RC}}$$

$$i(t) = \mathcal{L}^{-1}\{I(s)\} = \frac{V - \frac{q_0}{C}}{R}e^{-\frac{1}{RC}t} = 5e^{-2t}\ mA \quad t \geq 0$$

$\therefore i(t)|_{q_0=0} = \frac{V}{R}e^{-\frac{1}{RC}t}\ A \quad t \geq 0$ and $V_R = 10 * 10^3 * I(s) = \frac{50}{s+2}$

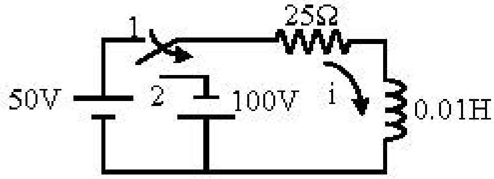
$$\therefore V_R(t) = \mathcal{L}^{-1}\{V_R(s)\} = (V - \frac{q_0}{C})e^{-\frac{1}{RC}t} = 50e^{-2t}\ Volt \quad t \geq 0$$

$V_C = \frac{1}{5s} - \frac{10^6}{50s} * I(s) = \frac{s-498}{5s(s+2)} = \frac{50}{s} - \frac{50}{s+2}$ with approximated values

$$\therefore V_C(t) = \mathcal{L}^{-1}\{V_C(s)\} = (V - \frac{q_0}{C})e^{-\frac{1}{RC}t} = 50(1 - e^{-2t})\ Volt \quad t \geq 0$$

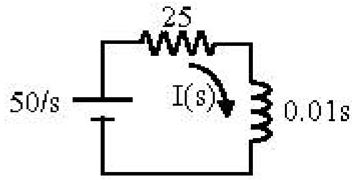
$$V_C(25msec) = 25 = 50\left(1 - e^{-\frac{25*10^{-3}}{R*50*10^{-6}}}\right) \quad \Rightarrow \quad R = 721.35\Omega$$

Example: In the circuit shown, if the switch was be in position "1" for a long time then at t=0 moved to position "2". Find the transient current equation in each case using L.T. method.



Solution:

The switch in position "1"; Apply L.T. to the circuit:



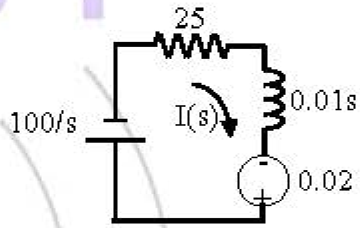
$$I(s)=\frac{\frac{V}{s}}{R+Ls}=\frac{\frac{50}{s}}{25+0.01s}=\frac{5000}{s(s+2500)}=\frac{A}{s}+\frac{B}{s+2500}$$

$$A=\frac{5000}{s+2500}\Big|_{s=0}=2\ \ and\ \ B=\frac{5000}{s}\Big|_{s=-2500}=-2$$

$$\therefore\ i(t)=\mathcal{L}^{-1}\{I(s)\}=2(1-e^{-2500t})\quad Amp.\quad t<0$$

$\lim_{s\to 0} sI(s)=2\ Amp.=\lim_{t\to\infty} i(t)$ steady-state

The switch in position "2"; Apply L.T. to the circuit:

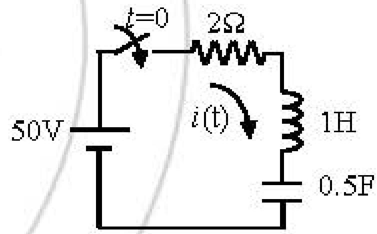


$$I(s)=\frac{\frac{V}{s}-Li_0}{R+Ls}=\frac{\frac{100}{s}-0.02}{25+0.01s}=\frac{-2s+10000}{s(s+2500)}=\frac{A}{s}+\frac{B}{s+2500}$$

$$A=\frac{-2s+10000}{s+2500}\Big|_{s=0}=4\ \ and\ \ B=\frac{-2s+10000}{s}\Big|_{s=-2500}=-6$$

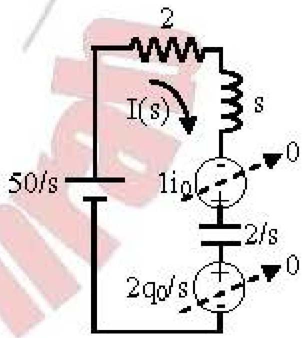
$$\therefore\ i(t)=\mathcal{L}^{-1}\{I(s)\}=4-6e^{-2500t}\quad Amp.\quad t\ge 0$$

Example: For the circuit shown, when the switch closed at t=0 and by assuming zero initial charge on the capacitor terminals. Find the transient current equation w.r.t. time and the voltage across each element.



Solution:

Apply L.T. to the circuit:



$$I(s)=\frac{V/s}{R+Ls+\frac{1}{Cs}}=\frac{50/s}{2+s+\frac{2}{s}}=\frac{50}{s^2+2s+2}=\frac{A}{s+1+j}+\frac{B}{s+1-j}$$

$$A=\frac{50}{s+1-j}\Big|_{s=-1-j}=25j\ \ and\ \ B=\frac{50}{s+1+j}\Big|_{s=-1+j}=-25j$$

$$\therefore\ i(t)=\mathcal{L}^{-1}\{I(s)\}=j25\left(e^{-(1+j)t}-e^{-(1-j)t}\right)$$

$$=50e^{-t}\left[\frac{e^{jt}-e^{-jt}}{2j}\right]=50e^{-t}sint\quad Amp.\quad t\ge 0$$

$$V_R=R*I(s)=\frac{100}{s^2+2s+2}=\frac{100}{(s+1)^2+1}$$

$$\therefore\ V_R(t)=\mathcal{L}^{-1}\{V_R(s)\}=100e^{-t}sint\ Volt\quad t\ge 0$$

$$V_L=Ls*I(s)=\frac{50s}{s^2+2s+2}=\frac{50(s+1)}{(s+1)^2+1}-\frac{50}{(s+1)^2+1}$$

$$\therefore\ V_L(t)=\mathcal{L}^{-1}\{V_L(s)\}=50e^{-t}cost-50e^{-t}sint=50e^{-t}(cost-sint)\ Volt\quad t\ge 0$$

$$V_C=\frac{2}{s}*I(s)=\frac{100}{s(s^2+2s+2)}=\frac{A}{s}+\frac{Bs+C}{(s+1)^2+1}=\frac{50}{s}-\frac{50s+100}{(s+1)^2+1}$$

$$\therefore\ V_C(t)=\mathcal{L}^{-1}\{V_C(s)\}=50(1-e^{-t}cost-e^{-t}sint)\ Volt\quad t\ge 0$$

Note:

$$\mathcal{L}\{sin(\omega t+\psi)\}=\frac{s*sin\psi+\omega*cos\psi}{s^2+\omega^2}\qquad ...(*)$$

$$\mathcal{L}\{cos(\omega t+\psi)\}=\frac{s*cos\psi-\omega*sin\psi}{s^2+\omega^2}\qquad ...(**)$$